PHYSICS

STEFAN MACHLUP

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Case Western Reserve University

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PHYSICS

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About the cover

How many "orders" of diffraction can you see? There is one order on the right and two on the left. On the vertical wall you see the "zero-order" white stripe, the image of the vertical slit through which the white light has passed before going through the diffraction grating. To the left of this "central maximum" there is first a dark region (call it UV) and then violet and blue, which are followed a few degrees further by orange and red. That is the first-order spectrum. The second-order spectrum starts with violet and blue again, but this time we see the green and the yellow, going over into orange and red (see Chapter 12).

S.C.E.R.T., West Bengal
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PREFACE

As you look at this book, your first thought might be, "How am I going to learn all this in one school year?" The answer is that every physics text has much more material than any student can digest in a one-year course. This gives the teachers who plan your course a lot of choices, and that is a good thing. So don't be disappointed if your instructor announces plans to skip chapter so-and-so or be surprised if chapters are assigned out of order. Courses in different colleges vary enormously in their content and in their style, even though instructors may be using the same text. Something must be omitted from every course, and the choice of what to omit does not make the course better or worse. Every physics course is designed to build your physical intuition. In the process it tries to make you comfortable with quantitative argument, and to help you build your skill at solving quantitative problems.

This book has been written as a text for the introductory physics course that is often entitled "College Physics." Although it is aimed primarily toward the student who has never studied physics before and who may plan to study no more physics after this course, it is sophisticated and its language is technical. You will find a conscious emphasis on applications to the human body. You will see a lot of arithmetic and "ninth-grade" algebra, but no calculus except in an appendix. Some chapters have sines and cosines. The book is intended to sound the way a physicist talks.

As you leaf through the book, you will notice that each chapter is equipped with a number of study aids. Each starts with a list of *Questions This Chapter Will Answer.* Some people might consider these as "goals." In fact, the goals are much loftier. But these questions tell you what to look forward to. At the end of each chapter there is a short *summary,* then a list of *key terms,* which are defined succinctly. This is followed by a set of *review questions.* The key terms list has definitions of the terms found in the chapter, both in the text and in the *problems,* and sometimes of terms that aren't used at all, but that you might run into in your reading of scientific literature. The first time one of the key terms appears in the chapter, it is printed in **boldface** type, so you know right away where you can find a definition. All of these terms also appear alphabetically in the *index* at the back of the book, with an *italic* page number referring to the appropriate key terms list, and roman-type page numbers referring to other appearances of the term. This feature makes the book a sort of physics encyclopedia, or quick-reference book. You will appreciate this feature especially when you consult it a few years from now.

On the other hand, the book is rather short, as physics texts go. It is purposely *not* encyclopedic. Its purpose is to be pedagogic, to teach as lucidly as possible. Many technical applications found in "engineering physics" texts have been left out. A lot of health-science applications have been put in, because so many students who study physics are planning to enter health-related careers. In taking a look at the problems at the end of each chapter, you will frequently notice some biological applications.

Each chapter of this book starts with the more familiar and works toward the more abstract. You will probably find that the chapters are easy at the beginning and get harder the further you read. In general, the more mathematical parts are further along in a chapter. This gives your instructor some choice about how mathematical your course should be. Very likely, the choice will be to be rather mathematical in some chapters and not in others. The chapter on entropy and information is an exception; it is mathematical throughout. The math is arithmetic and algebra. The only calculus you will see is there (in an appendix and in an occasional footnote) to help make connections for the reader who recognizes derivatives and integrals. But please do not let the rather elementary mathematical level of the book mislead you into thinking that the physics has been oversimplified.

The order of the *problems* at the end of each chapter tends to follow the sequence of topics in the chapter. For educational reasons the problems are not overtly keyed to specific sections. They do have names that give away what they are about. They also have stars, to grade them by difficulty:

* easy
** requires thought
*** challenging

Not surprisingly, two-star problems abound. Your instructor has been provided with a *Problem Table of Contents* to facilitate the planning of problem assignments. Quite often the problems go into areas not covered by the text at all and are therefore rather lengthy. This again is a feature that gives your instructor a choice of the topics your course will include. Remember, whatever form your course takes, it will assign only a fraction of the problems. But your best learning will take place while you are struggling with the tough ones. Don't expect to find all of the answers by leafing through the chapter. Numerical answers to the odd-numbered problems are in the back of the book. Your solution to a problem should, of course, include a lot of writing, so that you can go back later and follow the logic. A rather complete step-by-step solution also helps you to spot mistakes.

Should you try to memorize formulas? The answer is: Not before you understand them. Quantitative arguments are generally introduced with a numerical example so that you will immediately get a feeling for the size of the various physical quantities. You will need to follow the arithmetic to follow the logic. After you have worked through the numbers, the situation is generalized by substituting symbols for the numbers and following the argument algebraically, that is, doing a derivation. What comes out is a formula. You should try to learn the derivation. Then the formula is yours automatically. Occasionally, learning the formula as if it were a poem makes it a mnemonic—an aid that jogs your memory for the logical argument. It should never be a substitute for the logic.

Now you know why you don't see sections entitled "Worked Example" following the derivation of a formula. The worked example in fact, preceded the derivation. But you do see *exercises* right in the text. They will slow you down in your reading. They are intended to do this. They make parts of the book similar to a "programmed-learning" text. You have to do the work, to carry the ball for a while. You will also succeed in solving them quickly. Your new understanding will be quickly rewarded. Your instructor will let you know whether to hand in your solutions to the exercises. But handed-in or not handed-in, you will find it very useful to do them.

I owe huge debts of gratitude to a number of my colleagues who read and criticized early drafts of entire chapters. My students, who had to read rough, photocopied handout notes, have been kind and diplomatic, but did catch errors and ambiguities. Every year that I test-taught a chapter, the new version was typed and reproduced by Lucille Rosenberg, my wonderful secretary. Actually, I have enjoyed her help in many phases of the work. Most of the chapters have had the benefit of line-by-line editing by Professor James Lock of Cleveland State University. His keen insight, unflagging attention to detail, and gentle sensitivity to someone else's teaching style have had a profound impact—far beyond the spotting of mistakes. After the manuscript was "finished," it was sent by Wiley to six reviewers, college professors who, at my request, have remained anonymous. This anonymity was intended to encourage them to criticize freely, and they worked very hard at it. I also worked very hard at incorporating many hundreds of their suggestions.

Robert Quail solved all the end-of-chapter problems. In the process, he pointed out weaknesses in exposition as well as lots of embarrassing mistakes—fortunately just *before* they would have gone to the compositor. The book has been much improved by his advice. He is an excellent physicist and a thoughtful teacher, and on many occasions I fought or rejected his advice. For example, he thought it quite wrong to put so much didactic material in the problems rather than in the text. I believe that this feature not only makes the book shorter but gives your instructor a greater role in shaping the course to your common needs.

Two Cleveland artists, Linda Zolten and George Wang, created a bridge between my impossibly rough sketches and the Wiley Illustration Department by providing first drafts of the figures.

The College Division of John Wiley has an impressive concentration of creative and helpful people, many of whom contributed to producing this book. Here I mention only three, my heroes: Safra Nimrod (photo editor), whose artistic energy and impeccable taste you can see in the photos; Elaine Wetterau (copy editor), cultured, literate, a stickler for detail, yet so flexible as to allow me my idiosyncratic physicist's style; and Robert McConnin, formerly physics editor, who made it all happen. All the hard decisions were his.

Stefan Machlup

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Free fall - before the parachute is opened.

1

FORCE AND MOTION

SOME QUESTIONS THIS CHAPTER WILL ANSWER

Why don't you have to be far from the earth to experience "zero gravity"?

Why couldn't you play basketball in an orbiting space platform? To play tennis, what kind of strokes would you have to use?

What kind of experiment would you design to test Newton's First Law here on earth?

The Skylab astronauts used a vacuum cleaner rather than a broom for cleaning up. Why?

Under conditions of constant acceleration, what is the shape of the velocity-versus-time curve?

The area under such a velocity-versus-time curve is the ____________.

Accelerating uniformly from rest, an object goes 2 times / 3 times / 4 times as far in the first two seconds as it does in the first second.

Acceleration can be expressed in miles per hour per second. Is meters per square second also a unit of acceleration?

Under what conditions is it true that displacement goes as the square of the time?

Under what conditions is it true that displacement is directly proportional to the time? The proportionality constant is the ____________.

An upward acceleration of 9.8 meters per second per second doubles your weight here on earth. What effect does twice that much upward acceleration have on your weight?

By how much does a downward acceleration of 9.8 m/sec^2 reduce the apparent weight of a 150-pound man?

A force of 1 newton applied to a mass of 1 kilogram imparts to it an acceleration of 1 m/sec^2 in the direction of the force. What does the same force do to a 2-kg mass?

Here on earth, a 1-kilogram mass weighs 9.8 newtons when it is not accelerated. The same object weighs more / less / the same / on the moon.

The Principle of Equivalence states that the gravitational force on an object and the force needed to impart a given acceleration to it are proportional. The particular value of acceleration that makes these two forces equal is called ____________.

Here on earth its value is ____________. A hundred miles up its value is greater than / less than / the same as this.

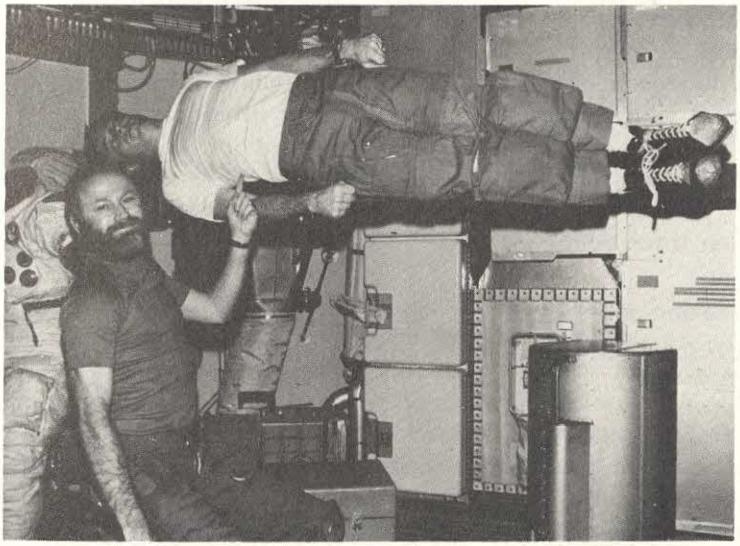
1.1 MOTION WITHOUT FORCE

This chapter begins the study of **classical mechanics.** That means it is about motion and about forces. It starts with an essay called "Space Travel" because you will actually find it easier to understand the motion of objects in an orbiting satellite than motions in a physics laboratory here on earth. The reason is that in an orbiting spacecraft, things have no weight.

Space Travel: An Essay on Weightlessness

Down here on earth, everything we touch has weight. If we drop something, it falls down. Not so for the orbiting astronaut. If he lets go of his toothbrush, it just stays floating in midair. If he gives it a little shove, it moves off in a straight line. That may be unfamiliar behavior, but it is pretty easy to visualize. The astronaut's world is amazingly symmetrical.

When his craft is in **orbit,** the astronaut loses all sense of the heaviness of his body. Raising an arm takes no more effort than lowering an arm. Here on earth, we always have that special direction we call down, and the opposite direction, up. For the astronaut in orbit, all directions feel alike. Here on earth, the falling toothbrush picks up speed as it falls down. For the astronaut, in an orbiting spacecraft, once an object is started moving, it just keeps going in a straight line until it bumps into something. Until something interferes with its motion, it neither speeds up nor slows down. Here on earth, when you throw a ball, it curves downward. Think about the curve traced by the basketball on a "rimless" shot. How much easier it is to describe the motion of a ball thrown in an orbiting spacecraft: It flies in a straight line at constant speed.



Weightlessness: Astronauts in orbiting spacecraft.

FIGURE 1.1 Astronaut at blast-off. (He looks pretty well squashed, lying on his support couch.)

To accustom your intuition to frames of reference in which your own weight changes, think about your experience with elevators. You have probably ridden on a fast elevator. Remember the feeling as the elevator starts down? Suddenly you seem to weigh less. The floor seems to fall out from under you, and your stomach seems to press up. As the "down" elevator slows down, you temporarily feel yourself geting heavier, feel the floor pushing up on you harder. The same thing happens as the elevator starts to move up: Your feet seem to press harder on the floor; your stomach seems to sink. Moments later, while the elevator is moving steadily upward, your weight is back to normal. But as the elevator stops its upward motion, the floor seems to drop away and your stomach feels like it is getting close to your throat. All this feels the same approaching the fifth floor as the twenty-fifth floor.

Now think about the space traveler. As his rocket blasts off and starts moving upward, he becomes very heavy. As long as the craft accelerates upward, the supporting couch pushes up on his body, tending to compress him from below. Perhaps we should say from *behind*. The push on his body is in the forward direction, the direction in which the rocket is accelerating. When the "burn" is over, the push stops. When his craft is "in orbit," he feels no upward push, no downward push. His body is weightless. (See Fig. 1.1)

Instead of talking about weightlessness, physicists also use the term **free fall.** For our space traveler, after the upward acceleration stops, the free fall is an upward fall, of course—at least for a while. The term *free fall* is used regardless of whether the fall is down or up.

You experience weightlessness every time you jump. When you lose contact with the floor, you are in free fall. To have time to savor it for more than a second, try a diving board or trampoline. If you close your eyes while you are in the air, you will notice that the feeling is quite the same on the way up, at the top, and on the way down: While unsupported, you are in free fall. When you reestablish contact with the trampoline, it pushes up on you. It slows down your downward motion, stops it, then starts your upward motion. During this time, you become a heavy-weight. Your feet press into that supporting surface. On the way up, that pressure lessens until you lose contact and "go into orbit," becoming temporarily weightless.

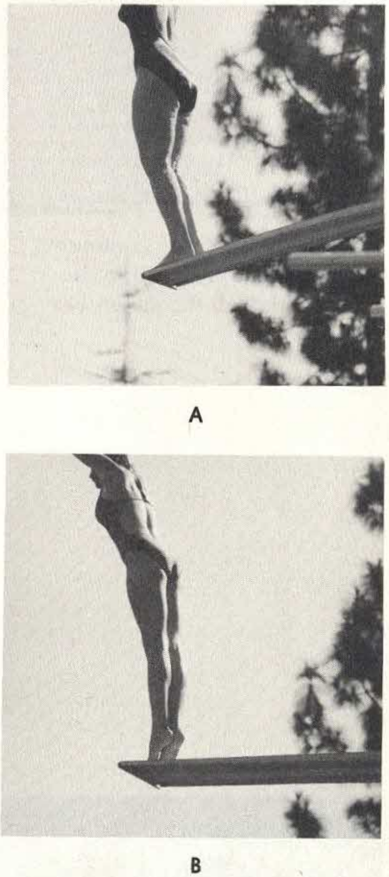
Think about our three examples:

The elevator starting and stopping.

The astronaut blasting off and finally going into orbit.

The trampoline.

In all three situations, the presence or absence of forces on the body does not depend on which way the body is headed. The elevator floor has to push harder than usual both to start the upward trip and to stop the downward trip. The same goes for the trampoline. The astronaut gets pushed hard from below at blast-off. That push gets less as the thrust of the rocket decreases. When the push stops altogether, the astronaut feels just like you would if someone cut the cable on the elevator. The book you hold in your hand would no longer be heavy. If you let go of it, it would stay right where you released it. In other words, it falls at the same rate you do. And that would be so whether the elevator was moving up or down before the cable was cut or wasn't moving at all.



(a) The board is at its lowest point, the downward motion has stopped, and the pressure of the feet on the diving board is at its maximum. (b) As the feet lose contact with the diving board, free fall (= weightlessness) begins.

Some Medical Aspects of Weightlessness

Here on earth, your head is higher than the rest of your body most of the time. It pushes down on your neck. When you lie in bed, that stress is relieved. Under conditions of weightlessness, all the stresses on bones, on joints, and on muscles that have to do with supporting parts of ourselves against gravity are relieved. So we have to worry whether, on a long space voyage, those bones will soften, those joints stiffen, those muscles weaken. These worries are quite similar to worries that arise in a long hospitalization. The astronauts return to earth a couple of inches taller, so that their trousers seem too short. That only lasts a day or two, until their weight shrinks their bodies back to size. But during the trip, their legs tend to get thin and their upper bodies seem bloated. The face is rounder than usual and flabbier. What happens is that body fluids that normally have a tendency to flow downward distribute more uniformly over the body. When you got up out of bed this morning, the fluid pressure in your legs increased and became higher than the pressure in your head. Under conditions of weightlessness, that pressure difference disappears.

You may have seen some of the television tapes made by the Skylab astronauts. One shows how they exercise under conditions of "zero gravity" in order to stay healthy. It is fun to see them floating around in the large cabin. They come to rest as easily on the ceiling as on the floor. If you ask "Which way is up?," the question has no answer. There is no "up" and no "down." That means there is no lifting. The concept of *carrying* has only a vague meaning. Even walking is impossible without hanging on: The slightest push of a foot against the floor sends the whole body moving off the other way.

Eating and Drinking

The astronaut does not heap peas on his plate and pick them up with a fork. Weightless peas just won't heap and won't stay put on a moving fork. (See Fig. 1.2.) Since everything in the spacecraft falls at the same rate, everything stays put relative to everything else unless it is pushed or pulled. Anything that is not fastened floats freely. Anything that starts moving just keeps going in the same direction at the same speed until it bumps into something else. A salt shaker would be hopeless. There would be salt all over. The **Skylab** crews used a squeeze bottle with salt water. Here is a report of an interview:

S.C.E.R.T., West Bengal
Date.............................
Acc. No.

S.C.E.R.T., West Bengal
Library
Calcutta
25/3, B. C. Road

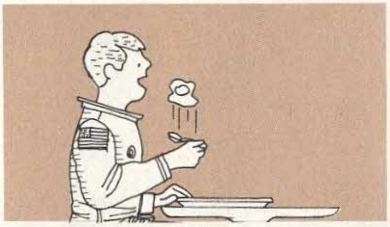


FIGURE 1.2 Bit of egg continuing on its straight-line path to the mouth although the spoon has stopped.

> Keeping food from flying off the silver was a real problem in Skylab. If an astronaut spooned up a bit of egg and then paused with his hand halfway to his mouth—to ask someone to pass the salt, say—the egg would leave the spoon and, continuing its trajectory, land on the astronaut's face.
>
> The trick was not to stop the spoon on its trip. . . . They had to have their mouths in exactly the right spot, and open, for there was no way to stop the spoonful of food once it had started on its way.[1]

Liquids do not have smooth surfaces. The slightest vibration will cause a liquid to fill its container by breaking into droplets. The concept of *pouring* loses all meaning in a world without an up and a down. When the force of gravity has been "canceled out" by giving in to it, that is, by falling, other forces ordinarily much smaller than weight become more noticeable. Although there is no such thing as "heaping" in a weightless environment, the forces of adhesion and surface tension now dominate. So liquids tend to form into spherical drops. They only flow if they are pushed. Anything that gets loose in the cabin, whether a cracker crumb or a drop of milk, just keeps floating until it adheres to some surface. If you spill, you can wipe, but you mustn't sweep.

Can you imagine what washing is like?

Newton's First Law

> An object persists in its state of rest or motion in a straight line at constant speed unless acted upon by an external force.

Hovercraft is supported on a cushion of moving air, just like the air-track gliders.

That is a formal statement of the First Law of Motion. In an orbiting spacecraft, that law is a generalization of easily observable behavior of objects loose in the cabin. In everyday life here on earth, on the other hand, it is a difficult law to verify. The motion of the objects around us is evidently affected by the force of gravity. For objects that are not falling, the gravitational force is canceled out by an opposing upward force, a supporting force. To verify Newton's First Law, should we concentrate on objects in horizontal motion? How about a toy car moving on a horizontal table or track? You note that it slows down unless it is pushed or pulled. That is because of the force of friction, you will be quick to point out. There is fiction from the track, friction from the bearings, even air friction. If we could remove all friction, the car would keep on moving at the speed it had when you stopped pushing. That is what the First Law would predict.

Try to experiment with an **air track** or air table. Like a **Hovercraft,** the gliders are supported on a cushion of moving air, so they don't touch the track at all. If the track is accurately level, you can check how well Newton's First Law is obeyed by timing various distances along the track. The photograph shown here was made with a stroboscope. Problem 1.1 asks you to use this multiexposure photo to find the glider's speed at various places

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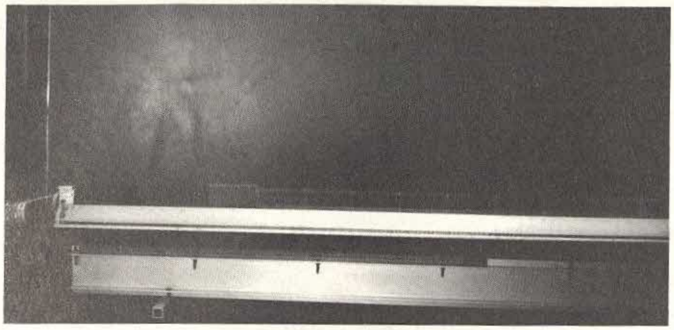
[1] H. S. F. Cooper, *The New Yorker*, **52,** 34 (1976).



Stroboscope flashed every 0.1 second in making this multiexposure photo of glider in uniform motion on air track.

along the track. Even a quick inspection will satisfy you that the glider goes approximately equal distances in equal times.

The air track represents our attempt to make measurements on motion with as little friction as possible. Making the track nicely horizontal allows us to forget about the force of gravity. The point is to create something close to the ideal situation that the First Law is about: motion without force. In our everyday lives, we are accustomed to motions that require force in order to persist. If we want something to go, we usually have to push. The First Law of Motion says that things go without pushing. No wonder it was not discovered until the seventeenth century! It is *not* intuitively obvious to someone living here on earth. In the orbiting space craft, by getting rid of weight, we get rid of most friction as well. For the astronaut in orbit, an object loose in the cabin is very close to the ideal "no-force" situation. For the astronaut, the First Law is intuitive.

In the next section, we take up the relation between force and motion. To make things simple, we start with situations in which the force is in the direction of the motion.

1.2 ACCELERATION

A drag racer might get up to 60 miles per hour in 6 seconds. That is an average acceleration of 10 miles per hour per second.

EXERCISE 3 What did we divide by what to arrive at that result?

EXERCISE 4 What average acceleration is needed to speed up from rest to 60 miles per hour in 5 seconds?

Acceleration is defined as the rate of change of the velocity. With Newton's First Law in mind, you know that acceleration requires force. Newton's Second Law says that the force and the acceleration are proportional.

KINEMATICS

A comfortable walking speed is about 1 meter per second (=3600 m/hour). Suppose that we plot speed versus time, say for 10 seconds, while the speed remains constant at 1 m/sec (see Fig. 1.3). Then the area enclosed in that rectangle (the "area under the curve") has an immediate interpretation:

$$\text{height of rectangle} \times \text{base of rectangle} = \text{area}$$

$$\text{speed} \times \text{time} = \text{distance covered}$$

$$1\frac{\text{m}}{\text{sec}} \times 10\ \text{sec} = 10\ \text{m}$$

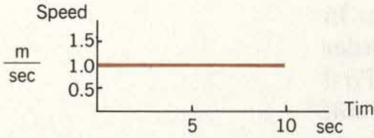


FIGURE 1.3

You may look at the graph and say to yourself, "That is about 1 square inch." But we are talking about the interpretation: The vertical axis is measured off in meters per second; the horizontal axis is measured in seconds. The product of a vertical distance times a horizontal distance (an "area") has to be measured in

$$\left(\frac{\text{meters}}{\text{second}}\right) \times (\text{seconds}) = \text{meters}$$

If the speed of walking changed from 1.0 m/sec to the 1.5 m/sec abruptly at $t = 10$ sec, then after 15 seconds the walker will have covered 17.5 meters (= 10 m in the first 10 sec + 7.5 m in the next 5 sec). Again, the area under the speed versus time curve represents the distance covered (see Fig. 1.4).

EXERCISE 1 Convince yourself that his average speed during those 15 seconds was 1.17 m/sec (not 1.25 m/sec). (See Fig. 1.5.)

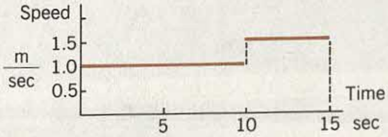


FIGURE 1.4

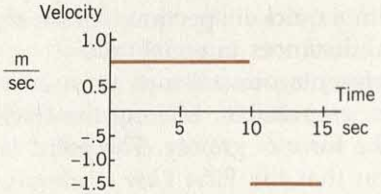


FIGURE 1.5

HINT: **Distance = rate × time.** If he had turned around after 10 seconds at 1 m/sec, and walked backward at 1.5 m/sec, we would have subtracted that area below the axis so that his final displacement at the end of the 15 seconds would be 2.5 m:

$$\underset{(0\ \text{to}\ 10\ \text{sec})}{10\ \text{meters}} - \underset{(10\ \text{to}\ 15\ \text{sec})}{7.5\ \text{meters}} = 2.5\ \text{meters}$$

In this way we give meaning to "negative areas" as "backward displacements," just as we plotted forward velocities as "up" and backward velocities as "down."

Constant Acceleration

Now suppose that the walker starts from rest and accelerates uniformly to 1.5 m/sec by the end of 10 seconds (see Fig. 1.6). His average velocity during those 10 seconds is 0.75 m/sec. So his displacement at the end is

$$0.75\,\frac{\text{m}}{\text{sec}} \times 10\ \text{sec} = 7.5\ \text{m}$$

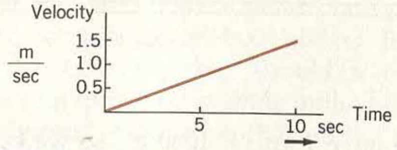


FIGURE 1.6

One way to see this geometrically is to sketch in a horizontal dotted line at $v_{\text{average}} = 0.75$ m/sec and see that the displacement is the area under the dotted line, or, what is the same thing, the area of the triangle under the solid line, as shown in Fig. 1.7. Can you see that the two shaded triangles have equal areas?

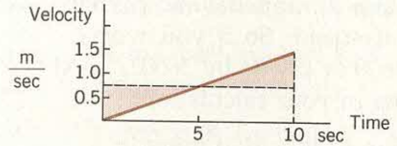


FIGURE 1.7

We now repeat the reasoning algebraically:

1. Uniform velocity v for the time t.

displacement $= v \cdot t =$ area of rectangle of height v and base t

2. Uniform acceleration, that is, velocity increasing in proportion to the time.

$$v = v(t) = a \cdot t$$

$$\text{average velocity } v_{\text{average}} = \tfrac{1}{2}\left(v_{\text{initial}} + v_{\text{final}}\right) = \tfrac{1}{2}(0 + a \cdot t) = \tfrac{1}{2}at$$

$$\text{in a time } t,\ \text{displacement} = v_{\text{average}} \times t = \tfrac{1}{2}at \times t = \tfrac{1}{2}at^2$$

That is the area of a triangle of height at and base t.

Putting in the numbers we had may give these formulas some life. The velocity went from $v_{\text{initial}} = 0$ to $v_{\text{final}} = 1.5$ m/sec in 10 seconds, so the acceleration was

$$a = \frac{v_{\text{final}} - v_{\text{initial}}}{t_{\text{final}} - t_{\text{initial}}} = \frac{1.5\ \text{m/sec}}{10\ \text{sec}} = 0.15\ \text{m/sec}^2$$

The average velocity over those first 10 seconds was

$$v_{\text{average}} = \tfrac{1}{2}a \cdot t = \tfrac{1}{2} \times 0.15\,\frac{\text{m}}{\text{sec}^2} \times 10\ \text{sec} = 0.75\,\frac{\text{m}}{\text{sec}}$$

and the displacement was

$$\text{displacement} = \tfrac{1}{2}at^2 = 0.75\,\frac{\text{m}}{\text{sec}} \times 10\ \text{sec} = 7.5\ \text{m}$$

If the same acceleration persisted for 20 seconds (twice as long), the final velocity would be double the 10-second velocity, and the final displacement would be four times the 10-second displacement: The velocity is proportional to the time; the displacement goes as the square of the time. To see the latter geometrically, we extend our triangle: The base doubles; the height doubles; so the area quadruples, as shown in Fig. 1.8.

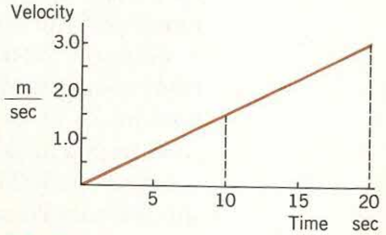


FIGURE 1.8

CONVERTING UNITS

The speedometer on a car gives speeds in miles per hour, or perhaps in kilometers per hour. You know how to convert such a reading to meters per second. But just to rehearse the technique of units conversion, we go through the method.

SAMPLE EXERCISE Convert 36 km/hr into m/sec.

Remember, we can always multiply anything by 1 without changing it. Since 1 hour = 60 minutes, and 1 minute = 60 seconds, we can multiply by

$$\frac{1\ \text{hr}}{60\ \text{min}} \times \frac{1\ \text{mm}}{60\ \text{sec}} = \frac{1\ \text{hr}}{3600\ \text{sec}} = 1$$

Here is how:

$$36\ \frac{\text{km}}{\text{hr}} \times \frac{1\ \text{hr}}{3600\ \text{sec}} = 0.01\ \frac{\text{km}}{\text{sec}}$$

(The hours canceled.)

Similarly, we have 1 km = 1000 m, so we can write 1 in the form 1000 m/ 1 km = 1.

$$0.01\ \frac{\text{km}}{\text{sec}} \times \frac{1000\ \text{m}}{1\ \text{km}} = 10\ \frac{\text{m}}{\text{sec}}$$

Repeat the whole thing:

$$36\ \frac{\text{km}}{\text{hr}} \times \frac{1\ \text{hr}}{3600\ \text{sec}} \times \frac{1000\ \text{m}}{1\ \text{km}} = 10\ \frac{\text{m}}{\text{sec}}$$

The trick is to end up with the units you want. If we had multiplied by unity in the form 3600 sec/1 hr, we would have had a correct answer, or course. But the units would have been m sec/hr^2. That is an unusual and perhaps even useless unit of speed. So if you worry, "Do I multiply by 3600 or divide by 3600?," just carry the units along in your calculation.

EXERCISE 2 Verify that 10.0 m/sec = 22.37 miles per hour.

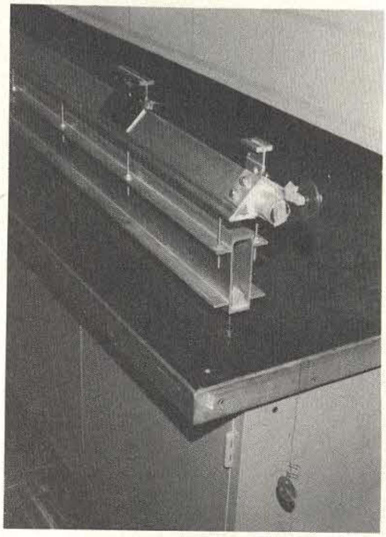
Newton's Second Law

Imagine a situation in which a constant force acts on an object. The Second Law then says that the object will have a constant acceleration. Double the force, and the acceleration doubles. To verify that law, you might use an air track again. You will want to have some arrangement for keeping the force from changing during a time long enough to make measurements of acceleration. In other words, you want to have some standard force, and it has to be reproducible. Suppose you had a device that could pull with a steady 1-pound pull. Then two such devices could provide a 2-pound pull. What the Second Law says is that two 1-pound pulls both pulling on the same glider in the same direction give it twice the acceleration it gets from one 1-pound pull.

If we have two identical gliders and each one gets a 1-pound pull, we get the same acceleration for both gliders. It should not matter whether the gliders are connected together, should it? That way we would have one double-glider with a force of 2 pounds pulling it. But what if we had a double-glider with only one 1-pound pull? Then the Second Law would predict an acceleration exactly half the old acceleration. Not only is the acceleration proportional to the force, but the acceleration is also inversely proportional to the number of gliders that force is pulling.

Here is a more general statement:

> The acceleration is proportional to the force and inversely proportional to the mass of the object.



Newton's-Second-Law experiment on an air track.

EXERCISE 5 A certain tug boat can pull with a steady force of 1 ton. The barge it is pulling accelerates at 0.2 miles per hour per second. A 2-ton pull would accelerate the barge at ______ miles per hour per second. Neglect the forces of friction in solving this and the following exercise.

EXERCISE 6 A 1-ton pull applied to two identical barges hitched together would give them an acceleration of ______ miles per hour per second.

EXERCISE 7 A 1-ton pull applied to 5 such barges hitched together would give them an acceleration of ____________.

EXERICSE 8 A 2-ton pull would accelerate the 5-barge train at ________.

EXERCISE 9 How hard would the tug have to pull to accelerate a single barge at 0.1 mile per hour per second? ______ tons

EXERCISE 10 to accelerate the 5-barge train at 0.1 mile/hr sec?

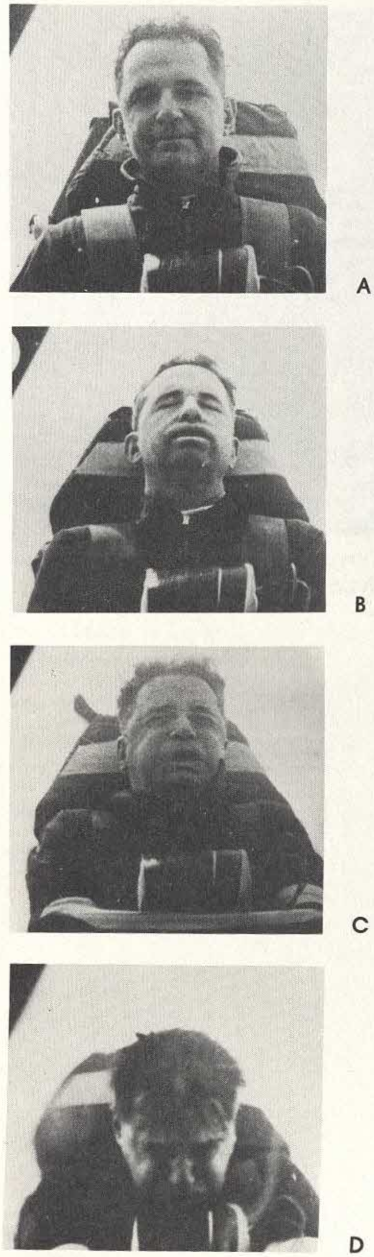
Things get a bit more complicated for vertical motion. We have to take gravity into account. In order to have a situation in which the force is in the direction of the motion, we analyze the forces on our astronaut as the rocket blasts off from earth.

Essay: The Spacecraft Blasts Off

During the lift-off and the almost vertical climb that starts the space voyage, the craft gains speed rapidly. It pushes on its passengers, who feel their stomachs getting heavy. To withstand their increased weight, they lie face up on a contoured support couch. That way, even though the blood gets very heavy, there is little pressure difference between the head and the heart.[2]

How much of this squeezing can the body stand? How many times heavier than our earthly weight can we get without being hurt? In other words, how much acceleration can we tolerate? In experiments preparatory to the U.S. space program, human subjects remained conscious and uninjured under accelerations that made them 9 times heavier than normal. So if we imagine a 165-pound man lying on a bathroom scale, the scale would register 1485 pounds—and he would be thinner too. Of course! He is being compressed.

We might ask how much acceleration gives rise to that much weight increase. An easier question is, how much acceleration does it take to just *double* the astronaut's weight? Before we answer, note that it is not rapid motion that makes his weight increase, but the *rate of change* of velocity. For the astronaut, it turns out that to double his usual weight requires an upward acceleration of 22 miles per hour per second. To triple his weight requires an acceleration of 44 miles per hour per second = 2 × 22 miles/hr sec. For every 22 miles/hr sec of acceleration, his weight increases by an amount equal to his normal weight on earth. So to get to 9 times his old weight requires an acceleration of 8 × 22 miles/hr sec (= 176 miles/hr sec). The engineer says simply "8 g's" or "8 times gravity."



Watching the face deform during acceleration. Bottom photograph was made while braking in a rocket sled (deceleration) at 22 g's.

EXERCISE 11 With an upward acceleration of 66 miles per hour per second the astronaut weighs ______ times his usual weight.

The numerical value of the acceleration required to double a weight (what we call 1 g) may seem more familiar if we express the miles per hour in feet per second. This exercise is left to the reader as Exercise 12.

EXERCISE 12 Remember that 1 mile = 5280 feet.
(HINT: How many seconds in an hour? Converting 22 miles/hr sec) to metric units, we get $g = 9.8$ meters per second per second.)

That number varies by less than 1% over the surface of the earth. Perhaps you recognize it as the rate at which a falling object increases its downward velocity: the acceleration of free fall. This is what you expect! The acceleration that adds a weight equal to your usual gravitational weight is the acceleration of gravity. Remember, when you accelerate *downward* at 1 g (= 9.8 m/sec^2), your weight decreases to zero. (See Sec. 1.1; also Problem 39.)

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[2] If the head were higher, the blood pressure in the brain would be much less than in the heart, resulting in possible loss of consciousness. You might get an idea of that effect when, after you have been squatting for a while, you get up fast, accelerating the head rapidly upward, and then feel giddy for a few seconds after.

The Acceleration of Gravity

The acceleration of gravity on the surface of the earth, that characteristic 22 miles per hour per second (= 9.8 m/sec^2), is a measure of the gravitational pull of the earth on objects here on the earth's surface. On other planets, the value is different. On the moon, the pull of gravity is about one-sixth of what it is here. Does it seem surprising to you that at any one location the value of g is the same for *all* objects, regardless of their size or composition? It really is the same for a feather as for a lead brick. Indeed, this intimate connection between the gravitational force on an object and the force needed to give it a certain acceleration is a profound truth about the physical world. It is called the **Principle of Equivalence.**

Let us review our definition of g. We assumed that the spacecraft blasts off vertically and asked what value of its acceleration would double the astronaut's weight. That value we called g. Such a definition implies that we have an instrument that measures weight. This "bathroom scale," or whatever you want to call it, supposedly reads correctly even under the unusual circumstances of blasting off on a space voyage. Assuming we know how to measure weight, we find that the extra weight our astronaut (or any object in the craft) gains by being accelerated straight up is proportional to the acceleration: Two g's triple the weight, three g's quadruple, and so on. Repeat: The extra push his support couch has to supply in order to accelerate him is proportional to his acceleration. This proportionality is a special case of Newton's Second Law: The force needed to accelerate an object is proportional to the acceleration. Earlier, we had that statement turned around and said that the acceleration of an object is proportional to the force applied to it. You might want to think about whether those two statements are equivalent.

EXERCISE 13 *Are they?* Support your answer with 10 or 20 well-chosen words.

EXERCISE 14 During blast-off, the vertical force supporting a 100-pound astronaut becomes 500 pounds.

(a) What is the upward acceleration of the rocket?

(b) Later on, in orbit, the astronaut is weightless. If his feet push on the floor of his cabin with a force of 100 pounds, then the floor pushes on him with a force of 100 pounds. Argue that he will accelerate away from the floor at 22 miles per hour per second.

(c) If he pushes on the floor with only a 50-pound force, what will be his acceleration?

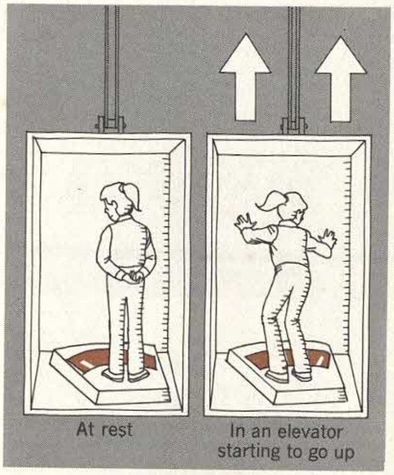
At rest
In an elevator starting to go up

FIGURE 1.9 Standing on that bathroom scale in the elevator: (a) at rest; (b) as the elevator starts to go up.

*Fictitious Forces

The astronaut feels heavier at blast-off. You feel heavier as the elevator starts on its "up" trip. This extra weight due to acceleration is sometimes called a "fictitious force." Don't misunderstand. The support couch is really pushing up on the astronaut's back. The floor of the elevator is really pushing up on the soles of your shoes. But you feel heavy all over. What force is pulling down on you, making you feel heavier? Some books call it a **D'Alembert force** to distinguish it from a real force.

If you are standing on a bathroom scale during the acceleration, does the scale register how hard *it* is pushing up on *you*, or how hard *you* are pushing down on *it?* There is no way to tell. The two pushes are equal and opposite. Both are real forces. One object is in contact with the other and pushing on it. Not so with the extra weight you feel, even though you are content to let the increased reading of the bathroom scale be the measure of that extra weight (see Fig. 1.9).

The terminology is a bit tricky here. If you took your scale to the planet Jupiter and weighed yourself there, the scale would register an increased weight. Jupiter, being a more massive planet than Earth, has stronger gravity at its surface. Your added weight there is a *real* force. It is Jupiter pulling on you. But the weight you gain in an upward acceleration is a fictitious force, even though it requires an equal and opposite real force to produce the acceleration.

How about weightlessness? Suppose I let go of all supports and allow the force of gravity to accelerate me downward. We called that free fall. The force of gravity is a real downward force. But this time there is a fictitious upward force due to my downward acceleration. Since the two forces are equal and opposite, there is no force at all from my point of view. The fictitious force is equal and opposite to the real force, and I feel no weight at all. Indeed in my accelerated frame of reference, nothing has weight. Again, note that we said nothing about how fast I am falling, or whether I am falling up or down, or moving sideways at the same time. All that is needed to make me weightless is that my acceleration be equal to the local value of g. Being unsupported guarantees that.

The fact that you can't tell "by feel" the difference between a gravitational force and a "fictitious force" is a consequence of the Principle of Equivalence. Standing on the surface of the earth, where g is 22 miles per hour per second, you have the familiar sensation of your usual weight. You would get the same sensation standing in a rocket far in outer space accelerating at 22 miles per hour per second in the direction of your head. We say "in the direction of your head" instead of saying "upward" to emphasize again that there is no "up" or "down" when you get away from gravity. Remember, the farther away you are from the earth, the source of the gravitational pull, the smaller is the local value of g.

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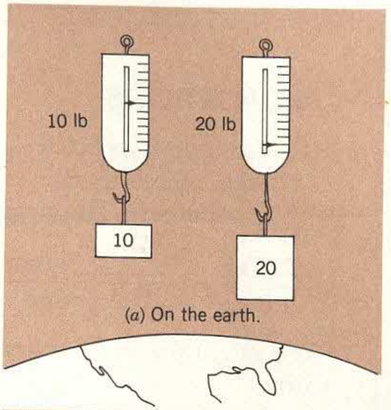
*This section is optional and can be omitted without loss of continuity.

1.3 DEFINITION OF MASS

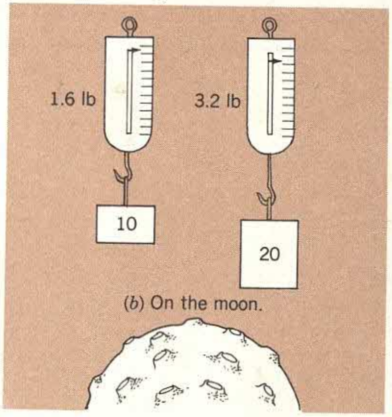
When we talk about the weight of an object, most people know what we mean. When I say I weigh 165 pounds, people know that I am talking about my weight here on earth. They may even have a mental picture of me standing on a bathroom scale. They assume that I will be standing still for the measurement and not jumping on the scale. You now know that I would weigh more on Jupiter, less on the moon. You also know I would weigh more on a rocket accelerating upward, less on a rocket near the end of its burn. But suppose you were really interested in whether I am fat or thin, *not* in how hard the bathroom scale has to push to support me. Then you need the concept of *mass*. My mass will not depend on where I happen to be, or on which way I happen to be accelerating. Here is one way to define mass.

Suppose that you weigh two objects, and one object weighs twice as much as the other—in your laboratory. In another laboratory, each object might have a different weight. For example, the second laboratory might be on the moon. (See Fig. 1.10.) The remarkable fact is that the first object still weighs twice as much as the second, regardless of where the laboratory is located or how it is accelerated. The weights may be different in the two laboratories, but the ratio of the weights will be the same. We say the ratio is *invariant*. The ratio of the weight of an object to the weight of some standard object is therefore an intrinsic property of that object. That property is called *mass*. A standard object, called the standard **kilogram,** is kept in a vault in a suburb of Paris.

In Paris, the value of g is 9.80943 meters/sec^2. There, the standard kilogram has a weight of 2.20462 pounds. Armed with these two numbers,[3] we could find the weight of a one-kilogram object at some other place if we were given the value of g there, just by proportions. [See the box called "Proportions."] We could also find the weight of another object whose mass we were given. For example, convince yourself that a 75-kilogram man weighs 165 pounds in Paris. On the moon, where g is 1.6 m/sec^2, that same man weighs 27 pounds. Of course, he is still a 75-kilogram man. His mass is invariant.



(*a*) On the earth.



(*b*) On the moon.

FIGURE 1.10 "The second laboratory might be on the moon." The value of g on the moon is 16% of g on earth, so a 10-pound weight only weighs 1.6 pounds there. The point is that the ratio of the two weights stays the same.

EXERCISE 15 Verify that 27 pounds.

It probably does not surprise you that mass is defined in such a way that it is measured by weighing. It should also not surprise you that it makes no difference whether the weighing is performed in a laboratory firmly attached to a planet like ours, or in an accelerating rocket in outer space. Remember that a given acceleration adds weight to an object in proportion to its mass.

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[3] Please don't memorize them! But if you insist, the first three digits are enough.

PROPORTIONS

If a 1-kilogram object weighs 2.2 pounds, then a 2.0-kg object in the same location weighs ______ pounds. The proportion reads

$$\frac{2.0\text{ kg}}{1.0\text{ kg}} = \frac{\boxed{}\text{ pounds}}{2.2\text{ pounds}}$$

You might prefer to write the proportion in the form

$$\frac{2.2\text{ pounds}}{1.0\text{ kg}} = \frac{\boxed{}\text{ pounds}}{2.0\text{ kg}}$$

You will not have trouble getting the 4.4 doing the arithmetic in your head. But make sure you know how to solve this kind of problem on your calculator.

If the acceleration changes rapidly, if there is jerking or rotation or vibration, that makes a measurement of mass difficult. It does not in principle upset our definition of mass.

In some books the concept we have defined is called *gravitational mass* when the weighing is performed in a laboratory anchored to the earth. This is to distinguish it from *inertial mass,* defined in such a way as to involve acceleration, but not gravity. What is important is that the different definitions result in the same numbers for the masses of objects. The Principle of Equivalence guarantees that.

EXERCISE 16 **Inertial mass**

In outer space, far away from the earth and other planets, a rocket is accelerating at 22 miles per hour per second (= 9.8 m/sec^2).

(a) How much force does the support couch exert on a 75-kilogram astronaut? In other words, how much does he weigh? [Answer: 165 pounds] Note that this weight has nothing to do with gravity.

(b) How much force is needed to support a 1-kilogram object and keep it from moving toward the back of the rocket? Answer in pounds. (HINT: Use proportions.)

EXERCISE 17 **Vertical Accelerations from Earth**

A laboratory weight labeled "1 kg" weighs 2.2 pounds as measured on a spring balance here on earth. In a rocket blasting off vertically with an acceleration of 66 miles per hour per second (= 3 *g*'s), its mass is still 1 kg. Remember, mass is an invariant.

(a) What is its weight in pounds?

(b) At what vertical acceleration would it weigh 4.4 pounds, that is, twice its earthly weight?
(c) At what acceleration would it be weightless?

A NOTE ABOUT MIXED UNITS Here we are purposely using metric units (kg) for mass and English units (pounds) for weight. We do this because the pound is familiar from bathroom scales. The **SI** unit of weight, the *newton*, is defined in Sec. 1.4.

EXERCISE 18 How much does a 5.0-kg object weigh at this location?

Now you can also verify the Paris weight of the 75-kg man. Did you get 165 pounds? On the moon, where g is 1.6 m/sec^2, the proportion for a 1-kg object is

$$\frac{\text{weight on earth}}{g \text{ on earth}} = \frac{\text{weight on the moon}}{g \text{ on the moon}}$$

$$\frac{2.2 \text{ pounds}}{9.8 \text{ m/sec}^2} = \frac{\boxed{} \text{ pounds}}{1.6 \text{ m/sec}^2}$$

You should have gotten a moon weight of 0.36 pounds for that kilogram. For the weight of a 75-kg man on the moon we have

$$\frac{\text{weight of 1.0 kg on the moon}}{1.0 \text{ kg}} = \frac{\boxed{} \text{ pounds}}{75 \text{ kg}}$$

Put in the 0.36 pounds and solve the proportion. Imagine how high he could jump!

1.4 DEFINITION OF THE NEWTON

Remember our definition of mass: The weight of an object anywhere in the universe is proportional to its mass. Recall also how we defined the local value of g: The weight of an object in a given location is proportional to g. Combining the two definitions, we can write

$$\text{weight} \propto \text{mass} \times g$$

The symbol $\propto$ means "is proportional to." This proportion can be written as an equation by finding the proportionality constant. Just because we chose to express weight in pounds and mass in kilograms, the proportionality constant is ugly. So the following exercise is optional, and you may want to skip it.

EXERCISE 19 In a place where the value of g is 9.8 m/sec^2, we said that a 1-kg mass weighs 2.2 pounds. The equation for the weight of any mass at any place can therefore be written

$$\frac{\text{weight}}{2.2 \text{ pounds}} = \frac{\text{mass}}{1 \text{ kg}} \times \frac{g}{9.8 \text{ m/sec}^2}$$

or

$$\text{weight} = \text{mass} \times g \times \frac{2.2 \text{ pounds}}{1 \text{ kg} \times 9.8 \text{ m/sec}^2}$$

That cumbersome 2.2/9.8 is just there in order to give the weight in pounds when the mass is given in kilograms.

If we prefer to write the simpler equation

$$\text{weight} = \text{mass} \times g$$

the physics is unchanged, but the units are different. In particular, if mass is in kilograms and g is in meters per second per second, then weight comes out in kg meters/sec^2. The price we pay for getting the simple formula "weight = mass × g" is that we get a less familiar weight unit. Instead of getting American bathroom-scale units, we get the **newton:**

$$\frac{1 \text{ kg meter}}{\text{sec}^2} = 1 \text{ newton}$$

To put a familiar fact in the new language, we now say that on earth, 1 kilogram weighs 9.8 newtons. The same fact in the same language is that, on earth, g is 9.8 newtons per kilogram.

EXERCISE 20 How many newtons does a 2-kilogram mass weigh? How big a mass weighs 1 newton?

Newton's Second Law: $F = ma$

Back in Sec. 1.2, the Second Law was stated: Acceleration is proportional to force and inversely proportional to mass:

$$a \propto \frac{F}{m}$$

Alternatively: The force required to impart to a mass m an acceleration a is directly proportional to the mass and to the acceleration:

$$F \propto ma$$

We did not ask about the value of the proportionality constant. Now if we express

F in newtons,
m in kilograms,
a in m/sec^2,

the proportionality constant turns out to be unity:

Newton's Second Law: $F = ma$

These are the units of choice in the SI (Système International). This system of units, based on the meter, the kilogram, and the second (MKS), has been officially adopted by various international organizations. Problems 1.34 through 1.40 show how to use "$F = ma$" in some other systems of units.

EXERCISE 21 **Accelerating Those Barges**

A 1-ton force (= 2000 pounds = 8.9×10^3 newtons) gives an acceleration of 0.2 miles per hour per second = 8.9×10^{-2} m/sec^2. Find the mass of the barge in kilograms. [Answer: 1.0×10^5 kg.] You may be worried that part of that mass is not barge but water carried along with the barge's motion. If so, you are right!

EXERCISE 22 **Free Fall**

A 1-kg mass on earth weighs 9.8 newtons. If dropped, its downward acceleration is

$$a = \frac{F}{m} = \frac{9.8 \text{ newtons}}{1 \text{ kg}} = \underline{\qquad\qquad}$$

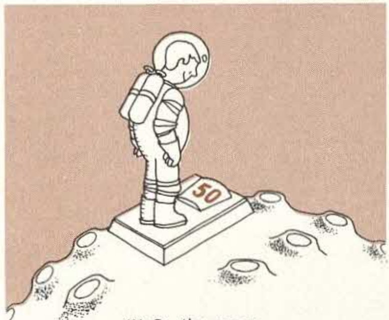
NOTE: From now on, SI units for force and speed (newtons, m/sec) will be used in preference to pounds and miles per hour.

1.5 THE POUND: MASS OR WEIGHT?

Using the pound as a unit of *weight* and the kilogram as a unit of *mass* is conventional usage, but it gives rise to a confusion: A laboratory balance is a *weighing* instrument, yet it reads in kilograms. In much of the world, our 165-pound man would be said to weigh 75 kilograms. Is the kilogram, then, a unit of weight? No. The confusion is one of words, but it is subtle: When

I want to lose weight, it is not in order to decrease the load on my legs, but in order to fit into my favorite clothes. I am concerned with how much of me there is, not with the pull of gravity. I am really concerned with my mass. But because I use a bathroom scale to measure it, what I talk about is my weight (weight = mass × g). Moving to a location with a smaller value of g would serve to decrease my weight, but would not serve my purpose. So whether I say 75 kilograms or 165 pounds, I really mean the same thing (mass). (See Fig. 1.11.)

(*a*) On the earth.



(*b*) On the moon.

FIGURE 1.11 "Moving to a location with a smaller value of g would serve to decrease my weight but would not serve my purpose." (a) On the earth. (b) On the moon.

Remember, we said that in Paris, 1 kilogram weighs 2.20462 pounds. We could use that as a conversion factor to make the pound a unit of mass. Indeed, the U.S. Bureau of Standards has adopted that definition of the *avoirdupois pound*. If you use it, you have to be careful when you get a place where g is not the same as in Paris. See Problem 1.28.

We don't usually have to worry about the distinction between pound (mass) and pound (weight), simply because the value of g varies so little over the surface of the earth. The variation is less than 1%. Bathroom scales and similar spring balances are not usually accurate to better than ±1%. That is one reason they are illegal in grocery stores and unfashionable in the doctor's office. A good scale compares the weight to be weighed with that of standard "masses" that are part of the instrument. That is what a beam balance does. So it really measures mass, not weight.

Because g varies so little here on earth, the mass-weight distinction is lost in popular usage. The metric system tries to avoid the confusion. One simply says, "One kilogram weighs 9.8 newtons" (on earth). The kilogram is the unit of mass; the newton is the unit of weight. Accordingly, a metric bathroom scale should be calibrated in newtons. Scientific supply houses sell spring balances calibrated that way. But European department stores sell their bathroom scales calibrated in kilogram weight![4]

1.6 SOME BOOKEEPING WITH F = MA DURING LIFT-OFF*

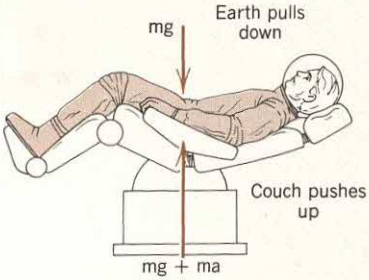


FIGURE 1.12 Forces on the astronaut during lift-off: The support couch pushes up with force $mg + ma$. The net force on the astronaut is ma.

An upward acceleration of 8 g gave our astronaut nine times his usual weight (Sec. 1.2). How did we interpret that? Well, with no acceleration, he had weight $m \times g$. (See Sec. 1.4; let mass = m.) Each "g" of upward acceleration added one "$m \times g$" to his weight. Adding $8(m \times g)$ to the $1(m \times g)$ gave him a weight of $9(m \times g)$. (The support couch has to supply an upward force equal to $(m \times g) + 8(m \times g) = 9(m \times g)$ pushing up on him to make his mass m accelerate with an acceleration $a = 8g$. Of that $9\ (m \times g)$, the $1(m \times g)$ part serves to cancel out the pull of gravity; the $8(m \times g)$ is the accelerating force. To decompose the force in this way into a gravitational force and an accelerating force helps clarify our thinking:

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[4] Please do not conclude that the French Revolution was fought for nothing. The metric system, which was one of its products, has some easy conversion factors. For example, one liter (= 10^{-3} m^3) of water has a mass of 1 kilogram.

*This section is optional and can be omitted without loss of continuity.

S.C.E.R.T., West Bengal
Date 27.2.91
Acc. No. 4920

force needed to overcome gravity $= mg$
accelerating force $= ma$

force exerted by couch $= mg + ma$

Both forces are proportional to the astronaut's mass m. Given this mass, the gravitational force mg depends only on the local value of g. The accelerating force ma depends only on the acceleration and is quite independent of location. Notice that both forces are independent of the velocity.

How about the *net* force on the astronaut? The couch pushes up on him; the earth pulls down on him:

force exerted by couch $= mg + ma$
force exerted by earth $= -mg$

net force $= ma$

The downward pull of gravity was written as $-mg$ because we arbitrarily called "up" positive. The result for the net force on the astronaut,

$$F = ma$$

is just what Newton's Second Law (Sec. 1.2) tells us.

SUMMARY

In free fall, the notions of "up" and "down" lose meaning.

Free fall, weightlessness, motion without force—they all mean the same thing.

Newton's First Law: Things don't speed up or slow down or curve unless there is a force on them.

Newton's Second Law: When a thing does speed up or slow down, its acceleration (+ or −) is proportional to the external force on it and inversely proportional to its own mass.

The upward acceleration needed to double the weight of an object here on earth is 22 miles per hour per second (= 9.8 meters per second per second).

An object dropped close to the earth gains speed at the rate of 22 miles per hour per second. Although that number is the same for all objects, it does vary somewhat with location on the earth and falls off with increasing distance from the earth.

The mass of an object can be defined as the ratio of its weight to the weight of a standard object.

Without acceleration, an object of mass m has weight mg.

Writing Newton's Second Law $F = ma$, the MKS unit of force is the newton. 1 newton $= 1$ kg m/sec^2.

The pound may be defined as a unit of force (= 4.45 newtons) or as a unit of mass (= 0.454 kg).

KEY TERMS

Mechanics The science of forces and motion.

Classical mechanics Same meaning as *mechanics*. The word *classical* is added to emphasize that the objects being studied are not so small (i.e., atomic size) that *quantum* effects would be important.

Orbit The path traveled by an object (a planet? a satellite?) without rocket power. We talk about the earth's elliptic orbit around the sun and the moon's almost circular orbit around the earth. A spacecraft is "in orbit" when the rocket engines are turned off.

Air track A smooth rail with many tiny holes through which a stream of air escapes. The gliders ride above the track with hardly any friction. This instrument is used in student laboratories.

Hovercraft A transportation vehicle that glides above the ground or water by emitting a downward jet of air.

Newton's First Law A body persists in its state of rest or motion in a straight line at constant speed unless acted upon by an external force.

Newton's Second Law The acceleration of a body acted upon by an external force is proportional to that force and inversely proportional to its mass.

Kinematics The science of motion without reference to force or mass. Being concerned only with the description of motion, its only variables are position (distance) and time.

Meter (metre) The unit of length (distance) in the SI (Système International). 1 meter = 39.37 inches = 3.28 feet.

g The acceleration of gravity. On earth, this is about 9.8 meters/sec^2 = 22 miles per hour per second. It varies by less than 1% over the earth's surface but decreases at high altitudes.

Weight What a bathroom scale is designed to measure. The unit of weight is a force unit, for example, the pound or the newton. Here on earth, a stationary 1-kilogram mass weighs 9.8 newtons; a 1-pound (avoirdupois) mass weighs 1 pound.

Acceleration Rate of change of velocity.

Skylab A U.S. manned spacecraft in which experiments were conducted during 1973–1974. It had three different three-man crews.

Principle of Equivalence Force on an object due to gravity is indistinguishable from force experienced by virtue of acceleration. Alternative statement: Gravitational mass is equal to inertial mass. A more careful statement has to specify that the object has to be small enough so that the difference in direction of gravitational force at different places on the object is too small to be measured.

D'Alembert force Fictitious force, proportional to the acceleration of your frame of reference.

Kilogram Standard of mass. The primary standard is a cylinder of platinum-iridium alloy kept at the International Bureau of Weights and Measures in Sèvres, France. According to the U.S. Bureau of Standards, 1 kilogram is 2.2046 pounds. Abbreviation: kg.

gram Unit of mass in the c.g.s. system. 1 gram = 10^{-3} kilograms.

Newton Sir Isaac Newton (1642–1727), British scientist and theologian, who formulated the laws of mechanics, discovered the universal law of gravitation, and invented calculus.

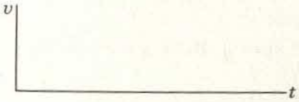
newton Unit of force in the SI (Système International). One newton is the force needed to accelerate a mass of 1 kilogram with an acceleration of 1 meter per second per second.

dyne Unit of force in the c.g.s. system. One dyne is the force needed to accelerate 1 gram with an acceleration of 1 centimeter per second per second. 1 dyne = 10^{-5} newtons.

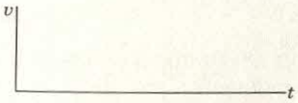
SI Système International. The system of units based on the meter-kilogram-second (MKS) adopted by all nations represented at the 15th General Conference on Weights and Measures (1975) and recommended for all scientific, technical, practical, and teaching purposes.

REVIEW QUESTIONS

- When the elevator starts down, you feel yourself getting temporarily: lighter / heavier.
- While the elevator is moving downward at constant speed, you feel that your weight is: the same as without any motion / heavier than usual / lighter than usual
- When the down elevator slows to a stop, you feel yourself getting temporarily: lighter / heavier
- What happens to an object released inside an orbiting spacecraft? Why could an astronaut in orbit not pour himself a glass of water? According to Newton's First Law, when an object has no external force on it, its acceleration / speed / position does not change.
- According to Newton's Second Law, when an object has a constant external force on it, its acceleration / speed / position does not change.
- When an object has a constant acceleration, we can conclude that it has: no external force on it / a constant external force / a uniformly changing external force
- On the speed-versus-time axes, sketch a graph corresponding to motion at constant speed.



- On the speed-versus-time axes, sketch a graph corresponding to motion under the action of a constant external force.



- If the external force on an object is doubled, its velocity / acceleration is doubled.
- For a given external force, the acceleration of a 1-kg object is half / double / equal to the acceleration of a 2-kg object.

- On the moon, where the acceleration of gravity is only $\frac{1}{6}$ of its value on earth, a 1-kg object weighs more than half / less than half / exactly half as much as a 2-kg object.
- If a 2-kg object experiences an acceleration of 2 m/sec^2, we can conclude that the external force on it is: 1 newton / 2 newtons / 4 newtons
- An object on a horizontal air track experiences an acceleration of 9.8 m/sec^2 when pulled with a force of 9.8 newtons. Its mass is:
  1 kg / 9.8 kg / 96.04 kg / 1 pound
- On earth, a 1-kg mass weighs: 1 newton / 9.8 newtons / 22 newtons
- In a rocket accelerating upward at 11 miles per hour per second, a 1-kg mass weighs: 1 newton / 1.5 newtons / 9.8 newtons / 14.7 newtons
- In a rocket accelerating downward at 4.9 meters per second per second, a 1-kg mass weighs: 1 newton / 0.5 newtons / 9.8 newtons / 4.9 newtons

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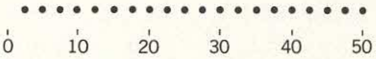
PROBLEMS

✶ **1. Uniform Motion on a Linear Air Track** The dots in the figure represent successive exposures made with stroboscopic lighting with 10 flashes per second. This means that the successive exposures you see are 0.1 second apart. (a) Use the centimeter scale below the track to find the glider's speed between the first two flashes (far left).



SOLUTION At time $t = 0$, position $x = 3.1$ cm. At time $t = 0.1$ sec, position $x = 5.5$ cm. The distance moved in 0.1 sec is

5.5 cm − 3.1 cm = ________ cm

Average speed during that $\frac{1}{10}$ second is

$$\frac{2.4 \text{ cm}}{0.1 \text{ sec}} = 24 \text{ cm/sec}$$

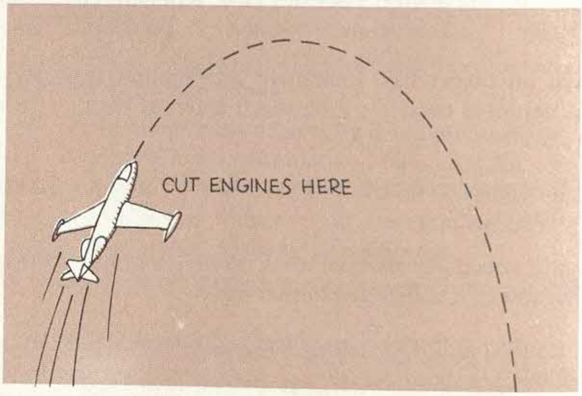
Since our position measurements are good only to the nearest millimeter, the distance moved (2.4 centimeters) has only two significant figures. It would not be correct to write the speed as 24.0 cm/sec. Always keep only as many digits in your answer as the measurements justify.

(b) Now find the speed between the last two flashes (far right) and

(c) between the 10th and 11th flashes.

(d) By measuring the distance between the first and last flashes, find the average speed during the entire 2-second interval. Note that, here, the distance measurement has three significant figures.

2. Weightlessness To prepare future astronauts for the experience of weightlessness, they are taken up in



PROBLEM 2. Half a minute of weightlessness: training astronauts. The engines are cut on the way up, and restarted as late as possible. While the airplane is on its parabolic trajectory, its occupants are weightless.

airplanes and the engines are suddenly slowed down while the airplane is climbing, so that its path is exactly like free fall in a vacuum (see the figure). It can remain in such a parabolic arc for as much as half a minute before the engines are turned to full power again to break the fall.

(1) Without looking out the window, would the astronaut be able to tell when the airplane had reached its greatest height?

(b) Why slow the engines while the airplane is climbing rather than in level flight?

✶ **3. Meters per Second, per Hour** Comfortable walking speed is a little over 2 miles an hour. Suppose that you kept up a steady speed of 1 meter per second.

(a) How many meters would you walk in an hour?

(b) How many meters would you walk in a minute?

(c) If your stride is 80 cm (= 0.80 meters), how many steps do you take per minute?

(d) How many seconds does each step take? Express in decimal form.

(e) How many steps do you take per second?

✶ **4. Percentage Change** If you increased your speed by 1%, your speed would be 1.01 m/sec. (See Problem 1.3) Suppose that you did this by increasing the length of your stride by 1%, to 80.8 cm.

(a) By what percentage is the distance covered in an hour changed?

increased / decreased by ________%

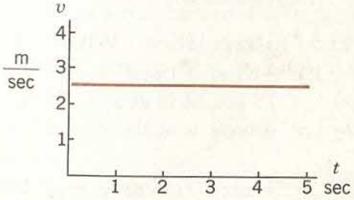
(b) The time per step?

increased / decreased by ________%

✶ **5. Constant Speed** The graph shows that the speed is constant as a function of time, at least for 5 seconds.

(a) How far does the object go in 1 second?

(b) In 5 seconds?



✶ **6. Changing Speed Suddenly** The graph shows the object changing speed suddenly at $t = 3$ seconds. How far did it move:

(a) In the first second?

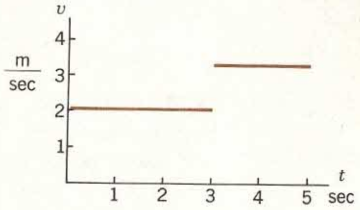
(b) In the first 3 seconds?

(c) In the fourth second?

(d) In the time interval between $t = 3$ sec and $t = 5$ sec?

(e) In the entire 5 seconds?

(f) What was the average speed over the 5 seconds?



NOTE: Such a graph is evidently an idealization. No object can really change speed suddenly. That would imply an infinite acceleration and therefore an infinite force.

✶ **7. Changing Speed Suddenly** At a speed of 2 meters per second, it takes 50 seconds to go 100 meters.

(a) How long does it take to go 100 meters at 4 m/sec?

(b) If I go the first 50 meters at 2 m/sec and the second 50 meters at 4 m/sec, how long does it take to go the whole 100 m?

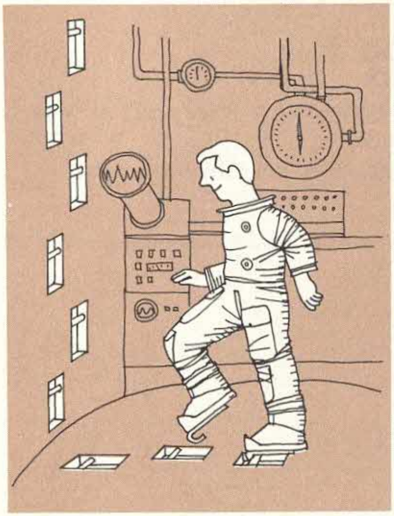
(c) What is the average speed during this time? HINT: The answer is *not* 3 m/sec.

(d) [Harder] If I divide up the 100 m so I spend half the time going at 2 m/sec and the other half of the time going at 4 m/sec, will the trip take more or less than 37.5 seconds? How long?

(e) How long after starting will I have gone halfway, that is, 50 meters?

(f) Make rough graphs of distance versus time for the situations of parts *(b)* and *(d)*. Does your graph agree with the answer to part *(e)*? (HINT: The graphs consist of two straight-line segments each.)

✶✶ **8. Walking in Skylab** The Skylab astronauts had shoes with hooks that fit into special slots in the floor, as shown in the figure. That way they could stay in one place or walk without losing contact with the floor. In normal walking here on earth, the impulse your foot gives the floor in a walking step is large enough to start you moving away from the floor at about 0.8 meters per second under conditions of weightlessness. Suppose that your hook missed the slot in the Skylab floor as you took a normal step. How fast must your reaction time be to enable you



PROBLEM 8. The Skylab astronauts had shoes with hooks that fit into special slots in the floor.

to grab hold of some fastened-down furniture before you are more than 1 meter off the floor?

✶✶ **9. Knowing What to Neglect** The speed of sound in air is 3.3×10^2 m/sec. The speed of light is 3.0×10^8 m/sec. If you hear the thunder 2.0 seconds after you see the lightning, how far away did the flash occur?

✶✶ **10. Bullet Goes Faster Than Sound** A bullet traveling at an average speed of 660 m/sec gets to the target before the sound of the gun, which travels at 330 m/sec,
(a) The sound gets there 0.2 seconds after the bullet. How far is the target from the gun?
(b) Why do soldiers say, if you hear the whine of a shell, it won't hurt you?

PROBLEM 10. "If you can hear the whine of a shell, it is not going to hit you."

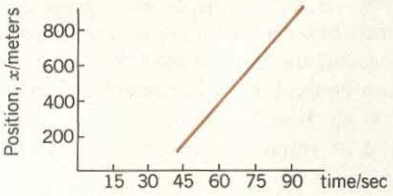
✶ **11. Kinematics** In an orbiting spacecraft, any free object travels in a straight line at constant speed.
(a) To bring the peas from the plate to the mouth—a distance of 40 cm—might take half a second. What is the average speed of the fork?
(b) The astronaut stops the fork, but the peas keep going. They miss his mouth. How long before they hit the wall 3.0 meters away?

Here on earth, a pea dropped from a fork has a chance of falling back on the plate. As it falls, it gains speed at the rate of 22 miles per hour per second (= 9.8 m/sec^2).
(c) If a pea hits the plate after 0.1 second, at what speed does it hit? Assume that it left the fork at negligible velocity.
(d) Since it started at zero speed, its average downward velocity during that one-tenth of a second is only half the final speed. How far did it fall? Answer either in meters (m) or centimeters (cm). (1 m = 100 cm.)
(e) If it had fallen for twice as long, a time 0.2 sec, it would have fallen ________ times as far. Prove your answer.
(f) To fall twice as far takes ________ times as long. Prove.

✶ **12. Average Speed** A car travels the first half hour at 30 miles per hour and the second half hour at 50 miles per hour.
(a) Write down a convincing argument that it traveled 40 miles, so that its average speed was 40 miles per hour.
(b) Another car traveled 20 miles per hour for a quarter of an hour and 60 miles per hour for three-quarters of an hour. Find the average speed during the hour. (HINT: Average speed × total time = total distance.)

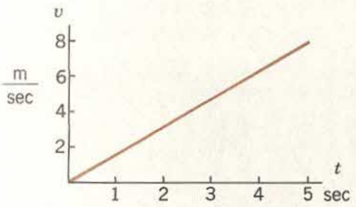
✶ **13. Average Speed** A car travels the first 10 miles at 30 miles per hour and the second 10 miles at 60 miles per hour.
(a) How long did it take to travel the 20 miles? $\frac{1}{3} + \frac{1}{6} = \frac{1}{2}$ hr.
(b) What was its average speed? (HINT: The answer is *not* 45 mph.)
(c) Plot the velocity versus time for this trip.

✶ **14. Interpolation and Extrapolation** When the clock reads 45 seconds, the car was at a position marked 200 meters in the figure. At $t = 75$ sec, it is at $x = 700$ m.
(a) Assuming constant velocity, where was the car at $t = 60$ sec?
(b) Where was the car at $t = 51$ sec? You may read the position off the graph or you may do the arithmetic, or better yet, do both.
(c) What is the car's speed?

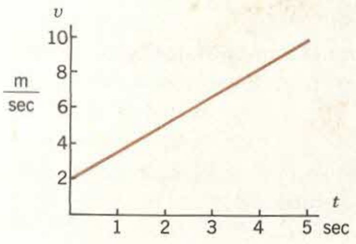


(d) When the clock read $t = 0$, the car was at a point marked with a negative x. Calculate it.

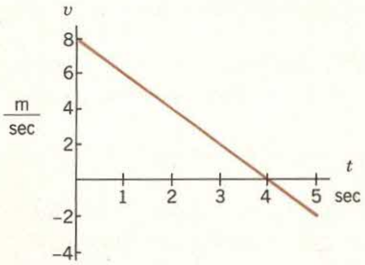
✳ **15. Uniform Acceleration** The graph shows a uniform acceleration from rest lasting at least 5 seconds.
(a) What is the speed at $t = 1$ sec?
(b) What is the acceleration? What tells you that it is the same in the first second and in the fifth second?
(c) What is the speed at $t = 5$ sec?
(d) What is the average speed over the 5 seconds?
(e) What is the distance traveled during those 5 seconds?
(f) Compare the distance traveled in 5 seconds with the distance traveled in the first second. Interpret those distances as the areas of triangles.



✳ **16. Uniform Acceleration** This graph shows a uniform acceleration starting at 2 m/sec and lasting at least 5 seconds. What is
(a) The speed at $t = 1$ sec?
(b) The acceleration?
(c) The speed at $t = 5$ sec?
(d) The average speed over the 5 seconds?
(e) The distance traveled in those 5 seconds? Interpret this distance as the area of a trapezoid.



✳ **17. Uniform Acceleration (Negative)** The graph shows a uniform negative acceleration starting at a speed of 8 m/sec and lasting at least 5 seconds. What is
(a) the speed at $t = 1$ sec?
(b) the acceleration?
(c) the speed at $t = 5$ sec?
NOTE: The velocity is -2 m/sec; the speed is 2 m/sec. After $t = 4$ sec, the motion is in the opposite direction from the initial velocity.
(d) the average velocity over the 5 seconds?
(e) the displacement at the end of the 5 seconds? NOTE: This is not the same as the distance traveled, since some of the travel was backwards.)



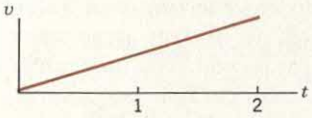
✳ **18. Stopping the Car** In order to stop a car going at 60 miles per hour in 3 seconds, the average acceleration has to be -20 miles per hour per second. You may prefer to call that a deceleration of 20 miles per hour per second. Conventionally, it is called an acceleration of -20 miles/hr sec. In order to keep from skidding, it is best to have a uniform acceleration.
(a) How far does the car go during the first second?
(b) How far does the car go during the second second?
(c) How far does the car go during the third second?
(d) What is the total stopping distance? Is the average speed during those three seconds 30 miles/hr?
(e) In order to stop the car in half the time, the acceleration would have to be ________ times as great.
(f) [Harder] In order to stop the car in half the distance, the acceleration would have to be ________ times as great.

✳ **19. Acceleration** A rifle bullet travels faster than the speed of sound. Suppose that it leaves the muzzle at 1500 miles per hour (= 668 m/sec). Then its average speed through the barrel of the rifle is half of that, or 750 miles/hr.
(a) How long does the bullet take to travel the 76-cm length of the barrel?
(b) What is the average acceleration during this time?

✶ **20. Drag Race** A certain drag racer is able to gain speed at a uniform rate of 8 miles per hour per second. At the end of the first second he is moving at 8 miles per hour. So his average speed during that time is 4 miles per hour. The distance he travels during that first second (= $\frac{1}{3600}$ hour) is

$$4\,\frac{\text{miles}}{\text{hour}} \times \frac{1}{3600}\,\text{hour} = 1.11 \times 10^{-3}\ \text{miles}$$

(a) During the next second he accelerates from 8 miles per hour to 16 miles per hour. Argue that his average speed during that second second is 12 miles per hour.
(b) Argue that he covers 3 times as much distance during the second second as he did during the first. (HINT: A geometrical argument counting triangles of equal areas on the graph is convincing.)
(c) During the third second he accelerates from 16 miles/hr to 24 miles/hr. How much distance does he cover during the third second?
(d) Make a table of distance from the starting point versus time for the first 5 seconds.
(e) Plot the values of distance versus time on a rough graph, and connect your six points with a smooth curve.

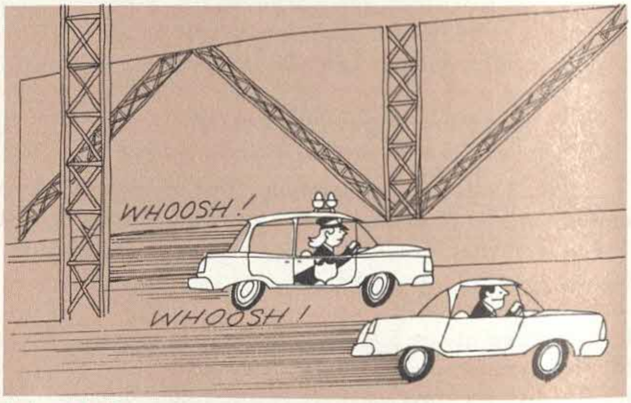


✶ **21. Drag Race with Speed Limit** Suppose that there is a 40-mph speed limit, so the drag racer only accelerates for 5 seconds. Then he travels at uniform speed for another 5 seconds.
(a) How far does he get from the starting point?
(b) Plot the speed versus time for the full 10 seconds and interpret the area under the curve in terms of displacement.

✶ **22. Pushing Off from the Floor of the Spacecraft** The kind of push-off you give yourself at the wall of a swimming pool might be enough to start you moving at 4 meters per second perpendicular to the floor of a spacecraft in orbit. Suppose that your feet stay in contact while your legs straighten out and your body moves 50 cm. At the beginning of the push your speed is zero; when you lose contact, your speed is 4 m/sec. Assuming a constant acceleration (i.e., a uniform push), how long does the push last? (HINT: Your average speed during the push is 2 m/sec.)

✶ **23. Slowing Down** A car being braked from 20 mph to a stop in 4 seconds has an average acceleration of −5 miles per hour per second during the braking.
(a) Could that acceleration be expressed in meters per second per second? If *yes,* do so. If *no,* explain.
(b) Could it be expressed in square miles per second? If *yes,* do so. If *no,* explain.

✶✶ **24. Catching Speeders** The police cruiser in the figure can accelerate at the rate of 5 miles per hour per second. The cruiser starts from rest just as the speeder passes the bridge abutment where it was hiding. The cruiser catches up with him in 15 seconds.
(a) How fast was the speeder going? (HINT: The answer is *not* 75 miles per hour.) Sketch a graph of the cruiser's speed versus time and of the speeder's constant speed.
(b) How far did the cruiser travel before overtaking the speeder?
(c) This time the speeder going 80 mph notices the police cruiser right away and slows down at the rate of 5 miles per hour per second until he is down to 30 mph. Then he stays at 30 mph. How far did the cruiser travel at a constant acceleration of 5 mph per second before overtaking the speeder? Plot!


WHOOSH!
WHOOSH!

PROBLEM 24. "Starting from rest, the cruiser catches up with him in 15 seconds."

✶✶ **25. Uniform Acceleration Algebraically** The displacement x in a time t at average speed $\bar{v}$ is

$$x = \bar{v}t$$

Starting from initial velocity v_i and accelerating to a final velocity v_f, the average acceleration is

$$\bar{v} = \tfrac{1}{2}(v_i + v_f)$$

(a) Now we can write

$$\begin{array}{ll} x = \tfrac{1}{2}(v_i + v_f)\, t & \\ \phantom{x = {}}(v_i + v_f)\, t & \\ \phantom{x = {}}2(v_i + v_f)\, t & \text{(Circle the correct one.)} \end{array}$$

(b) At uniform acceleration

$$a = \frac{v_f - v_i}{t}$$

that displacement can be rewritten

$$\begin{aligned} x &= v_i t + \tfrac{1}{2}at^2 \\ &= \frac{\tfrac{1}{2}(v_i + v_f)\,(v_f - v_i)}{a} \\ &= \frac{\tfrac{1}{2}(v_f^2 - v_i^2)}{a} \qquad \text{(Prove all three forms.)} \end{aligned}$$

All these formulas are good also for negative values of the variables. Thus if positive x means displacement to the right, then negative x means displacement to the left. Similarly for velocities. A negative acceleration may mean that a motion to the right is slowing down or that a motion to the left is speeding up.

(c) Suppose that $v_i > 0$ and $a < 0$ are given. Starting at $x = 0$, how long does it take until $x = 0$ again?

(HINT: Solve the equation $x = 0$ for t in terms of the given v_i and a.)

(d) Solve also for v_f at that instant, again in terms of v_i and a. Does it depend on both?

✶ **26. Free Fall on Earth** The dots in the right-hand margin represent the position of a falling object at intervals of one-fiftieth of a second. The top dot is the position at the instant the object is dropped. Call that time $t = 0$.

(a) How much time has elapsed at the eleventh point, that is, the bottom dot?

(b) Measure the distance y fallen during this time t.

(c) Divide y by t to find the average speed $\bar{v}$ during this time.

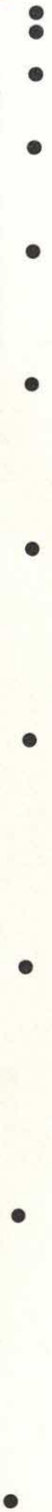
(d) Argue that the speed at the eleventh point should be $2\bar{v}$.

(e) Calculate the acceleration by dividing $2\bar{v}$ by t.

(f) Measure the distance Δy_{10} fallen in the last $\frac{1}{50}$ second, that is, between dots 10 and 11. The "final speed" v_{10} may be taken to be $\Delta y_{10}/0.02$ sec. Is this the speed at time = 0.20 sec? or 0.18 sec? or 0.19 sec?

(g) Argue that the acceleration may be calculated as $v_{10}/0.19$ sec. What is your result this way?

(h) Which value of the acceleration is more precise? That means, which value is likely to be more trustworthy. Which value is closer to the value you expected to find, based on your reading? (HINT: If they are both "right on," that just means you have not measured to enough significant figures.)



✶ **27. Falling Up, Falling Down** Object A is shot vertically upward from the earth at 29.4 m/sec (= 66 miles per hour). In Table 1.1, the column labeled y_A gives its height above the starting point as a function of time. The object is in free fall throughout. Note that it reaches its maximum height in 3.0 seconds. (HINT: 66 miles per hour/22 miles per hour per second = ________ seconds.) Column v_A gives its velocity. Plus means up; minus means down. Column y_B is for object B launched at 49.0 m/sec (= 110 miles per hour), and v_B is its velocity.

(a) When will B reach its maximum height?

(b) Sketch a graph of v_A versus time t for the full 6 seconds.

(c) Sketch a graph of y_A versus time t.

(d) Predict when B will be back to its starting point.

(e) At what time are A and B moving at the same speed? NOTE: The word *speed* means the absolute value of the *velocity*.

(f) Suppose that you are riding on A and observing how high B is above you. Plot $y_{\text{relative}} = y_B - y_A$ as a function of time.

(g) Plot B's velocity as measured in A's frame of reference, $v_{\text{relative}} = v_B - v_A$, as a function of time. Why would you be inclined to say that B is weightless if you observe it in A's frame of reference?

[Harder] (h) Write down a formula for y_B as a function of time t. Use it to verify the last entry in column y_B.

✶ **28. Force of Gravity Falls Off with Distance** The farther we get from the earth, the lighter we get. Near the earth, the pull of gravity decreases about 1% for every 32 km of altitude. So 100 km up, the value of g is about 3% less than at sea level.

(a) At sea level, g is 22 miles per hour per second. What is it at 200 km above sea level?

(b) Many artificial satellites are in orbits 100 to 200 kilometers above the earth's surface. Comment on the mistaken notion that such satellites are "beyond the pull of the earth's gravity."

✶✶ **29. Escape Velocity** If you throw an object straight up, it comes back down after a while. The faster it is launched, the longer it takes to return. Is there a speed at which it will never come back? After all, the strength of the earth's gravity gets weaker farther from the earth. [*Answer:* Yes, things launched at 1.12×10^4 m/sec

TABLE 1.1 Objects A and B are both launched at $y = 0$. B, the faster one, reaches a higher altitude (Problem 27).

T	Y_A	V_A	Y_B	V_B
0.0	0.00	29.40	0.00	49.00
0.2	5.68	27.44	9.60	47.04
0.4	10.98	25.48	18.82	45.08
0.6	15.88	23.52	27.64	43.12
0.8	20.38	21.56	36.06	41.16
1.0	24.50	19.60	44.10	39.20
1.2	28.22	17.64	51.74	37.24
1.4	31.56	15.68	59.00	35.28
1.6	34.50	13.72	65.86	33.32
1.8	38.04	11.76	72.32	31.36
2.0	39.20	9.80	78.40	29.40
2.2	40.96	7.84	84.08	27.44
2.4	42.34	5.88	89.38	25.48
2.6	43.32	3.92	94.28	23.52
2.8	43.90	1.96	98.78	21.56
3.0	44.10	0.00	102.90	19.60
3.2	43.90	−1.96	106.62	17.64
3.4	43.32	−3.92	109.96	15.68
3.6	42.34	−5.88	112.90	13.72
3.8	40.96	−7.84	115.44	11.76
4.0	39.20	−9.80	117.60	9.80
4.2	37.04	−11.76	119.36	7.84
4.4	34.50	−13.72	120.74	5.88
4.6	31.56	−15.68	121.72	3.92
4.8	28.22	−17.64	122.30	1.96
5.0	24.50	−19.60	122.50	0.00
5.2	20.38	−21.56	122.30	−1.96
5.4	15.88	−23.52	121.72	−3.92
5.6	10.98	−25.48	120.74	−5.88
5.8	5.68	−27.44	119.36	−7.84
6.0	0.00	−29.40	117.60	−9.80

or more do not return. Call that the escape velocity]

(a) Suppose that you could accelerate a rocket at 100 g's ($= 980\ \text{m/sec}^2$). How long would it take to reach the escape velocity?

(b) How much thrust does the rocket engine have to provide? Answer in the form, "_______ times the weight of the rocket."

(c) Conventional rocket boosters give initial accelerations closer to 10 g. Argue that the height attained in reaching the escape velocity is so great (a reasonable fraction of the earth's radius) that the escape velocity up there is actually a bit less.

★★ **30. Fictitious Force** In a spacecraft in orbit, objects are completely weightless. But during a small "course correction," a period of slowing down, for example, the astronauts would feel pushed toward the front of the craft. For the duration of the burn of the retrorockets, they would feel some weight, with the "forward" direction being the "down" direction. If the slowing down is at the rate of 1 mile per hour per second and the craft's speed is 17,000 miles an hour, it would take almost 3 minutes to slow the speed down by 1%. (Verify!) During these 3 minutes, what fraction of her earth weight does the astronaut have? [Answer: Less than 5%]

HM-THIS IS EASY...
UGH!

PROBLEM 31. At the bottom of the pushup the force on the hands is maximum.

✱✱✱ **31. Pushups** Slow pushups take about 2 seconds, down and up, and have an amplitude around 50 cm top-to-bottom, so the average speed is 0.5 m/sec. The speed is certainly not uniform, since the motion stops at the top and at the bottom. A measurement shows that the force of the hands against the floor increases by 25% at the bottom turnaround (see the figure).

(a) What is the maximum acceleration of the shoulders during this phase of the pushup?

(b) Estimate how long this acceleration lasts.

(c) How far do they move—down and up—during this time?

(d) The force of the hands against the floor can be almost zero at the top. If so, how long can that last?

✱ **32. $F = ma$** To accelerate a 100-pound vehicle at an acceleration of 22 miles per hour per second along a level road requires a push of 100 pounds. This assumes that friction can be neglected.

(a) Suppose that the horizontal accelerating force is 25 pounds. In how many seconds does the vehicle accelerate from rest to a speed of 11 miles per hour?

(b) You were able to solve this problem by proportions without ever calculating the mass. Here is the same problem in SI units: mass = 45.36 kg; accelerating force = 111 newtons. How long will it take to attain a speed of 4.9 m/sec?

✱ **33. Accelerating the Car** For acceleration on a smooth, horizontal road, the bookkeeping is easy. A horizontal force of 3000 pounds applied to a car weighing 3000 pounds will impart to it an acceleration of 22 miles per hour per second (= g). (See the accompanying figure.) For other accelerations, just use proportions. Remember, no horizontal force is needed to support the weight.

(HINT: A 1500-pound push will accelerate it at 11 miles/hr sec.)

(a) How much horizontal force is needed to accelerate the 3000-pound car at 1 mile per hour per second? Answer in pounds.

(b) Estimate how much force is needed to *keep* a car rolling at 1 mile per hour on a smooth road. The answer is not likely to be more than 100 pounds. All that force does is to overcome friction in the bearings and rolling friction in the tires. At such slow speeds, air drag is negligible. Whatever your answer, if you pushed with a horizontal force 50 pounds harder than that, what would be the acceleration of a 3000-pound car?

(c) What would be the acceleration of a 2000-pound car? Feel free to leave your answer in miles per hour per second.

(HINT: It is not necessary to convert the mass to kilograms and the force to newtons. "Proportional reasoning" is easiest here. Just don't make the mistake of trying to apply the formula $F = ma$ with F in pounds *and* m in pounds.)

PROBLEM 33. Accelerating the car takes more force than keeping it rolling at constant speed.

✱ **34. Accelerating Elevator** An elevator starts up (up) from rest and accelerates uniformly at an acceleration of 4.4 miles per hour per second (= 0.2 g).

(a) How fast is it going after 1 second?

________ miles/hour

(b) How fast is it going after 0.5 sec?

(c) What is its average speed during the first second?

(d) What is its average speed during the first half-second?

(e) How fast is it moving after 0.25 sec?

(f) How fast is it moving after 0.75 sec?

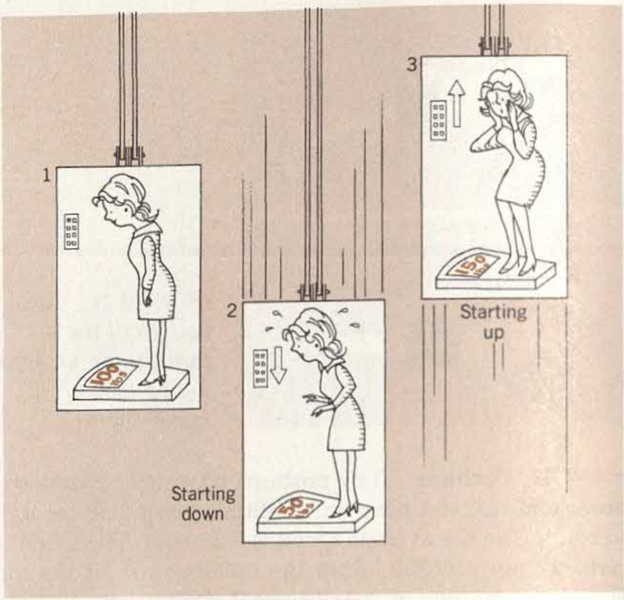
(g) What is its average speed during the second half-second?

(h) In order to double the tension in the elevator cable, the elevator would have to be accelerating upward at 22 miles/hr sec. By what percentage is the tension increased by an acceleration of 4.4 miles/hr sec?

35. Accelerating Elevator (Using Different Units) The elevator accelerates uniformly upward at 6.4 feet per second per second.
(a) How fast is it going after 1 second? _______ ft/sec
(b) How fast is it going after 0.5 sec?
(c) How far did it move during the first second? _______ ft
(d) How fast was it moving after 0.25 sec?
(e) How far did it move during the first half-second?
(f) How far is it moving after 0.75 sec?
(g) How far did it move during the second half-second?
(h) After 1 second it moved _______ times as far as after half a second.

(HINT: Does this kind of arithmetic lead you to conclude that the distance moved is proportional to the square of the time?)

✶ **36. More About the Car** At faster speeds, air resistance becomes increasingly important. At 30 miles per hour it takes about 100 pounds to overcome the frictional force of the air, the bearings, and the tires.
(a) How much extra force does it take to accelerate that 3000-pound car from 30 miles/hr to 31 miles/hr in 1 second? Convince yourself that when the car is traveling under its own power, all that forward force is provided by the road pushing forward on the tires. Indeed, for a conventional car with transmission in the rear, all that force is on the rear tires.
(b) A good tire on dry pavement can stand a horizontal (i.e., tangential) force about 3 times the vertical (i.e., normal) force before it starts to skid. Argue that this makes it theoretically possible to brake a car from 66 miles/hr to a stop in 1 second. But please don't try to do it!
(c) What would be the force on the seat belt of a 150-pound passenger during such an acceleration (deceleration)?
(d) How far does the car travel during the braking?
(e) Suppose that it really takes twice as long (i.e., 2 seconds) to brake to a stop. Argue that the tangential force on the tires is only half as great.
(f) A realistic value for a reliable braking acceleration is −11 miles/hr sec. How far does the car travel in stopping from 66 miles per hour? _______ miles. Is your answer around 300 feet? Remember that you have neglected reaction time of the driver.
(g) List a few reasons why the car might start to skid with accelerations close to 22 miles per hour per second, even though 3 times that is theoretically possible without skidding.



PROBLEM 37. Bathroom scale on the elevator (1) at rest (2) starting down (3) starting up.

✶✶ **37. Bathroom Scale on the Elevator** The bathroom scale reads 100 pounds just before the elevator starts down (see the accompanying figure). The elevator starts suddenly, getting to a downward velocity of 11 miles per hour in 1 second.
(a) Argue that the person standing on the scale weighs only 50 pounds during the acceleration.
(b) Argue that the person's weight is back to the usual 100 pounds as soon as the elevator has attained a uniform downward velocity.
(c) If the acceleration from speed 0 to 11 miles per hour had lasted only half a second, what would have been the downward acceleration?
(d) What would have been the weight during the downward acceleration in part (c)?

NOTE: Conventional bathroom scales do not respond fast enough to make experiments on elevators practical. They usually oscillate for a second or more.

✶✶ **38. Bathroom Scale on the Elevator** The bathroom scale reads 100 pounds with the elevator at rest. We say that the person standing on the scale weighs 100 pounds. The elevator starts to move up, achieving an upward velocity of 11 miles per hour in 1 second. During this second, its average acceleration is 11 miles per hour per second. Call it half a g.

(a) Argue that the bathroom scale should read 150 pounds during the 1 second that the acceleration lasts.

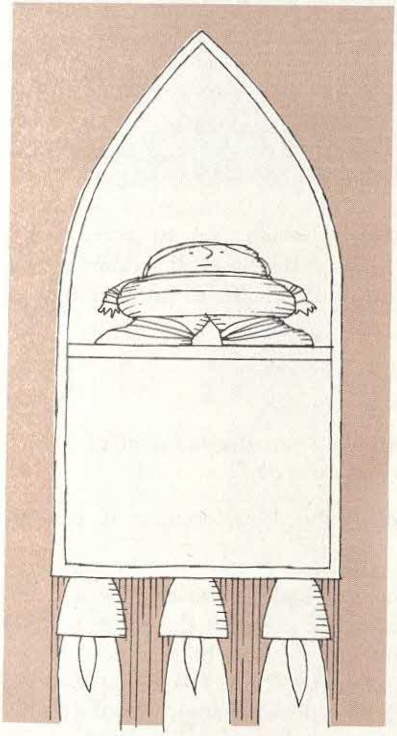
(b) Argue that the reading should drop back to 100 pounds while the elevator is moving up at a steady 11 miles per hour.

✶ **39. Your Weight on the Elevator** If the up elevator gained speed at the rate of 22 miles per hour per second, your weight would increase by 100%.

(a) If the up elevator gained speed at the rate of 2.2 miles per hour per second, your weight would increase by ________%.

(b) At what rate would the down elevator have to gain speed in order to give you a 1% weight loss during the acceleration?

✶✶ **40. Blood Pressure at Blast-Off** When you stand erect, the pressure difference between the blood in your head and the blood in your toes is approximately 0.2 atmospheres. This is just because of the weight of the blood; the pressure is proportional to the weight of the blood. If an astronaut were to be situated vertically during blast-off (see the accompanying figure), how great would that pressure difference become at an acceleration of 10 g's? (HINT: The answer is not 2.0 atm. It is even greater. Now you appreciate the need for lying flat at blast-off.)



PROBLEM 40. Blood pressure at blast-off. What would happen if the astronaut were positioned vertically?

✶ **41. Towing Icebergs** The suggestion has been made to bring icebergs from the Arctic down south to shores close to desert areas, as sources of fresh water. Consider an iceberg of mass 10^{14} kg. Suppose that tugboats can provide a towing force of 10^{8} newtons.

(a) How long would it take to accelerate the iceberg to a cruising speed of 1 m/sec? Express your answer in seconds, also in days.

(b) How far would the iceberg move during that time?

(c) FERMI QUESTION[5] Estimate the diameter of an iceberg whose mass is 10^{14} kg, good to the nearest power of 10 only. (HINT: It makes little difference whether you assume that it is a sphere or a cube.)

✶ **42. Accelerating That Barge** (a) How much force is needed to give a 10^{5}-kg barge an acceleration of 10^{-1} m/sec^2? Can one dock worker possibly pull that hard on a tow line?

(b) How long does it take to get up to a speed of 1 m/sec with that much pull on the tow line?

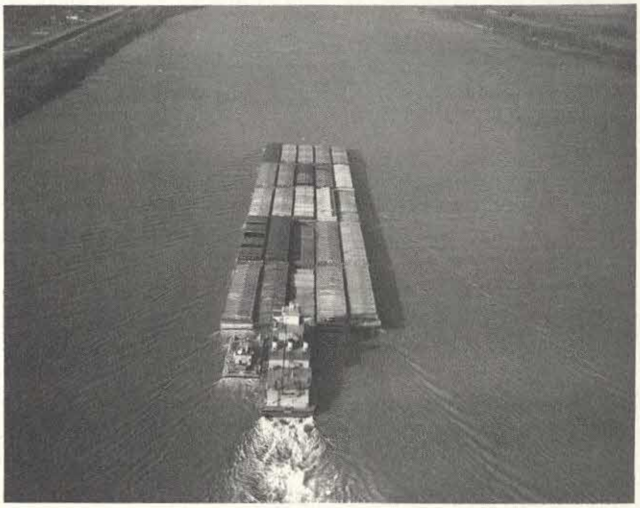
(c) How long does it take to move the barge 1 m from rest with that much pull? How fast is it going after moving 1 m?

✶✶ **43. Accelerating That Barge, in English Units** The 1-ton force will be called 2×10^3 pounds. The 0.22 miles per hour per second will be called 0.32 ft/sec^2. If we want to write the formula $F = ma$, then we solve for the mass as follows:

(a) $m = \dfrac{2 \times 10^3 \text{ pounds}}{0.32 \text{ ft/sec}^2} =$ ________________

---

[5] Enrico Fermi (1901–1954) was one of the twentieth century's great physicists. Among his discoveries was the prediction of the neutrino in beta radioactivity, and the "quantum statistics" of electrons. Having moved from his native Italy to the University of Chicago, he directed the building of the first "atomic pile" under the stands of the football stadium, and achieved a chain reaction, showing the feasibility of a nuclear-fission bomb. He was a great teacher and delighted in order-of-magnitude questions, like "How many Ping-Pong balls would it take to fill up a telephone booth?" There is no exact answer, of course.



Barges on the Mississippi River.

This unit of mass also has a name:

1 pound sec^2/ft = 1 slug

But if you want to know how much the barge would weigh (remember, weight = mg), you must multiply that mass value by $g = 32\ \text{ft/sec}^2$.

(b) weight = ________ pounds

Some people prefer to say that the barge has a *mass* of 2.2×10^5 pounds. If they also want to write $F = ma$, then the 2000-pound force becomes

$F = 2.2 \times 10^5$ pounds mass $\times\ 0.32\ \text{ft/sec}^2$ = ________.

Again a new unit:

1 pound mass ft/sec^2 = 1 poundal

A 1-poundal force gives a 1-pound mass an acceleration of 1 foot second per second. Conversion: 32 poundals = 1 pound force.

(c) Another choice is to write the mass in pounds *and* the force in pounds. Then the formula has to be rewritten

$F \times 32\ \text{ft/sec}^2 = ma$

Verify that the acceleration a 2×10^3-pound force gives a 2.2×10^5-pound mass barge is what you got before.

★★ **44. $F = ma$ in Nuclear Physics** The uranium-238 nucleus is radioactive and emits an alpha particle, thereby transforming into a thorium-234. The force on the alpha as it crosses the surface of the Th-234 is 560 newtons. Find the instantaneous acceleration of the alpha. [HINT: You can calculate the mass of an alpha particle (a helium-4 nucleus) by knowing that 6.02×10^{23} (Avogadro's number) alpha particles weigh 4 grams.]

★★ **45. Sudden Stop** An ice hockey player can achieve speeds of 15 m/sec and can stop in as little as 2 meters.

(a) How long does he take to cover that 2 meters?

(b) What is his average acceleration during this time?

(c) How many times his weight do his ankles have to support during this time? Neglect the mass of his feet.

(d) What angle does this force make with the vertical?

★ **46. Catcher's Mitt, $F = ma$** Try to calculate the force on the catching hand of the baseball catcher during the time the ball is slowing down in his glove. Here are the data:

Mass of the baseball = 0.15 kg

Speed of a fast pitch = 40 m/sec

Distance his hand recoils from the moment of contact until it stops = 0.5 m

Assume that the acceleration (deceleration) is uniform over those 50 cm.

★★ **47. Rolling Hospital Bed** A certain hospital bed has a mass of 250 kg, which includes the patient. If the nurse pushes it horizontally with less than 50 newtons (= 11.2 pounds) of force, static friction keeps it from moving. As soon as it does start to move, though, about 10 newtons (= 2.25 pounds) is enough to keep it moving at constant speed. (Rolling friction is less than starting friction.) So if a 50-newton force is applied, that leaves 40 newtons to accelerate the bed.

(a) How fast is it moving after 2 seconds?

(b) How far has it moved in 2 seconds?

(c) To avoid crashing into a wall, the nurse now pulls back on the bed and stops it in 10 centimeters. How many *g*'s is that acceleration (deceleration)?

(d) How hard did she have to pull? (HINT: This time the 10 newtons of frictional resistance help her.)

(e) Comment on the effect of the bed springs on the answers to parts (c) and (d).

✶✶ **48. Detached Retina: Tennis, Nodding, and Boxing** A patient with a detached retina is warned by the opthalmologist not to shake the eye. The doctor recommends that backward acceleration not exceed 1 or 2 *g*'s. The danger is that the vitreous humor—with the retina adhering to it— might pull away from the sclera, to which the retina is normally attached. "Doctor, can I play tennis?"

(a) Use estimated values of running speeds and the time it takes a player to stop, and calculate the maximum acceleration of the body in a fast game of tennis.

(b) A vigorous "yes" nod of the head might take a quarter of a second and have an amplitude of as much as 10 centimeters up and down. Estimate the maximum acceleration of the eye, making reasonable assumptions about the nature of the motion.

(c) Invent some reasonable values for times and amplitudes of head motion in boxing, and show that the acceleration of the head might well exceed 2 *g* (see the figure).

PROBLEM 48. How many *g*'s does the head experience in a well-aimed punch on the jaw?

✶✶ **49. Golf Ball** A golf ball might leave the tee at 50 meters per second, having been in contact with the surface of the "wood" for about 10^{-4} second.

(a) Calculate the average acceleration of the ball during this time.

(b) Calculate the average force on the ball during this time. Express it in the form "_______ times the weight of the ball."

(c) Argue that, in fact, the force is less than this value near the beginning and near the end of the contact, and greater while the elastic deformation of the ball is greatest.

✶✶ **50. Fly Landing** A fly can cross a big room in about a second. Cruising speeds must be around 10 m/sec. Does it use its wings to slow down before landing or do its legs exert all the deceleration force? Would the legs be strong enough? They are about 5 millimeters long, and the fly's mass might be estimated at 50 milligrams. Let's calculate how much force those six legs would have to withstand to stop the fly in 5 mm. That force would be least if the acceleration from 10 m/sec to zero were uniform. At an average speed of 5 m/sec, that would take _______ seconds, giving an average acceleration of _______ m/sec^2 or _______ *g*. Can you believe that the legs could support a thousand times the weight of the fly? That would be the weight of _______ grams. Are the legs that strong? Would a fly be crushed under that much weight?

✶✶ **51. Acceleration of the Housefly** High-speed stroboscopic photography shows how a fly takes off. It jumps backward while starting to flap its wings. It is up to its maximum flying speed (around 10 m/sec) in about one-quarter of a second.

(a) Calculate its average acceleration during that time.

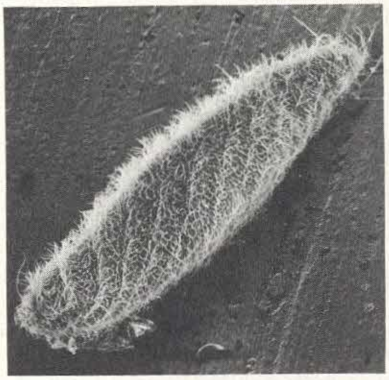
(b) How large a force is needed to produce that acceleration? Express it in the form "_______ times the fly's weight." (HINT: Why don't you have to know the fly's weight?)

NOTE: Comparisons between man and insect are discussed in Chapter 3, "Size, Strength, and Scaling."

✶✶ **52. Swimming Protozoa, $x = \frac{1}{2}at^2$, and $F = ma$** A paramecium is a one-celled animal about 0.2 millimeters long, with a mass around 10^{-10} kg (= 0.1 microgram). Using its hairlike cilia, it looks like a fast swimmer. Starting from rest, it can move its own length in one-fifth of a second.

(a) Assuming that it accelerates uniformly during that time, calculate its acceleration in m/sec^2 and translate that into *g*'s. (HINT: $1\ g = 9.8$ m/sec^2.)

(b) Calculate the force needed to produce that acceleration. Neglect the mass of water the animal pulls along as it moves.



Paramecium can coordinate its cilia to work like flippers.

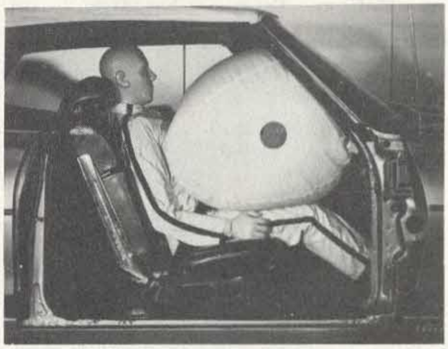
(c) What fraction is that force of the animal's weight in air?
(d) About how long does it take you to swim your own length starting from rest?

✶✶ **53. The Shoes Stayed Behind** In auto accidents one sometimes reads that a survivor was "pulled right out of his shoes." Let's estimate how many *g*'s of vertical acceleration the foot would have to suffer to pull out of its shoe. It takes perhaps 10 pounds of force (= 44.5 newtons) to pull a firmly laced shoe off a foot. (Try it.) A shoe might weigh half a pound. How many *g*'s of acceleration would it require for that shoe to be pulled with a 10-pound force?

✶✶ **54. Seat Belts—How Strong Do They Have to Be?** The maximum braking acceleration a car can achieve is about 3 *g* (= 66 miles per hour per second = 30 m/sec^2). To be sure, this is a rare happening, requiring good tires and dry pavement. During such an acceleration of 3 *g* the horizontal force against the seat belt is 3 times the usual weight of the passenger. If that is all we worried about, 600-pound-test belts would suffice even for 200-pound passengers. But consider a head-on collision, say at 30 miles per hour (= 13 m/sec).
(a) If the car is stopped within 1 meter, show that the average acceleration during the collision is 89 m/sec^2 (= 9 *g*), so the seat belt needs to hold 9 times the passenger's weight.
(b) Show that a 60-mph collision requires belts 4 times as strong as a 30-mph collision.

✶✶ **55. Air Bags** One way to protect the driver and passengers in a car against injury in a collision is with a device that inflates when the deceleration of the car exceeds a certain value (perhaps 5 *g*'s). Suppose that a car traveling at 20 m/sec (= 45 miles per hour) has a head-on collision, so that its forward motion is stopped in about

A



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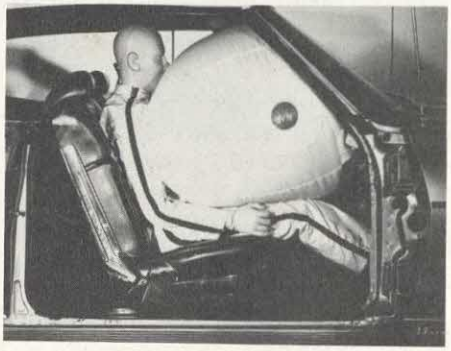


C

D

1 meter. That 1 meter is the amount by which the front of the car may be expected to deform.

(a) Show that the average acceleration of a passenger is -200 m/sec^2 in this case. (HINT: How long does it take to travel that 1 meter?)

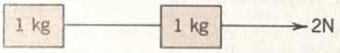
(b) The horizontal force on the driver during the acceleration is ________ times his or her normal weight.

(c) If that much force were applied to a few ribs, they would surely break. The point of the air bag is to distribute the force over a larger area of the body. But the bag also may allow more time for the acceleration. Argue that an air bag 0.5 m thick extends the stopping time to 0.15 sec and decreases the average acceleration by about 33%.

✶✶ **56. Train** A pull of 2 newtons to the right on the 2-kilogram "train" results in an acceleration of 1 m/sec^2 to the right.

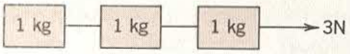
(a) What is the tension in the left-hand string?

(b) What is the net force on the right-hand mass?



(c) With a 3-kg "train," what is the tension in the middle string?

(d) What is the net force on the middle mass?



✶✶ **57. Falling Without Accelerating** A speck of dust is seen to fall at a constant speed. It does not accelerate at the familiar 22 miles per hour per second (= 9.8 m/sec^2). We conclude that its weight is exactly compensated by air resistance. With zero net force, its acceleration is zero. It accelerates only during the first fraction of a second of its fall, because air drag increases with speed v.

Suppose that the air drag is given by the formula

$$\text{retarding force} = \left(10^{-6}\,\frac{\text{newton sec}}{\text{m}}\right) v$$

and the speck weighs 10^{-8} newton.

(a) At what speed will it stop accelerating?

(b) What is its acceleration when it is falling at half of this "terminal speed"?

NOTE: The air-drag formula, called Stokes' Law, depends on the shape of the object. For a spherical speck, the drag force is proportional to the diameter.

✶✶ **58. On Avoirdupois Pound** If we decide to measure force in pounds and mass in avoirdupois pounds, we must be careful with the formula

$$\text{weight} = \text{mass} \times g.$$

In Paris, the formula would say

$$1 \text{ pound weight} = 1 \text{ pound mass} \times g.$$

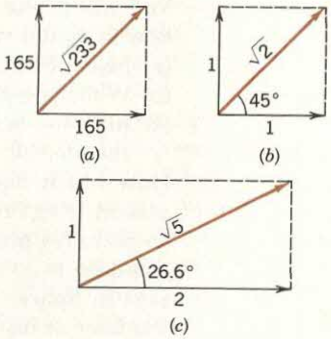
That may seem strange, since it makes g equal to one. One what? The obvious answer is "one pound weight per pound mass." But that is the g-value for Paris. In a different place with a different value of g, the same mass has a different weight. In a location in which g is one-tenth of a percent greater than in Paris, a 1-pound mass would weigh 1.001 pounds. Just to make sure that you understand this piece of sophistry, verify that, on the moon, a 1-pound "mass" weighs only 0.16 pounds (weight). (HINT: On the moon, g is 1.6 m/sec^2.)

✶✶✶ **59. On Vector Addition of Forces** (Not discussed in the text; see Appendix A.) As the airplane accelerates on the runway, the back of your seat cushion pushes on you. If the speed increases at 22 miles per hour per second (= g), that forward force is equal to your weight. If you normally weigh 165 pounds, the bottom of the seat cushion pushes up with a force of 165 pounds, while the back of the seat cushion pushes forward with a force of 165 pounds. The accompanying figure shows how to add two forces that are not in the same direction. The rule is called the "vector-addition rule" or the "parallelogram rule."

(a) Argue that the resultant force on you makes a 45° angle with the horizontal.

(b) Argue that the resultant force is 233 pounds. (HINT: 165 pounds × $\sqrt{2}$. Pythagoras!)

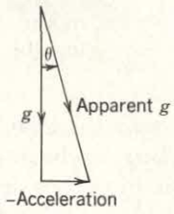
(c) If the airplane's acceleration were 44 miles per hour per second (= 2g), the back of the seat cushion would push with a force of 330 pounds. Argue that the direction of the resultant force would then be 26.6° above the horizontal. Find its magnitude.



PROBLEM 59. Vector addition of forces.

✶✶✶ **60. They Always Collect Your Tray Before Landing.** (Not discussed in the text. See Appendixes A and B.) What if the airlines didn't remove your tray? Would food and drink spill on your lap? Assume that you don't spill unless the cup deviates from the vertical by more than 10°. Certainly, waiters wobble more than that. Note that tan 10° = 0.176. So you could stand acrobatics up to $0.176\ g$ = 4 miles per hour per second.

From your experience with airplanes, estimate the maximum acceleration (deceleration) of the airplane after touching the ground (see the figure). How much would the surface of your coffee tilt?



PROBLEM 60. "They always collect your tray before landing."

✶✶ **61. Speeding Up Feels Like Driving Uphill** (see Appendix A) According to the Principle of Equivalence, "gravity" and "acceleration" feel the same. If your car starts going uphill, the back of the seat cushion starts to push on your back. That is exactly what happens when your car speeds up (see the figure). Argue that an acceleration of 2.2 miles per hour per second (= one-tenth the acceleration of gravity) on a horizontal road feels like driving at constant speed up a 1-in-10 incline. One-in-ten means the road rises 1 foot in every 10 feet of road.

PROBLEM 61. Speeding up feels like driving uphill.

✶✶ **62. Least Noticeable Difference in g** If the local value of g were to increase by 2%, you might not notice the increased downward pull on all parts of your body without a scale. But if the direction of the apparent "vertical" were to change because the floor under you started to accelerate horizontally at 0.02 × 22 miles per hour per second, you might well notice. How sensitive is your balancing organ? Could it detect an incline that rises 1 foot for every 50 feet along the floor?

☐ QUESTIONS FOR FURTHER THOUGHT

α. **Blood Pressure During Jump** If you don't have access to a trampoline or diving board, just try jumping straight up from the floor while standing with closed eyes. Can you tell the instant you are at the highest point, the point of zero velocity? Can you feel the motion of your stomach?

Now concentrate on the difference in blood pressure between head and heart. You will probably not be able to feel this during the jump. Try to reason out how it should vary:

(a) Just before take-off: greater than usual / less than usual
(b) While you are in free fall: greater than usual / less than usual
(c) Just after landing, while your knees are flexing: greater than usual / less than usual

β. **How Much Squeezing Can We Stand?** In centrifuge experiments a human subject lying on his back (see Figure 1.13) became unconscious between 14 and 16 times his normal weight when he had his knees slightly bent. Subjects could stand up to 20 times normal with straight legs. Try to explain the difference.

γ. **Lost in Space** An astronaut in orbit gives a slight push with his foot against the floor of his space craft so that he loses contact. Slowly he floats upward in

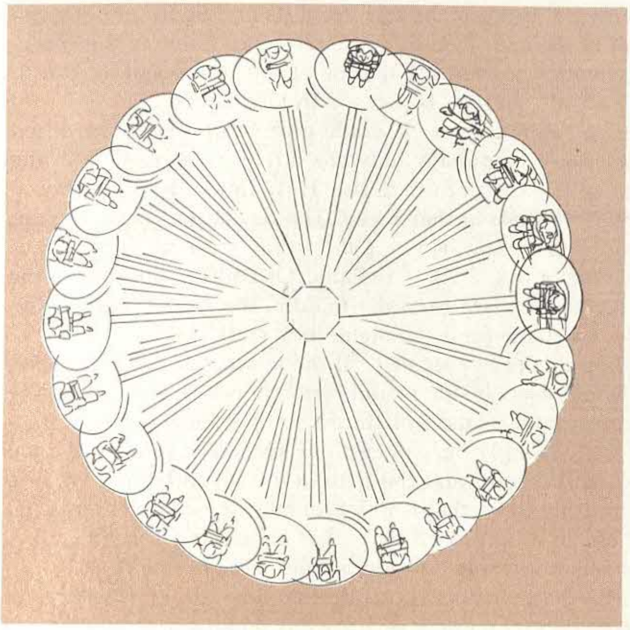


FIGURE 1.13 Human subject in a centrifuge.

the large cabin. He is far from the walls or anything to hang on to. His upward velocity is so slow that he would not hit the ceiling for some time. Invent two or three different ways he can get back faster. One obvious way is to have his roommate throw him a rope and pull him back. (HINT: You will know some answers after reading Chapter 2.)

δ. **Weightless Water**
(a) Describe what would happen if an orbiting astronaut tried to drink a glass of water the way we do here on earth?
(b) How would she get the water into the glass in the first place?

ε. **Constancy of *g*** Give evidence for your belief that the value of g in your house does not change from day to day or even from year to year. Would you be able to detect a 1% change?

ζ. **Bathroom Scale, Doctor's Scale** If you used a bathroom scale to weigh yourself, your weight on the surface of the moon would read one-sixth its usual earthly value. But if you used a doctor's scale ("No Springs—Honest Weight"), the reading would be unchanged from its usual value. Both instruments measure gravitational pull. Explain why the readings are so different.

η. **How Straight is That Straight-Line Path in the Spacecraft?** The text says that in an orbiting spacecraft, an object keeps moving in a straight line until it bumps into something. How accurate is that statement? You may argue as follows: The center of the earth is not in exactly the same direction when you are in the middle of the spacecraft as when you are at the rear of the spacecraft. If the craft is 5 meters from middle to rear, and it is 6400 kilometers from the center of the earth (tight orbit), those directions make an angle of $(5/6.4 \times 10^6 =)\ 0.78 \times 10^{-6}$ radians. The argument continues: A freely flying object experiences a downward acceleration of 9.8 m/sec^2. The spacecraft is freely flying, so it experiences that acceleration, in the downward direction, as seen from its middle (center of gravity). An object at the rear of the craft would, in the reference frame of the spacecraft, experience an acceleration toward the middle of the craft equal to $0.78 \times 10^{-6} \times 9.8 \text{ m/sec}^2 = 7.7 \times 10^{-6} \text{ m/sec}^2$. Is that worth talking about?

Let's see how much of an "aiming error" it would give. Suppose you launched an object "upward" (meaning in a radially outward direction, as seen from the spacecraft) near the rear of the craft at 0.1 m/sec. How much does it curve in moving 1 meter? It takes 10 seconds at that speed. The deviation from a straight line, that is, the forward displacement, is $x = \frac{1}{2} at^2 = 3.8 \times 10^4$ m, corresponding to an angular deviation of 3.8×10^{-4} radians. Of course, the longer the spacecraft, that is, the further you are from the middle, the bigger this acceleration toward the middle. Calculate for a 100-meter-long craft.

θ. **What About the Tumbling of the Spacecraft?** If the spacecraft were tumbling end over end, a freely flying object inside would certainly not be seen as having straight-line motion. Seen from a rotating platform, straight lines become circular. So we are tempted to say simply, let's not let our spacecraft tumble. But we might want the front to remain front and the bottom to remain bottom while it circles the earth. The only way to do that is to let it make one complete rotation every complete revolution. In a tight orbit that takes about 5000 seconds. Does such a slow rotation give us noticeable deviations from straight-line flight?

Again, let the object fly 1 meter in 10 seconds. In that time, the craft will rotate through an angle of $2\pi \times 10/5000$ radians $= 1.2 \times 10^{-3}$ radians. so the aiming error will be 1.2 millimeters.

Notice that the larger the orbit, that is, the farther the craft is from the earth, the slower this rotation, the smaller the aiming error. Look at the previous Question for Further Thought. Comment on how that deviation from straight-line motion depends on the distance from the earth.

ι. **Overcoming Classical Misconceptions** To keep a sled moving on a flat snowfield, you have to push. To make it go faster, you have to push harder. Twice as fast, twice as hard? It looks as though the "law" might be that the velocity is proportional to the applied force. But Galileo and Newton wrote—about 300 years ago—that an object maintains its state of uniform motion when *no* force acts on it. That does not seem to conform to our experience with sleds!

Argue that when the sled has achieved a constant speed, the net force on it is indeed zero. Remember, your push is not the only force on the sled. The snow also exerts force. Which way? How does the snow know which way its frictional force should be exerted?

Argue that the snow's retarding force is greater the faster the sled moves. But be careful: Just because you know that it is so doesn't mean you understand why. Note: That part is quite complicated. And the "law" that the frictional force is proportional to the velocity is very approximate indeed. In fact, for the friction of the *air* at automobile speeds, the force is approximately proportional to the square of the velocity.

κ. **Frames of Reference—Air Track Accelerating** The linear air track is a device for eliminating most of the friction we usually associate with the motion of vehicles here on earth. If we can make our track straight enough and level enough, the motion of the gliders is measured to have constant velocity, that is, no measurable acceleration, over several meters of motion along the track.

Suppose that such an air track is carried on a cart moving at constant velocity along the corridor. Will the glider still move at constant velocity

(a) in the reference frame of the air track?

(b) in the reference frame of the corridor? Present a brief argument for your answers to parts (a) and (b).

(c) Now let the cart be accelerated. Answer the same two questions.

NOTE: This is strictly a thought experiment. If you try it with a real cart on a real corridor, you will find you can't achieve sufficiently constant velocity of the cart down the corridor. What you need is an air track carried on an air track!

λ. **Sketch the Graph** An air-track glider is outfitted with a battery-powered model-airplane motor and propeller capable of providing a constant forward force for several seconds.

(a) Argue that, starting from rest, the velocity-versus-time curve for the glider has to have the straight-line shape shown in Figure 1.14.

(b) A remote-control switch turns off the motor, so the glider continues under no force. Sketch the v-versus-t curve in Figure 1.14 for the next few seconds.

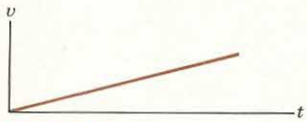


FIGURE 1.14 Sketch the graph.

μ. **Constant Speed? Analyze Some Data** Timing a bicyclist, we find that he travels 100 meters in 10 seconds. At the end of the 11th second, he reaches the 110-meter mark. At $t = 15$ seconds, he is at the 150-meter mark.

(a) Calculate his average speed (in meters per second) during the first 10 seconds

(b) Calculate his average speed during the 11th second.

(c) Calculate his average speed during the last 4 seconds.

(d) Is it possible that he traveled 60 meters in the first 5 seconds and 40 meters in the second 5 seconds?

(e) Is it possible that he traveled 90 meters in the first 5 seconds and 10 meters in the second 5 seconds?

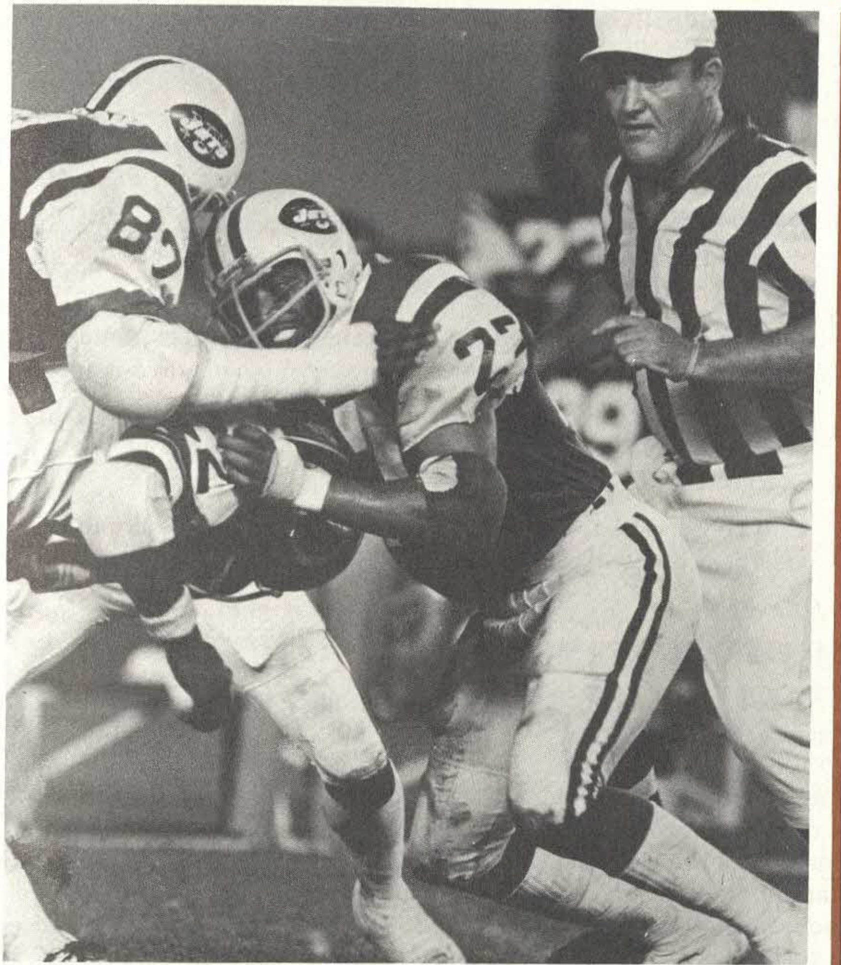
(f) Why do you think that part (e) is unlikely?

(g) Have you ever observed a cyclist accelerating from 2 m/sec to 10 m/sec in 1 second?

(h) What average acceleration would that require during that 1 second?

(i) Is the part (e) record possible without such a large acceleration at any time? If so, sketch a possible graph of speed versus time for those 10 seconds in which there are no accelerations as great as 8 m/sec^2.

(j) Estimate how long it takes a good cyclist to accelerate from rest to 22 miles per hour on a flat road.



A somewhat inelastic collision.

2

MOMENTUM CONSERVATION

SOME QUESTIONS THIS CHAPTER WILL ANSWER

Is the momentum of two colliding objects always conserved in the collision?

What is the "law of action and reaction"? Can it be used to derive the conservation of momentum?

If two objects are at rest and an explosion between them sends one to the right, the other to the left, argue that their center of mass will stay put or will move after the explosion.

Why is it so hard to step out of a canoe onto a dock without hanging on?

Argue that it is possible (or impossible) to "pull yourself up by the bootstraps." The expression comes from a story about being stuck in quicksand.

Can conservation of momentum be violated by a description in an accelerated frame of reference?

2.1 ESSAY: SPACE WALK

What is it like to open the hatch of an orbiting spacecraft and step out into space? (Fig. 2.1)

Of course, you'd better open slowly, because the air in the cabin will rush out, making a gust of wind. Also, you'd better have on that space helmet and carry your own air for breathing, because out there, there is nothing. There is nothing, in fact, to give you any sense of your motion. From the standpoint of observers on earth, you may be falling through space at many thousands of miles per hour. Yet you have no sense of speed because no air is rushing past you. You have no sense of acceleration because you are not

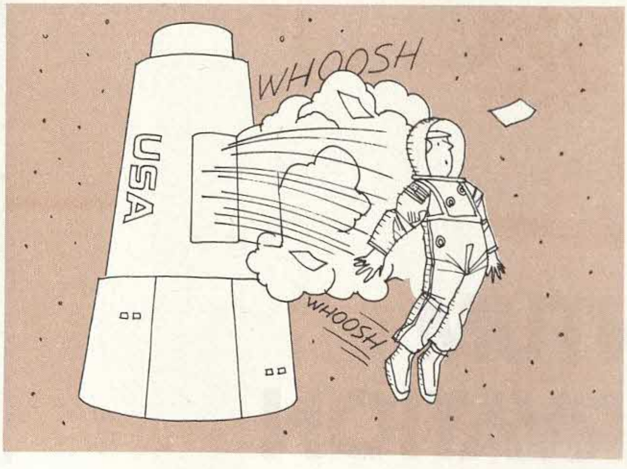
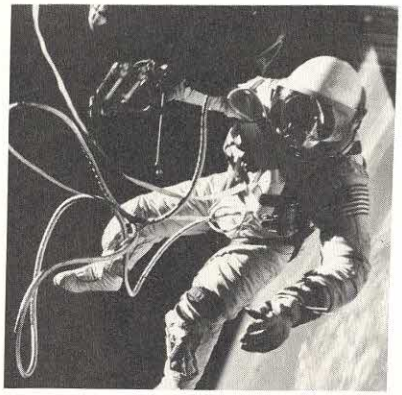


FIGURE 2.1 Stepping out of the spacecraft.

resisting the gravitational forces on you. You are falling freely, and so is the spacecraft you have just left. Unless you happened to push off hard as you got out, you'll be floating along right close to the craft. Any push you give as you get out will start you moving away from the craft. You'll just float off at constant speed in the direction your initial push gave you. You might prefer to say that it is the *craft* that is moving away from *you* at constant speed because of the initial push *you* gave *it*. What you realize here is that the question of which one is moving is a matter of point of view. In the *frame of reference* of the craft, you are moving one way. In your frame of reference, the craft is moving the opposite way. In any other frame of reference, both are moving. For example, if you and the craft are in orbit around the earth, then in the frame of reference of the earth your motion while you are moving away from the craft is quite complicated. To be pragmatic, when we want to describe a motion we usually choose a frame of reference that gives a simple description of that motion. Here is an example:



Astronaut leaving his spacecraft on a "space walk."

Equal Masses

You and your identical twin are floating just outside the craft, as shown in Fig. 2.2, with no perceptible motion. You give your twin a shove. Afterward–you see him receding from you at 2 meters per second, while the craft is receding from you at 1 meter per second.

What is the simplest description here? Nobody pushed on the craft; why should it have started to move? A description in the frame of reference of the craft says that you and your twin, after pushing off, moved off in opposite directions at 1 meter per second. The symmetry between yourself and your (equally massive) twin is preserved in this description. Each experiences the same acceleration during the push, or, more accurately, equal and opposite accelerations. Which of you actually exerted the muscular effort is not important for the subsequent motion.

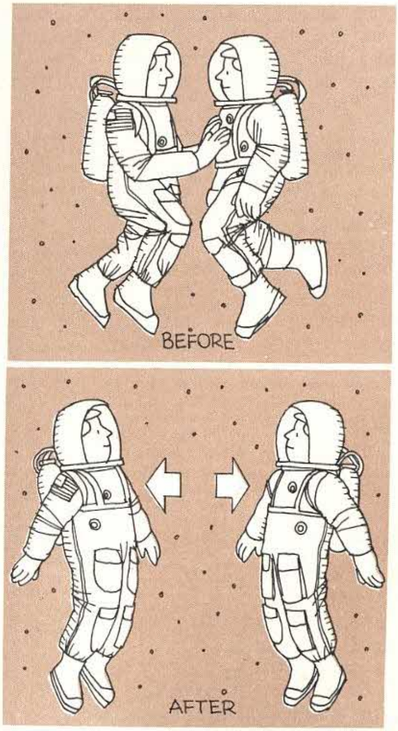


FIGURE 2.2 Identical twins pushing off in space.

2.2 DEFINITION OF MOMENTUM

How do you find the simplest frame of reference for describing a motion? In our example of the twins, there is an appealing symmetry in choosing the spacecraft frame of reference. In this frame both were at rest before the push-off. It is in this frame (also called the **center-of-mass frame of reference**) that you both receded at the same speed, but in opposite directions. If you call your velocity +1 m/sec, you could call your twin's velocity −1 m/sec. The product of mass times velocity is called **momentum.** If your mass is 75 kilograms, your momentum after the push-off is 75 kg × 1 m/sec = 75 kg m/sec. Your twin's momentum is −75 kg m/sec. The sum of your momenta is zero, just what it was before the pushing. We can say that, in the spacecraft frame of reference, the total momentum is *conserved:* It was zero before the interaction (the pushing); it is zero after. Note that we chose the rest frame of an object on which nobody was pushing. The spacecraft as a massive object was irrelevant to the situation. It just provided an imaginary place to sit and measure velocities of the objects on which forces were exerted—the twins.

EXERCISE 1 Calculate the momentum of your twin in your own frame of reference. Note that, in this frame, your own momentum remains zero. This is an "accelerated reference frame" (Remember the feeling of weight during the push?); total momentum is not conserved when measured in such a frame.

A harder push would give you more momentum, and your twin more momentum in the opposite direction. Again, the sum of the momenta would be zero. But what if the other fellow were more massive? We might think of a 75-kg twin with a 75-kg pack, total mass = 150 kg. If, in the spacecraft frame of reference, your speed after the push is 1 m/sec, his speed in the opposite direction turns out to be exactly half of that, or 0.5 m/sec. How come? Because that is the speed that conserves momentum. Your momentum is

$$(75\ \text{kg}) \times \left(1\frac{\text{m}}{\text{sec}}\right) = 75\ \text{kg m sec}^{-1}$$

the same as in the previous example; his momentum (pack and all) is

$$(150\ \text{kg}) \times \left(-0.5\frac{\text{m}}{\text{sec}}\right) = -75\ \text{kg m sec}^{-1}$$

Total momentum is conserved: zero before, zero after. In order to gain a certain amount of momentum, you have to give an equal and opposite amount of momentum to another object, regardless of its mass. In order to impart a certain amount of momentum to an object, you have to recoil with an equal and opposite momentum.

Now we can instruct our astronaut how to maneuver in space without hanging on to anything: Throw or shoot or somehow project an object in the direction opposite to the one he wants to go. If it is a rocket, he has to aim it in back of him. To stop, he has to aim his rocket forward; hence the word "retrorocket." What the law of conservation of momentum tells us is that the change in momentum of the astronaut plus the rocket engine is numerically equal to the momentum of the ejected rocket fuel, and, of course, it is in the opposite direction.

EXERCISE 2 How many kilograms of rocket fuel does a 75-kg astronaut have to eject at a speed of 300 m/sec in order to recoil at 1 m/sec, measured in the center-of-mass frame of reference?

HINT: total momentum remains zero.

$$75\ \text{kg} \times 1\frac{\text{m}}{\text{sec}} - \text{m}_{\text{fuel}} \times 300\ \frac{\text{m}}{\text{sec}} = 0$$

It does not matter which velocity you call positive, which negative.

EXERCISE 3 A 75-kg skater shoots a rifle. The 10-gram bullet (= 10^{-2} kg) is shot at 7.5×10^2 m/sec. At what speed does the skater recoil?

EXERCISE 4 Twin A moving at 1 m/sec collides with twin B, who is at rest. They hang on. At what speed do they move off together?
[Answer: 0.5 m/sec]

EXERCISE 5 The 75-kg astronaut moving at 1.5 m/sec collides with the 150-kg astronaut at rest and hangs on. At what speed do they move off together?

EXERCISE 6 The 150-kg astronaut moving at 1.5 m/sec collides with the 75-kg astronaut at rest and hangs on. At what speed do they move off together?

2.3 MOMENTUM AND FORCE

In the story of the twins, we did not say how hard you pushed. When we described your pushing your equally massive twin away, we only said that each of you ends up moving backward at 1 m/sec after the shove. That was the description of the interaction in the center-of-mass frame of reference. But it was only a before-and-after description. It did not specify the force or how long the push lasted. Suppose that the strength of the push was a steady 150 newtons. Then your 75-kilogram twin would accelerate away at a uniform acceleration of 2 m/sec^2.

$$150\ \text{newtons} = 75\ \text{kg} \times 2\frac{\text{m}}{\text{sec}^2}$$

To get up to a speed of 1 m/sec, the push would have to last 0.5 sec.

$$2\frac{\text{m}}{\text{sec}^2} \times 0.5\ \text{sec} = 1\frac{\text{m}}{\text{sec}}$$

In momentum language, we would say that a steady 150-newton force has to last half a second to impart a momentum of 75 kg m/sec:

$$150\ \text{newtons} \times 0.5\ \text{sec} = 75\ \text{kg}\ \frac{\text{m}}{\text{sec}}$$

All we did here was to multiply the previous equation by 75 kg.

Here is the same argument generalized. Suppose that a uniform acceleration a from rest to velocity v takes time t. We can write

$$v = at$$

The momentum of a mass m so accelerated would be

$$\text{momentum} = mv = mat$$

You can interpret the ma on the right-hand side using Newton's Second Law,

$$F = ma$$

to get

$$mv = Ft \qquad \text{(Eq. 1)}$$

In words: The momentum attained by being pushed with force F for time t is equal to the product of the force times the time.

Now back to our astronaut. Call M the mass of the astronaut plus the engine and m the mass of a small amount of rocket exhaust negligible compared with M. The momentum of the rocket exhaust is mv; the momentum of the recoiling astronaut is MV. The ejection velocity v and the recoil velocity V are both measured in the astronaut's resting frame, the frame of reference in which he was at rest before the mass m of exhaust gas was ejected. Of course V and v are oppositely directed, so if we call v positive (forward), then V will have to be negative (backward). The conservation of momentum says

$$mv + MV = 0$$

since we call the astronaut's momentum before the ejection zero. Repeat: The recoil momentum acquired by the astronaut as a result of ejecting that jet of gas is equal and opposite to the momentum of the gas:

$$MV = -mv$$

In the language of forces, this can be restated using Eq 1:

$$F_{\text{man}\leftarrow\text{gas}} = -F_{\text{gas}\leftarrow\text{man}} \qquad \text{(Eq. 2)}$$

To see that the last two equations are equivalent, simply write each momentum as a force times time product, and then cancel out the equal times. Equation 2 states that the force exerted by the exhaust on the man is equal and opposite to the force exerted by the man on the exhaust. We have really proved a general law of physics here, called the Law of Action and Reaction, or **Newton's Third Law.** (See Horse-and-Wagon Paradox, page 50.) Evidently, it is logically equivalent to the Law of Conservation of Momentum, once you are given the Law of Motion $F = ma$.

It may be instructive to go back over some of the physical situations introduced in these first two chapters to see how "action equals reaction" applies. You may find you "knew it all the time." Let's start with the support couch for the astronaut at blast-off (Sec. 1.2). We asked about the force tending to compress him, the upward force exerted by the couch. Surely that is equal to the downward force his body exerts on the couch. Indeed, substituting a bathroom scale for the couch (a thought experiment!) in order to measure that force implies our faith in the equality of the downward force *on* the scale and the upward force exerted *by* the scale. The same goes for the elevator: While the down-elevator is slowing to a stop, the extra upward force the floor is exerting on the passenger is equal to the extra downward force the passenger's shoe soles are exerting on the elevator floor. How about the accelerating drag racer? The road exerts a forward force on the tires, and that has to be exactly equal to the backward force the tires exert on the road. Also, in the case of the twins pushing off against each other in space, remember, it doesn't matter which one is doing the work. If you are the one pushing with your hand against the other fellow's chest, you know that the effort is all yours, but the force his chest exerts on your hand is equal (and opposite) to the force with which your hand pushes on his chest.

You may have heard the expression "pulling yourself up by the bootstraps." The Law of Action and Reaction is the physical principle that forbids that miraculous feat in the real world. (Fig. 2.3) So don't expect to go anywhere by blowing on the sails of the sailboat you are sitting in.

2.4 CENTER OF MASS

For the two identical masses (the twins) their **center of mass** is the point located exactly midway between them. That is just a definition of a mathematical point. Even though it is called the center of mass, there is, in general, no mass there. If B has twice the mass of A, the center of mass is half as far from B as it is from A; that is, it is one-third of the way from B, or two-thirds of the way from A.

FIGURE 2.3*a* Pulling yourself up by the bootstraps.

FIGURE 2.3*b* Blowing at the sails of your own sailboat.

HORSE-AND-WAGON PARADOX

According to the Law of Action and Reaction, the wagon pulls back on the horse just as hard as the horse pulls forward on the wagon. How can they ever get anywhere?

This apparent paradox is resolved by analyzing separately the forces on the horse and the forces on the wagon.

Forces on the wagon:

The earth (gravity) pulls down on it. The ground supports it, that is, pushes up, with an equal and opposite force, assuming that the ground is level. The horse pulls forward on the collar. The ground exerts frictional resistance backward.

As long as the frictional resistance is equal to the forward pull, the wagon does not accelerate. If it is stuck, it stays stuck. If it is moving forward, it continues to move forward at constant speed. Only if the horse's pull exceeds the frictional force exerted by the ground can the wagon accelerate forward.

Forces on the Horse:

Gravity pulls down. The ground pushes up. These two forces are equal and opposite. The wagon pulls backward. Why doesn't the horse accelerate backward? Because it is digging its hoofs into the ground, pushing backward on the ground. According to the Law of Action and Reaction, the ground pushes forward on the horse's hooves (Fig. 2.4). If this forward force exceeds the pull of the wagon, the horse accelerates forward.

The force of the horse on the wagon and the force of the wagon on the horse do add to zero. To determine the acceleration of the wagon, what we need is the sum of all the forces *on the wagon*.

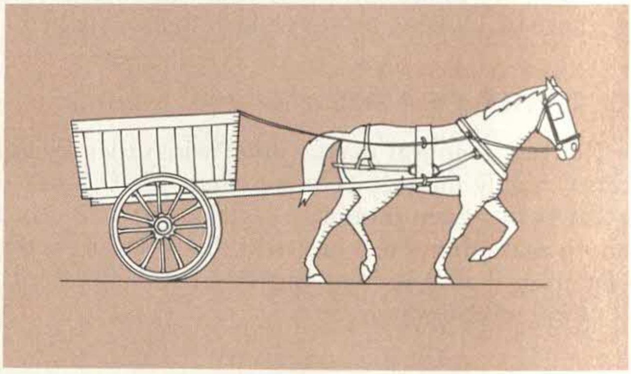


FIGURE 2.4

The beautiful property of this mathematical point is that conservation of momentum guarantees that the center of mass remains stationary. Consider our twins. After they push off, one goes off to the right at 1 m/sec, the other off to the left at 1 m/sec. They go equal distances in equal times, so their midpoint—their center of mass—stays put. We said that they have equal and opposite momenta. Calling momentum to the right positive, momentum to the left negative, we can add the momenta algebraically and say that the two momenta add up to zero. And we can use the center-of-mass concept to restate that: Zero total momentum means that the center of mass remains stationary.

That whole idea becomes more interesting for unequal masses.

EXAMPLE Let B have twice the mass of A.

$$m_A = 75 \text{ kg} \quad m_B = 150 \text{ kg}$$

After the push, A is moving to the right at 1 m/sec, B is moving to the left at 0.5 m/sec (see Fig. 2.5). So at any time after the push, A will have moved twice as far as B. The center of mass—that point half as far from B as it is from A—will stay right where it was. Let's calculate the momentum.

$$\text{momentum of A} = 75 \text{ kg} \times 1\frac{\text{m}}{\text{sec}} = 75 \text{ kg}\frac{\text{m}}{\text{sec}}$$

$$\text{momentum of B} = 150 \text{ kg} \times \left(-0.5\frac{\text{m}}{\text{sec}}\right) = -75 \text{ kg}\frac{\text{m}}{\text{sec}}$$

$$\text{total momentum} = 75 \text{ kg}\frac{\text{m}}{\text{sec}} - 75 \text{ kg}\frac{\text{m}}{\text{sec}} = 0$$

Repeat: Saying that the center of mass remains stationary and saying that the total momentum is zero are logically equivalent statements.

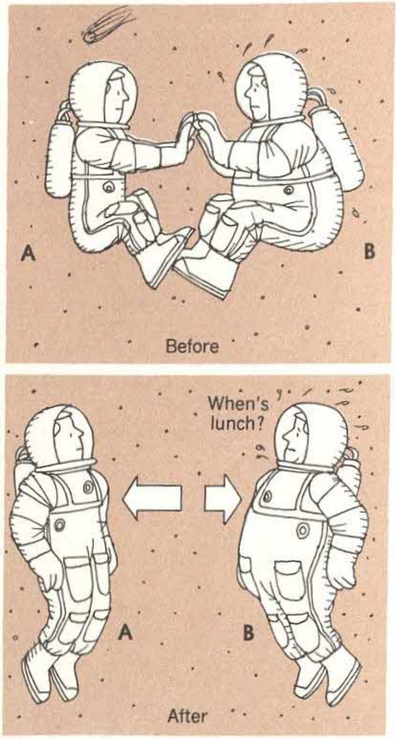


FIGURE 2.5 Pushing off in space. B has twice the mass of A.

But it is time for a proper definition. The center of mass is defined as the weighted average position of the two (or more) masses. The weighting is in proportion to the masses. Look at the previous example, say 10 seconds after the push. A has moved 10 meters to the right; B has moved 5 meters to the left. Suppose that they both started out at $x = 0$. Their weighted-average position is now

$$\begin{aligned} x_{\text{CM}} &= \frac{(75 \text{ kg}) \times (10\text{m}) + (150 \text{ kg} \times (-5\text{m})}{75 \text{ kg} + 150 \text{ kg}} \\ &= 0 \end{aligned}$$

Notice that we multiplied each position (the +10 m and the −5 m) times the mass located there and then divided the sum of those products by the total mass. That is what we call a weighted average.

There is no magic in calling the initial position $x = 0$. Suppose that the initial position is called $x = 20$ meters.

EXERCISE 7 After 10 seconds, A is at $x =$ ________ m, while B is at $x = 15$ m. Now do the weighted averaging.

$$x_{\text{CM}} = \frac{(75 \text{ kg} \times 30 \text{ m}) + (150 \text{ kg} \times 15 \text{ m})}{75 \text{ kg} + 150 \text{ kg}} = \underline{\hspace{2cm}}$$

Were you surprised at the answer? You should have known it even without doing the arithmetic (HINT: Note that the kilograms cancel and only the meters are left.) When we take a weighted average, the dimensions of the "weighting function" always have to cancel out. If you worry about A and B both starting out at $x = 0$, that is obviously an idealization. They would have to be awfully thin. We did that just to make the numbers come out pretty. The next exercise is more realistic.

EXERCISE 8 Start A out at $x = 20.50$ m, and start B at $x = 19.75$ m. (a) Where is their center of mass? (b) Where is each one after 2.00 seconds? (c) Calculate the weighted average of their two positions at this time.

Writing down a general formula for the position of the center of mass of two point masses,

$$x_{CM} = \frac{M_1x_1 + M_2x_2}{M_1 + M_2}$$

shows how the concept can be extended to any number of point masses:

$$x_{CM} = \frac{\Sigma M_i x_i}{\Sigma M_i}$$

The symbol Σ just means "sum of," and the *index i* takes on as many values as there are mass points. Of course, "position" implies three coordinates x, y, z, with analogous formulas for y_{CM} and z_{CM}. In this chapter we shall limit ourselves to motions that can be described in one dimension.

Why has the center-of-mass concept been introduced here? Well, for one thing, center of mass is usually identical with center of gravity. But it also has the following convenient property: For any set of objects, if you multiply the velocity of their center of mass by the sum of the masses the product is the total momentum of all the objects. Many discussions about the movement of athletes involve the motion of their center of mass. Examples are high jump (Problem 33), broad jump, and diving.

EXERCISE 9 Prove the convenient property. (HINT: What is meant by the velocity of the center of mass?)

EXERCISE 10 What was the velocity of the center of mass of the twins in our example?

That convenient property makes it possible to talk about the momentum of a complex thing like a human being, full of internal motions in diverse directions. The sum of the momenta involved in all those motions is the momentum of the center of mass, a point usually located somewhere in the pit of the stomach. Of course, if you hold your arms out in front of you, your center of mass will be further forward. If you sit straight-legged on the floor, your center of mass is likely to be several centimeters in front of your trunk.

EXAMPLE You (mass = 75 kg) throw a 750-gram (= 0.75 kg) ball in such a way that its velocity in the center-of-mass frame of reference is 10 meters per second. At what speed do you recoil? Remember, you are not standing on or hanging on to anything; this is a space walk. Call your initial momentum (before the throw) zero.

ball's momentum (let "forward" be positive) = (0.75 kg) (10 m/sec)
= 7.5 kg m/sec

your momentum = (75 kg)v_{recoil}
sum of the momenta = (75 kg)v_{recoil} + 7.5 kg m/sec = 0

Solve for v_{recoil}:

$$v_{\text{recoil}} = -0.1\frac{\text{m}}{\text{sec}}$$

Interpret: Your speed or recoil is $\frac{1}{100}$ the speed of the ball, since your mass is 100 times the mass of the ball. The center of mass is $\frac{1}{100}$ as far from you as it is from the ball.

That was analyzed in the frame of reference in which you were at rest before the throw. Now look at it in your own frame of reference after the throw. The ball is moving away at a speed of 10.1 m/sec (= 10 m/sec + 0.1 m/sec); the center of mass is moving away at a speed of 0.1 m/sec. In that frame, the total momentum is not zero but

$$(0.75\text{ kg})\left(10.1\frac{\text{m}}{\text{sec}}\right) = 7.575\text{ kg}\frac{\text{m}}{\text{sec}}$$

This looks unfamiliar. But if you look at the total mass (75 kg + 0.75 kg) before the throw, its velocity in that moving frame of reference was 0.1 m/sec, so its momentum was

$$(75.75\text{ kg})\left(0.1\frac{\text{m}}{\text{sec}}\right) = \boxed{}\text{kg}\,\frac{\text{m}}{\text{sec}}$$

and momentum *is* conserved. Note that this frame of reference was *not* attached to you before the throw but moved in the backward direction. In this frame, the velocity of the center of mass did not change during the throw. A frame attached to you during the throw would be an accelerated frame of reference. In such a frame, momentum would not be conserved.

If the last paragraph seemed difficult, try the following exercise.

EXERCISE 11 Moving sidewalk

Two people are tossing a baseball back and forth (Fig. 2.6.). They are 20 meters apart on a north-south line, and the ball's speed is 20 m/sec. Here is a description in the reference frame of a moving sidewalk traveling north at 2 m/sec. (a) The northward throw: During the 1 second the ball is in the air, the players both move south a distance of (2 m/sec) × 1 sec = 2 meters. This means that the ball only has to fly 18 meters in that second. So the

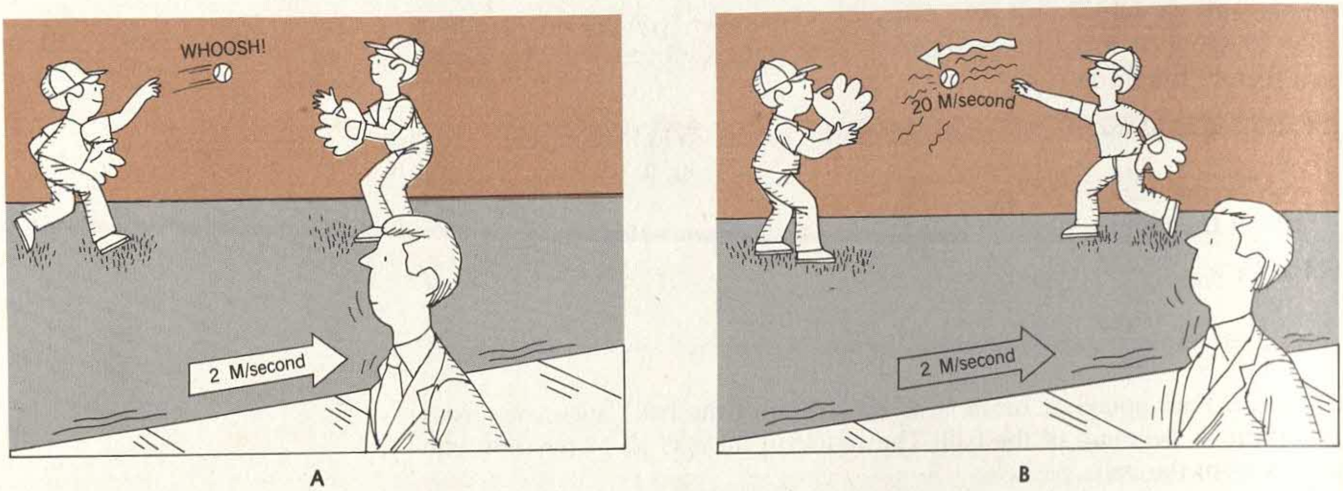


FIGURE 2.6 *(a)* Baseball players seen from a moving sidewalk, northward throw; *(b)*, southward throw.

speed of the ball is ______ m/sec. (b) The southward throw: During the 1 second the ball is in the air, the players move south a distance of 2 m. This means that the ball has to travel 22 meters during that second. So the speed of the ball is ______ m/sec.

Evidently, the speed of the players' reference frame is subtracted from the ball's speed in that reference frame for the northward throw, added for the southward throw. More generally, we can say that the velocities are added. We might, for example, write northward velocities as positive, southward as negative. Transforming velocities from one frame of reference to another by adding the relative velocity of the frames of reference is called a **Galilean transformation.**

*2.5 INERTIAL FRAMES OF REFERENCE

We distinguish between frames of reference in which momentum is conserved and those in which it is not. We use the term **inertial frame of reference** for any of the former class. More simply, an inertial frame is defined as one in which Newton's Laws of Motion are obeyed. It is actually easier to think of examples of noninertial frames: In the rocket blasting off, free-floating objects do not travel in straight lines but are accelerated toward the back of the rocket. Certainly, their momentum is not constant.

It is not hard to see that any vehicle in which you feel yourself spinning or turning corners is not an inertial frame. But that brings up a difficulty: We like to think of our earth as providing a privileged frame of reference.

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*This section can be omitted without loss of continuity.

Certainly, the laws of physics were formulated to apply here on earth. Doesn't the diurnal[1] rotation of the earth and its orbital revolution around the sun spoil it as an inertial frame? Well, yes it does. Fortunately, it doesn't spoil it very much, on the scale of most everyday phenomena. To be specific, the earth's rotation gives New York City an acceleration of 2.5 millimeters per second per second toward the axis of the earth. If you are concerned with weight, that acceleration makes things in New York lighter than they would be without any rotation by one-fortieth of 1%. In the section on "Why We Get Dizzy on a Merry-Go-Round" (Sec. 5.3), you will learn how to do this kind of calculation.

Evidently, the concept of an inertial frame of reference is an idealization. In any practical situation, there will be forces on objects that cause the objects to accelerate; sometimes those forces will be known or predictable, and other times they will be unknown or unpredictable. Here on earth, free-falling objects suffer an acceleration of 9.8 m/sec^2 toward the center of the earth. We don't ascribe that to making our measurements in a noninertial frame; we ascribe it to the gravitational pull of the earth. An electrically charged object experiences electrical and magnetic forces. We say that we are surrounded by a gravitational field, an electric field, a magnetic field. All of these are continually changing. The concept of inertial frame of reference allows us to distinguish between the "real" forces associated with these fields and the "fictitious forces" associated with the accelerations of our transportation vehicles, our earth, our solar system, or our galaxy.[2]

SUMMARY

When you impart momentum to another object, you impart an equal and opposite momentum to yourself.

When you exert a force on another object, it exerts an equal and opposite force on you.

The Law of Conservation of Momentum and the Law of Action and Reaction are logically equivalent once $F = ma$ is assumed.

The sum of the momenta of two objects measured in their center-of-mass frame of reference is zero. If there are no external forces, the combined momentum remains zero.

Newton's Laws of Motion are not valid in an accelerated frame of reference.

A frame of reference in which they are valid is called an inertial frame of reference.

In an inertial frame of reference, when an object is accelerated we say that it is being acted on by a field of force. Gravitational fields, electric fields, and magnetic fields are familiar.

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[1] Once-a-day.

[2] Some readers may now want to skip to Sec. 5.3, "Why We Get Dizzy on a Merry-Go-Round."

KEY TERMS

Momentum The product of mass times velocity. Plural: momenta

Retrorocket A rocket aimed forward in order to slow down.

Newton's Third Law For every action there is an equal and opposite reaction:

$$F_{A\leftarrow B} = -F_{B\leftarrow A}$$

Center of mass The weighted average position of all the masses:

$$x_{CM} = \frac{\Sigma m_i x_i}{\Sigma m_i}$$

For two masses, the product of the mass times the distance from the center of mass is the same for both. The center of mass is the same as the center of gravity. The product of the total mass times the velocity of the center of mass is equal to the total momentum.

Center-of-mass frame of reference In this reference frame, the total momentum is zero.

Galilean transformation Velocity referred to a moving frame of reference is obtained by adding the relative velocity of the reference frames. Named after Galileo Galilei (1564–1642), Italian astronomer and physicist.

Impulse Product of force times the time during which it acts. If the force is not constant, its impulse is the average force times the time.

Inertial frame of reference A reference frame in which Newton's Laws are valid. Conventionally, the reference frame of the fixed stars is assumed to be inertial. Accordingly, every frame of reference moving at constant velocity relative to it is also inertial.

Elastic collision The relative speed of the two objects after the collision is the same as before.

Inelastic collision In a perfectly inelastic collision the two objects stick together, so their relative speed after the collision is zero.

REVIEW QUESTIONS

- When you push on him, only he moves. When he pushes on you, only you move. True / False
- A mass of 1 kg moving at 1 m/sec has a momentum of ______ kg m/sec.
- Two 1-kg masses moving in opposite directions at 1 m/sec have a total momentum of ______.
- If these two masses collide, their total momentum: / can increase / can decrease / must stay the same
- If an object changes its velocity at the same rate in one second as in the next, we say it has a constant acceleration. Is its momentum also changing at a uniform rate?
- When a skater bumps into a stationary skater of the same mass, and hangs on, conservation of momentum predicts that the skaters move off at half the incident velocity. True / False

- A 1-kg mass moving at 1 m/sec collides with a stationary 1-kg mass, and they stick together. The momentum of the 2-kg object is _______ kg m/sec. Its speed is _______ m/sec.
- A 1–kg mass moving at 2 m/sec catches up with a 1–kg mass moving at 1 m/sec in the same direction. They collide and stick together. The momentum of the 2–kg object is _______ kg m/sec. Its speed is _______ m/sec.
- When a skater bumps into two stationary skaters of the same mass, and they all lock arms, conservation of momentum predicts that they move off together at one-third the incident velocity. (See Figs. 2.7*a* and 2.7*b*.) True / False (Neglect friction on the ice.)
- From the stern of the canoe you throw a heavy pack toward the bow, as shown in Fig. 2.8. Conservation of momentum predicts that the canoe will move—forward / backward—only while the pack is in flight / only after the pack is caught.

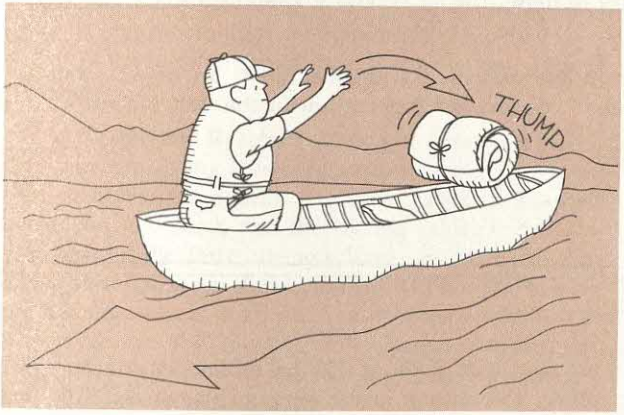


FIGURE 2.8 The pack is thrown from the stern toward the bow. Which way does this make the canoe move?

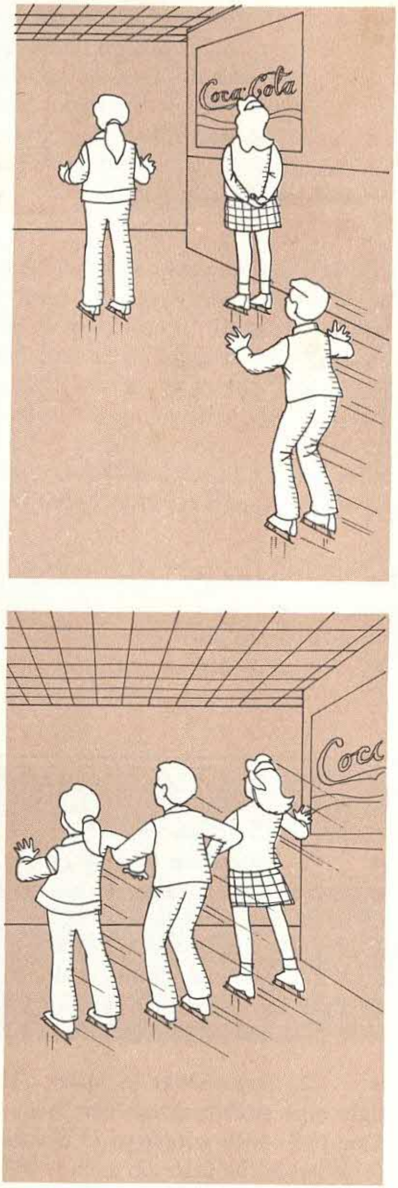


FIGURE 2.7 "Skater bumps into two stationary skaters and they all lock arms." How fast does the trio move off?

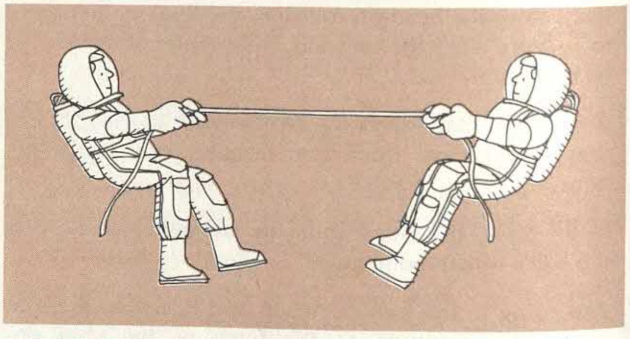
- In a perfectly inelastic collision, the relative speed of the two objects after the collision is zero. This implies that momentum cannot be conserved in the collision. True / False
- In elastic collisions, momentum is conserved. In inelastic collisions, some momentum is lost. True / False
- Just before the head-on collision, the total momentum of two objects is measured to be zero. After the collision, their center of mass: / must be at rest / may be in motion
- If the tennis ball leaves the racquet with its velocity exactly reversed, the momentum it received from the racquet was equal to twice its final momentum. True / False
- In this case, the momentum the racquet received from the ball is equal to twice the ball's initial momentum. True / False
- According to the Law of Action and Reaction, the force of the fist on the chin is equal in magnitude to the force of the chin on the fist. True / False (How about their directions?)

- According to the Law of Action and Reaction, the force exerted by A on B is equal and opposite to the force exerted by B on A: provided neither is accelerating / provided both are accelerating / provided the forces are measured in an inertial frame of reference
- In an accelerated frame of reference, the Law of Conservation of Momentum can be violated. True / False
- In an inertial frame of reference the total momentum of two isolated objects is conserved / is always inertial
- While swimming, you meet a large ore boat. By pushing on the freighter, you manage to make it move aside 2 feet. This is a description: in an inertial frame of reference / in an accelerated frame of reference
- In this frame of reference, momentum is / is not conserved.
- Did the shoreline move too, as you pushed?
- When a bird accelerates in flight, the air behind it has to start moving backward to conserve momentum. True / False
- Falling from a tower in the earth's gravitational field, an object gains as much / half as much downward momentum in the first second as in the second second.

PROBLEMS

✶ **1. Definition of Momentum** A force of 1 newton acts on a mass of 1 kg.
(a) What is the acceleration of the 1-kg mass?
(b) What speed does it attain in 1 sec?
(c) What is its momentum at the end of 1 sec?
(d) Compare its momentum change during the first second with its momentum change during the second second.

✶ **2. Tug-of-War in Space** The 75-kg twins have a light rope stretched between them, as shown in the figure. One pulls with a force of 75 newtons.
(a) What is the force on each twin?
(b) Argue that each accelerates toward the other at an acceleration of 1 m/sec.
(c) What is the speed of the first twin after 1 sec of pulling?
(d) What is the momentum of the first twin after 1 sec of pulling?
(e) What is the total momentum of both twins after 1 sec of pulling?
(HINT: Don't forget to add the momenta with proper + and − signs.)
(f) Now change the problem: The 75-kg space athlete pulls on one end, while a 150-kg athlete is attached at the other. The tension in the rope is 75 newtons. What is the acceleration of the more massive athlete?
(g) What is his momentum after 1 second?
(h) After 1 second, what is the combined momentum of both?
(i) Is there any way one could pull harder than the other?



PROBLEM 2 Tug-of-war in space.

✶✶ PUZZLE: Tug-of-War on Earth

According to the Law of Action and Reaction, both teams in a tug-of-war pull equally hard on the rope. Indeed, the tension in the rope is a measure of how hard each team is pulling. Nevertheless, one team wins. An acceleration in the direction of the winning side ensues.

Analyze all the forces on both teams, using a diagram including forces exerted by the ground on which they stand. State what it is that the winning team has more of. Remember, both teams are accelerated. Is it true that the force each player exerts on the ground is equal and opposite to the force the ground exerts on him or her?

✶ **3. Twins in Space** Two 75–kg twins are holding on to each other as they push off from the space craft at 1 m/sec. A few seconds later, twin B pushes twin A so that A is heading back toward the craft at 1 m/sec.

(a) What was the momentum of the clinging twins? How does their total momentum change during the push?

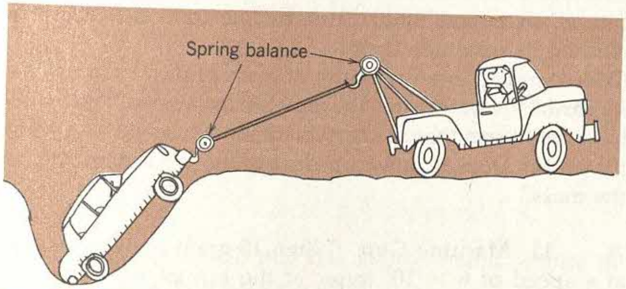
(b) What is the momentum of twin B after he has pushed his brother?

✶ **4. Action and Reaction** A simple spring balance is just a calibrated spring whose extension registers the force stretching it (see the figure). If such a balance is inserted between the tow truck and the towing cable, it presumably measures the force the tow truck exerts. If another one is inserted between the car in the ditch and the towing cable, this one should measure the force exerted by the car in the ditch on the towing cable. Argue persuasively whether these two readings should be equal or unequal:

(a) while the car is still stuck.

(b) while the car is moving uniformly out of the ditch.

(c) while the car is being accelerated.

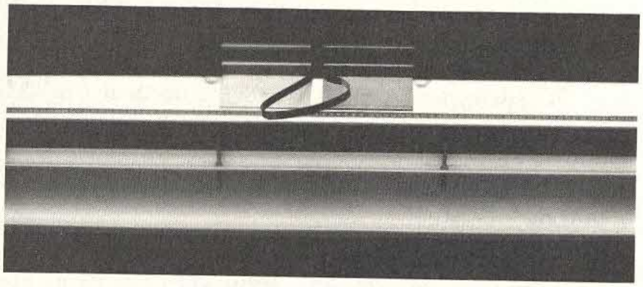


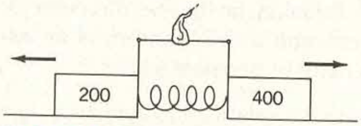
PROBLEM 4 Action and reaction. One spring balance is attached to the tow truck; the other one is attached to the car in the ditch.

✶ **5. Car on a Level Road** Why doesn't the car keep moving without power? Because through friction against the tires, the road exerts a backward force on a car moving forward. If the car is speeding up on a level road, the road must be exerting a forward force on the tires. State carefully which way the road pushes on the front tires and the rear tires of a car speeding up. Use the Law of Action and Reaction to state which way the tires push on the road. (HINT: Assume that the transmission is connected only to the rear wheels.)

6. Pushing That Stalled Car Do you really believe Newton's Third Law? Does the car really push back on you as hard as you push forward on it? To start it rolling might take a 150-pound push. But suppose your feet are on an icy spot. As you push harder, suddenly they go out from under you. Instead of the car accelerating forward, you accelerate backward. If you are pushing forward on the car with a force of 140 pounds and it is still stuck, what is the horizontal force of the ground on the car? Forward or backward?

✶✶ **7. Explosion on the Air Track** On an air track, an explosion can get two gliders in motion in opposite directions (see the figure). A spring tied with thread can be placed between them. The thread is burned to release the spring.





PROBLEM 7 Explosion on the air track. The thread is burned to release the spring, so the two gliders move off in opposite directions.

(a) Both gliders are at rest initially, one 200-gram, one 400-gram. After the explosion, the 200-gram glider moves north at 1 m/sec. At what speed does the 400-gram glider move south?

(b) Different gliders: 200-gram and 300-gram. After the ex-

plosion, the 200-gram glider moves north at 1 m/sec. At what speed does the 300-gram glider move south?
(c) Two 200-gram gliders are moving north together at 1 m/sec initially. After the explosion, one is moving north at 2 m/sec. What is the velocity of the other? State speed and direction.
(d) The 200-gram and the 400-gram glider are moving north together at 1 m/sec initially. After the explosion, the 200-gram glider is moving north at 2 m/sec. What is the velocity of the 400-gram glider?

✶✶ **8. Transforming Reference Frames** Here is a description of the twins pushing off from the point of view of someone traveling south relative to the space craft. Both twins are moving north at 1 m/sec. Twin A gives twin B a push toward the north. As a result, twin A stops moving, while twin B moves north at 2 m/sec.
(a) Verify that momentum is conserved, that is, the total momentum of both twins is the same before and after the push.
(b) Write a description in the frame of reference in which both twins are moving south at 1 m/sec before the push. Verify the conservation of momentum.
(c) Write a description in a frame of reference in which both twins are moving north at 2 m/sec before the push. Verify the conservation of momentum.

NOTE: The difference between A's velocity and B's velocity is independent of reference frame.

✶✶ **9. Symmetry** Twin A is moving north at 1 m/sec. Twin B is moving south at the same speed. They collide and hang on to each other.
(a) Use the Conservation of Momentum to argue that they both come to rest.

NOTE: You would feel justified in arguing that this follows from a simple symmetry argument. The twins are identical. How could the collision favor one direction or the other? Indeed, Conservation of Momentum is an assertion about the homogeneity of physical space.

(b) If you played this scene backward, it would look like the pushing-off scene we described in Sec. 2.1 You might feel justified in arguing that reversing the time should not change a law of physics. Indeed, the time-reversal invariance of physical laws is never violated, as far as we know today. Describe the time-reversed scene corresponding to the 75–kg astronaut pushing off his 150–kg companion.
(c) Twin A moving at 2 m/sec collides with twin B at rest and hangs on. At what speed do they move off together, to conserve momentum? Choose a frame of reference in which this scene becomes familiar and describe it in that frame.

✶✶ **10. Impulse** When a constant force F acts for a time t, the product Ft is called the **impulse** delivered by that force. From Sec. 2.3, you know that that product represents the change in momentum of the object on which the force is acting.
(a) A 1–kg mass has a weight of 9.8 newtons. Falling to the ground from a height of 2.0 meters takes 0.64 sec. Verify the arithmetic. (HINT: $\frac{1}{2}gt^2 = 2.0$ meters.)
(b) In that time, it attains a speed of 6.3 m/sec, or a downward momentum of 6.3 kg m/sec. What impulse does the earth's gravity impart to the mass during its fall?

The interaction with the ground when the mass hits is complicated. Things tend to deform. Certainly, the upward force slowing down the mass is large, brief, and not constant over any appreciable time period. But the product of the *average force* over the time of the interaction is still defined as the impulse of that force. Again, it is equal to the momentum change of the object on which the force acts.
(c) Verify that the change in the momentum of the mass upon striking the ground is 6.3 kg m/sec upward. Is this equal to the impulse applied to it by the ground?
(d) If a 1–kg mass is launched vertically upward at 6.3 m/sec, what impulse is needed to bring it to rest?
(e) How long does that take, given a constant gravitational force of 9.8 newtons?
(f) What height does the mass attain in that time? (HINT: $v_{\text{average}}t$.)
(g) What impulse did the launching device deliver to the mass?
(h) Assume that the launching device gave up all its downward momentum to the earth. Argue that the earth gets all that momentum back by the time the mass has reached its maximum height.
(i) How much does the center of mass of the system "mass + earth" move during the upward motion of the mass? Does this mean that the earth actually moves down during that time? Does it move up during the downward trip of the mass?

✶ **11. Machine Gun** When 15-gram bullets are shot at a speed of 6×10^2 m/sec at the rate of 8 per second, what is the average backward force on the gunner?

✶✶ **12. Collision** For the collision of two air-track gliders, the Conservation of Momentum may be written

$$m_1v_1 + m_2v_2 = m_1\dot{v}_1 + m_2\dot{v}_2$$

Velocities after the collision are marked with an asterisk. An alternative way of stating the conservation law is

$$m_1v_1^* - m_1v_1 = m_2v_2 - m_2v_2^*$$

The loss of momentum of the first glider is equal to the gain of momentum of the second glider.

(a) Assume that the collision lasts for a time t. Then $(v_1^* - v_1)/t$ represents the average acceleration of the first glider during the collision. Show that the less massive glider suffers the greater acceleration. (HINT: The word "greater" refers to absolute value.)

(b) Argue that this is why the occupants of the heavier car in a two-car collision are less likely to be injured.

✶✶ **13. Elastic Collision** If they are equipped with spring bumpers, air-track gliders will collide elastically. In an **elastic collision,** by definition, the relative speed of the two colliding objects is the same after the collision as before. Suppose that a glider collides elastically with a stationary glider of the same mass. The incident glider will then stop and impart all of its momentum to the struck glider. Evidently, the relative speed is then the same as before. Evidently momentum is conserved.

(a) Use the Conservation of Momentum to prove that, if the struck glider has less mass than the incident one, it ends up moving forward faster than the other one.

(b) Which way is the incident glider moving after the collision? Prove.

✶✶ **14. Ping-Pong** Someone claims that it is impossible for the Ping-Pong ball to move faster than the paddle that struck it. Respond to this claim. Remember, the paddle has not only its own mass, but also the mass of the hand, arm, and so on, carrying momentum.

✶✶ **15. Elastic Collision** Definition:

$$|v_1 - v_2| = |v_1^* - v_2^*|$$

The * marks the velocities after the collision. This relation between absolute values (relative speeds) can be satisfied in one of two ways: Either

$$v_1 - v_2 = v_1^* - v_2^* \quad \text{or} \quad v_1 - v_2 = v_2^* - v_1^*$$

Which is right? Argue that one choice would require the objects to pass through each other.

✶ **16. Elastic Collision** Remember the definition: $|v_1 - v_2| = |v^*_1 - v^*_2|$. Remember the Conservation of Momentum:

$$m_1v_1 + m_2v_2 = m_1v^*_1 + m_2v^*_2.$$

Let $v_2 = 0$ and $m_1 < m_2$.

(a) Prove that v_1 and v^*_1 have opposite signs. Does this mean that the incident object bounces back?

(b) Is $v^*_2 < v_1$? Prove your answer in words.

✶✶ **17. Elastic Collision** If $m_1 = m_2$, prove that an elastic collision results in an exact interchange of momenta.

✶ **18. What Kind of Collision?**

$$\text{Given: } m_1 = 0.2 \text{ kg}, \quad m_2 = 0.3 \text{ kg}$$

$$v_1 = 0.3\frac{\text{m}}{\text{sec}}, \quad v_2 = -0.6\frac{\text{m}}{\text{sec}}$$

$$v_2^* = +0.12\frac{\text{m}}{\text{sec}}$$

Is the collision elastic?

✶✶ **19. Elastic Collision** A 100-gram glider moving at 10 cm/sec collides elastically with a 300-gram glider at rest. Find the velocity of each glider after the collision.

✶✶ **20. Inelastic Collision** A perfectly **inelastic** collision is defined as one in which the colliding objects stick together. That is, the relative speed after the collision is zero: $v^*_1 = v^*_2$. This is actually closer to what happens in an automobile collision. If the masses are equal and $v_2 = 0$, show that $v^* = \frac{1}{2}v_1$.

✶✶ **21. Symmetric Perfectly Inelastic Collision** Let $v_1 = -v_2$.

(a) If the masses are equal, what is v^*?

(b) If $m_1 = 2m_2$, what is v^*?

(c) What is the greatest value that v^* can possibly have? What mass ratio would give that value? Is there more than one correct answer?

✶✶ **22. It Does Not Matter Who Is Doing the Work** Twin A ($m = 50$ kg) pushes on twin B ($m = 50$ kg) with his hands, exerting a constant force of 25 newtons for 1 second.

(a) At the end of 1 second, contact is lost, and each one moves off backward at a speed of ______ m/sec. A says to B: "Next time I want you to help push too."

(b) Next time, A pushes against the flat of B's hands with a constant force of 25 newtons, while B pushes back with a force of 25 newtons, again for 1 second. All four of their elbows are straightened as they push. Argue that the final velocities are the same as last time. Note: This used to bother me when I first learned about momentum. It just didn't seem fair that the end result does not depend at all on whether B exerts himself or not. If you are worried about who is doing the work, come back to this problem when you study Chapter 4 on Work and Energy. You will see that they share the work when both twins' arms straighten during the pushing.

✶ **23. Weighted Average** A class of 10 students shows the following scores:

100	1
80	4
60	5

What is the average score? (HINT: Do not make the mistake of just adding 100 + 80 + 60 and dividing by 3 to get 80. You have to add the 100 in once and add the 80 in 4 times and add the 60 in 5 times, and then divide by 10.)

NOTE: Why were you assigned this problem? Just to help understand the definition of center of mass as a *weighted average* position.

✶ **24. Center of Mass** A 1-kg mass and a 2-kg mass are 3 meters apart. Where is their center of mass? Answer in the form " ________________ from the 1-kg mass and ________________ from the 2-kg mass."

✶✶ **25. Moving Center of Mass** A 1000-kg car is moving north at 10 m/sec. A 2000-kg car is moving north at 20 m/sec.
(a) What is their total momentum? ________ kg m/sec
(b) At what speed is their center of mass moving north? (HINT: Divide their total momentum by their total mass.) Does it matter how far apart they are?

✶✶ **26. Moving Center of Mass** A 1000-kg car is moving north at 10 m/sec. A 2000-kg car is moving south at 20 m/sec.
(a) What is their total momentum? Don't forget to state the direction.
(b) What is the velocity of their center of mass?

✶ **27. Center of Mass** A 50-kg astronaut and a 75-kg astronaut are at rest 20 meters apart.
(a) Where is their center of mass? They are attached by a light rope. One pulls on the rope, keeping a tension of 150 newtons.
(b) Find their respective accelerations.
(c) Find their respective velocities after 1 second of pulling.
(d) How far is the 50-kg astronaut from the center of mass after 1 second of pulling? Remember, the center of mass does not move.

✶✶ **28. Baseball** The pitched ball arrives at the plate at 40 m/sec and is hit straight back toward the pitcher at the same speed.
(a) How much momentum does the bat impart to the 150-gram ball?
(b) The bat (mass 2 kg) is only moving at 10 m/sec toward the pitcher at the instant of impact. Which way is the center of mass of the ball and bat moving? At what speed?
(c) How much velocity does the bat lose while it is in contact with the ball?
(d) Suppose that the contact lasts 1 msec. Calculate the average force of the bat on the ball during the contact.
(e) Why does the batter not feel that enormous force?

✶✶ **29. Rifle Recoil** A 10-gram bullet leaves the muzzle at 668 m/sec.
(a) Calculate the momentum it imparts to the rifle.
(b) In Problem 1.19, you found that the bullet travels the length of the barrel in 2.28 msec. What is the average force it exerts on the rifle during that time?
(c) In that time, the bullet travels 0.76 m. If the rifle's mass is 4 kg, how far does it recoil in that time?
(d) Why do you think the rifleman's shoulder feels a much smaller force than the answer to part (b) (HINT: Estimate how long it takes to bring the rifle to rest? That is, how long does it take for the body to absorb the rifle's momentum?)

✶✶ **30. Space Walk** When the astronaut leaves the space craft for his space walk, he remains attached by a rope.
(a) Explain why, if he pushes off from the craft, he would have difficulty getting back without the rope. Your explanation should either use the term *Conservation of Momentum* or the term *center of mass*.
(b) Horrors! The knot attaching the rope to the craft came undone! He decides to pull on the rope anyway. Does this help him get back to the craft? Explain. Does the mass of the rope play a role in your answer?
(c) His initial push-off was too vigorous, and he finds himself receding from the craft. In desperation he throws his bag of tools in the direction away from the craft. Write down an inequality relating the momentum of the tools and his own initial momentum if he is ever to get back to the craft. Work in the reference frame of the craft.
(d) Why would the same inequality not work in the astronaut's own frame of reference?
(e) Come back to this problem after reading Sect. 2.4, and answer part (c) using the center-of-mass concept. How far away are the tools by the time the astronaut gets back to the craft? Assume reasonable values for the various masses.

✶✶ **31. Adrift** The astronaut finds himself floating 1 meter from the "floor" of his spacecraft, slowly moving toward the ceiling at 1 cm/sec (= 10^{-2} m/sec). There is nothing in reach to hang on to. In order to get back to the floor, he takes off his jacket (mass 1 kg) and throws it at the ceiling at a speed of 5 m/sec.

(a) How do you know that the center of mass of the astronaut (75 kg) plus jacket (1 kg) still moves toward the ceiling at 10^{-2} m/sec?

(b) The astronaut imparts a momentum of 5 kg m/sec to the jacket. How much momentum does the jacket impart to him?

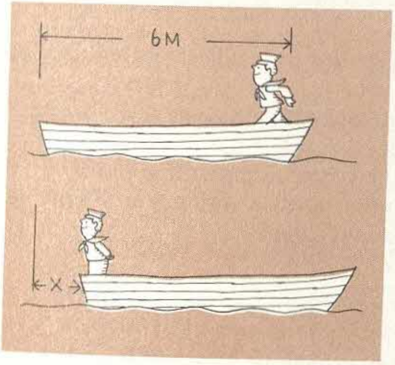
(c) How much momentum does that leave him? Don't forget to say in what direction. (HINT: Remember his initial momentum of 0.75 kg m/sec toward the ceiling.)

(d) Which way is he moving after the throw, and how fast?

(e) How far off (%) would your answer be if you neglected his initial velocity?

(f) When he hits the floor and hangs on, the center of mass of the system astronaut plus jacket starts to move faster. Explain.

✶✶ **32. Walking in a Boat** A 75-kg sailor walks from the bow to the stern of a 150-kg boat. (See the figure.) The boat is 6 m long.



PROBLEM 32
Walking in a boat.

(a) How far does the boat move?
(HINT: If the boat moves a distance x, then the sailor moves 6m − x, measured in the center-of-mass frame of reference. Neglect any friction between the boat and the water. Use the fact that the center of mass remains stationary.)

(b) If the sailor walks at 1 m/sec, as measured in the center-of-mass frame, how fast does the boat move?

(c) How fast is the sailor walking, as measured in the boat's frame of reference?

NOTE: Have you ever tried to step out of a canoe onto a dock? Did you miss?

✶✶ **33. High Jump** An object launched vertically upward at 4.43 m/sec can gain an altitude of 1 meter before starting to drop.

(a) Verify. (HINT: $g = 9.8$ m/sec^2.) How long does the object rise? A useful property of the center of mass is that its acceleration is given by dividing the sum of all the external forces by the sum of all the masses.

(b) A highjumper, whose center of mass is normally 1 meter above the ground, needs to raise it approximately another meter in order to clear the 2-meter bar. How long after take-off does his center of mass start to drop?

High-jumping technique is such that the center of mass does not have to be raised quite to the level of the bar. Parts of the body roll over the bar while others remain at lower altitudes.

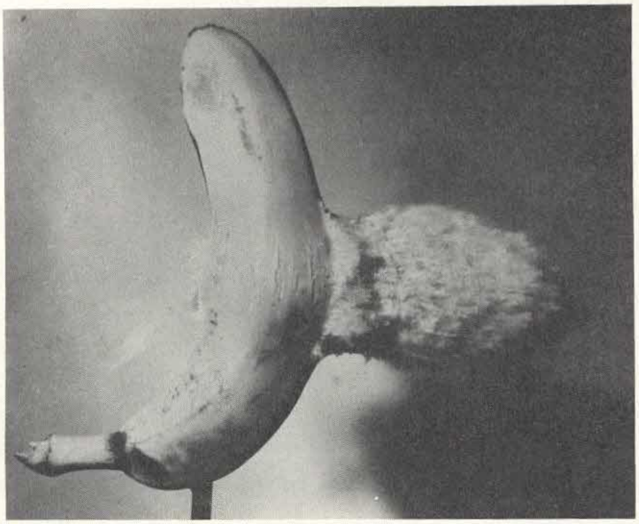
✶✶✶ **34. Why Did the President's Head Snap Backward After the Fatal Shot?** The movie film taken of President Kennedy at the moment of his assassination shows his head moving backward at 0.5 m/sec after being hit from behind. Some people said this proved that there had to have been another bullet entering from the front. But only one bullet, one that had entered the back of the head, was found inside. A jet of blood and soft tissue had emerged toward the front.

The one-bullet explanation of the backward recoil of the head is that the momentum of the jet exceeded the incident momentum of the bullet. Both the jet and the recoil of the head occurred in response to the large compressive stresses inside. Here are some order-of-magnitude numbers:

mass of bullet	0.01 kg
mass of head	10 kg
mass of jet	0.1 kg
velocity of bullet	6×10^2 m/sec
velocity of head	-0.5 m/sec

(a) Find the velocity of the jet that conserves momentum.
(b) How fast was the center of mass moving forward before the impact?
(c) In the center-of-mass frame of reference, how fast was the head moving backward before the impact?

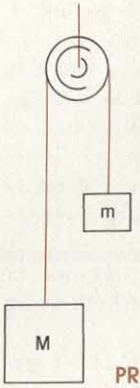
NOTE: Fruit hit by rifle bullets show the backward recoil. This problem demonstrates the usefulness of the conservation of momentum in analyzing a situation in which the internal forces are not known.



The bullet enters the banana from the left, causing a jet of fruit pulp to emerge toward the right. To conserve momentum, the rest of the banana recoils toward the left.

✶✶ 35. **Atwood Machine** When the two weights are released the heavier one accelerates downward, the lighter one accelerates upward (see the figure).
(a) Show that the center of mass of the two masses M and m accelerates downward.

NOTE: The center of mass accelerates as though all the external forces were applied to it and the entire mass of the system were concentrated there.
(b) Here is a solution for the accelerations: Let upward forces and accelerations be positive.



PROBLEM 35 Atwood machine.

Mass M	Mass m
Rope pulls up with tension T	Rope pulls up with tension T
Gravity pulls down with force Mg	Gravity pulls down with force mg
Downward acceleration a	Upward acceleration a
$T - Mg = -Ma$	$T - mg = ma$

Subtract the two equations: $(M - m)\, g = (M + m)\, a$
Solve:

$$a = \frac{M - m}{M + m}\, g$$

Solve for the tension T in the rope. (Neglect the mass of the pulley.)
(c) Show that if $M = m$, your solution gives $T = Mg$. Why is this what you expect?
(d) Now worry about the center of mass. If m moves up a distance y, then M moves down a distance y. Then the center of mass moves down a distance

$$\frac{M - m}{M + m}\, y$$

Show that, if $m = \frac{1}{2}M$, the center of mass moves only one-third as far as the weights.
(e) According to part (d), the acceleration of the center of mass is

$$\frac{M - m}{M + m}\, a$$

Let us add up the external forces on the system: the weight $(M + m)\, g$ acting downward, and the supporting force on the pulley ($2T$) acting upward. Writing $F = ma$ for the center of mass, we have

$$-(M + m)\,g + 2T = -(M + m)\frac{(M - m)}{(M + m)}\;a$$

Compare with the equations of part (b).
(f) Let $m = \frac{1}{2} M$. Solve for the acceleration a and for the tension T.
(g) Find the tension in the rope holding up the pulley. Note that it is less than $(M + m)\,g$. Explain in terms of the motion of the center of mass.

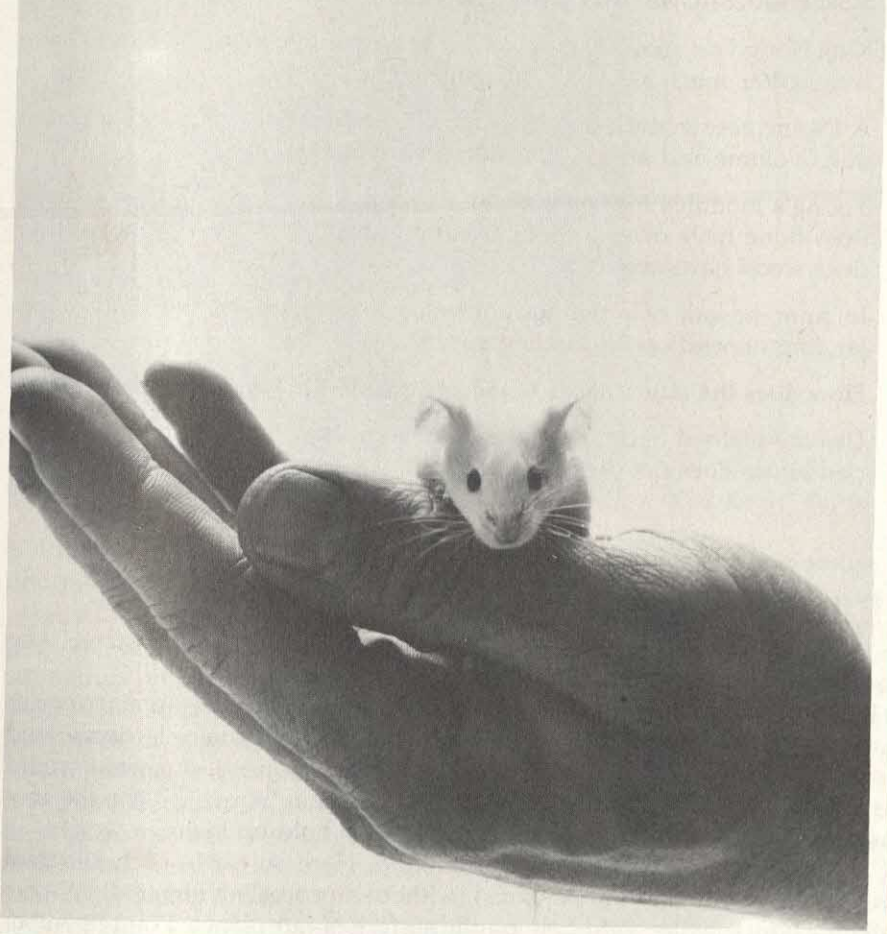
✶✶ 36. **Ballisticardiogram** Some people can stand so still on the bathroom scale that they can see the dial jiggle in rhythm with their heartbeat. It seems that the pumping action of the heart perturbs the weight reading periodically by about an ounce above and below its average value.

What is going on in the circulatory system? Does the center of mass of our blood move up and down during one heartbeat? Yes. It appears that more blood flows down than up during that phase of the cardiac cycle in which the left ventricle is contracting and pushing blood out through the aorta into the arterial system. You may object that the circulatory system is a closed network of tubes, so as much blood as goes down in the arteries must come back up in the veins. How can more go down than up? The answer is that the blood vessels are elastic; they expand and contract in response to the rhythmic pressure changes due to the pumping action of the heart. The arterial blood flow pulsates; the venous flow is much more uniform. For about one-sixth of the cardiac cycle most of the blood squeezed into the aorta (one "stroke volume") is moving through it at a speed of about 1 meter per second, much faster than it flows during the remaining part of the cycle. Since the end of the aorta points down, this downward acceleration of the arterial blood results in an upward recoil of the rest of the body: conservation of momentum, or, if you prefer, action–reaction. During this part of the cycle, then, the body will be accelerated upward and will "weigh less" on the bathroom scale.

We can get an idea of how much the apparent weight is reduced if we assume that the entire stroke volume, about 65 cubic centimeters in an average adult, has to be accelerated to 1 m/sec in 0.13 seconds (= one-sixth of a cycle at 77 beats per minute), an acceleration of ________ m/sec^2.
(a) The mass of 65 cm^3 of blood is about 65 grams (= 0.065 kg). Calculate the force needed to give that mass the acceleration just estimated. (HINT: $F = ma$. Express both in newtons and in pounds. Your answer is actually a bit more than the amplitude of the jiggle usually observed. For one thing, not all the blood moves toward the feet. The aorta has a branch to the arteries that go up to the head. For another thing, the heart is not firmly anchored. It moves back and forth as it beats.)
(b) If the center of mass of the blood moves toward the feet, the center of mass of the rest of the body has to move in the direction of the head. The physiologists tell us that, during that left-ventricular contraction, the center of mass of the 65 cc of blood that leaves the ventricle is shifted about 7 cm closer to the feet. How far does the rest of the body have to recoil in order to leave the center of mass of a 75-kilogram patient unchanged? In the ballisticardiogram, the patient is placed horizontally on a light bed mounted on frictionless bearings. Minute displacements of the bed can be monitored sensitively.

☐ QUESTION FOR FURTHER THOUGHT

α. **Throwing** Do you think you can give more momentum to a light object like a baseball or to a massive object like a 100-pound bag of flour?



The mouse is smaller than your thumb. Compare your abilities in supporting your own weight, running, jumping, and keeping warm.

3

SIZE, STRENGTH, AND SCALING

SOME QUESTIONS THIS CHAPTER WILL ANSWER

King Kong was about 20 times as tall as an average man. He would have weighed as much as ______ average men.

A 1% increase in the diameter of a sphere results in a ______% increase in its volume and a ______% increase in its area.

Young's modulus for bone is only 5% of that for steel. What advantage does bone have over steel for making human bodies? What advantage does wood have over both?

In jumping out of a window, how does the force on the body upon landing depend on the landing speed?

How does the time it takes to fall depend on the height of the window?

Use the method of dimensional analysis to show that the frequency of a pendulum does not depend on its mass.

3.1 THE SIZES OF MAN AND ANIMALS

In our fairy tales we have giants. They are tall; they take huge steps; they have enormous strength. What advantages!

If it really were an advantage to be taller, why would the human species not have evolved taller? That is what the doctrine of natural selection would predict. There must be drawbacks to being big. Of course, our giant would be heavier. But isn't that compensated for by being stronger? For the moment, just think about the strength he needs to hold up his own weight.

This discussion is going to be earthbound. Here on earth, each kilogram weighs 2.2 pounds (= 9.8 newtons), without any acceleration at all. We are talking about gravity, right here on the surface of our planet. You remember all about *g* from Chapter 1. What if we were twice as tall?

I am about 1.7 meters tall and weigh 165 pounds. Suppose that our giant were $2 \times 1.7\text{ m} = 3.4\text{ m}$ tall and built to my proportions. That means every linear dimension would be doubled. Every inch on me would be 2 inches on him. So every cubic inch of volume in my body would be $(2\text{ inches})^3 =$ 8 cubic inches of volume in his body. His body volume would be 8 times mine. If he were made of the same materials, he would weigh 8 times my weight: 8×165 pounds = 1320 pounds. Could he hold himself up? Would his muscles be strong enough?

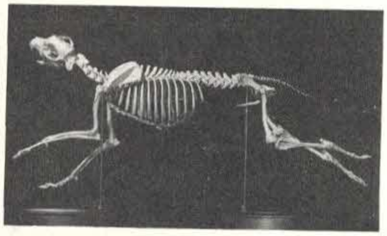
The strength of a muscle is proportional to its cross section (Fig. 3.1). You can see that by thinking of a muscle as a bundle of fibers. Its strength is proportional to the number of identical fibers. The number of fibers contained in a bundle is proportional to the cross-sectional area of the bundle. That area goes as the square of the diameter. (Remember πr^2?) If the giant's leg is twice as thick as mine, it ought to be $2^2 = 4$ times as strong. The giant

Scene from the original "King Kong" movie.

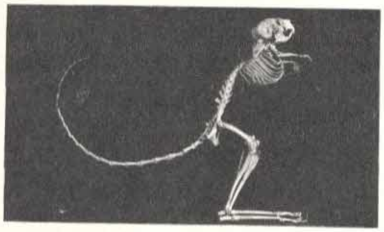
is 4 times as strong, but 8 times as heavy! When I get up in the morning, I find my 165 pounds enough. Our giant might not make it out of bed at all.[1]

Here is the same thing said more generally: The weight of a body is proportional to the cube (= third power) of its dimensions, while the structural strength is proportional to the square (= second power) of the linear dimensions. This scaling principle has put a natural limit on the size of animals here on earth. You rarely see an elephant lying on its stomach. The pressure of the ground on its internal organs must be intolerable. Indeed elephants only copulate in water, where the weight of the male need not be borne by the female partner. The larger whales do not leave the water at all. Their bones and muscles are not strong enough to support their weight. A stranded whale's ribs break.

Comparing species of animals of different sizes, we find that the proportions change. The larger the animal, the thicker are the bones relative to their length. The photographs show analogous bones enlarged to be the same length. As an exercise, compare a dog and a mouse. Suppose that the dog is 10 times as long as the mouse. It might well weigh 10^3 = a thousand times as much, for example, 50 pounds versus 0.05 pounds. That means each leg of the dog needs to support 1000 times as much weight as each leg of the mouse. In order for a bone to have 1000 times the strength, it has to have 1000 times the cross-sectional area, or $\sqrt{1000} = 31.6$ times the diameter. Repeat: the dog's leg bone may be expected to be 31.6 times as thick, but 10 times as long as the mouse's. So the ratio of bone diameter to bone length is likely to be more than 3 times (31.6/10) greater for the dog than for the mouse.



A



B

Skeleton of wolf and skeleton of kangaroo mouse. Notice how much smaller the thickness-to-length ratio of the bones is for the mouse. See also Prob.3.11, "Elephant" and Prob.3.12, "Dinosaur".

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[1] Try to see the film "Change of Scale," PSSC Physics Film No. 0106, which includes an excerpt from the original "King Kong."

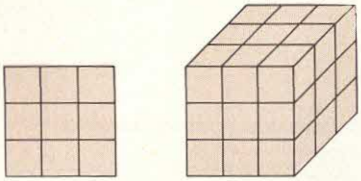


FIGURE 3.2 A 3×3 square has an area of 9 little squares. A $3 \times 3 \times 3$ cube has a volume of 27 little cubes.

EXERCISE 1 On Scaling. (See Fig. 3.2.)

Arguments based on **scaling** are used throughout physics and biology. Here are some, using 1-centimeter cubes each having a mass of 1 gram.

The area of each face is 1 cm^2. If we build a 3×3 square, that takes _______ (how many?) little cubes. The face-up area is 9 cm^2.

If we build a $3 \times 3 \times 3$ cube, that requires _______ (how many?) little cubes. The volume of the big cube is _______ cm^3.

A cube has _______ (how many?) faces. So the total surface area of each little cube is _______ cm^2. The surface area of the big cube is _______ cm^2, that is, 9 times as great.

The weight of each little cube is 10^{-3} kg × 9.8 newton/kg = 0.0098 newton = 980 dynes. NOTE: 1 newton = 10^5 dynes. The normal stress on the bottom surface of each little cube standing alone is therefore 980 dynes/cm^2. The stress on the bottom surface of the big cube is _______ times as great. (HINT: The weight that has to be supported is _______ times as great, and the area over which it is distributed is _______ times as great, so that the

$$\text{stress} = \frac{\text{force}}{\text{area}}$$

is $3^3/3^2$ = _______ times as great.)

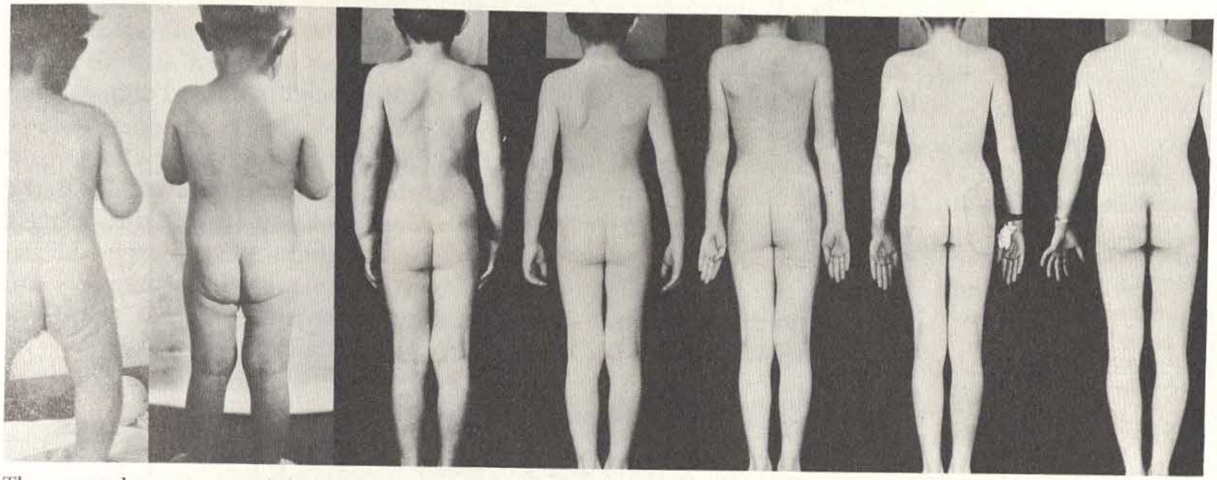
What if we had started with some other shape, some shape other than cubes, and scaled it up by a factor of 3? The arguments would have been the same. Just imagine the new shape built up out of lots of tiny cubes. All the areas multiply by 3^2; all the volumes multiply by 3^3.

In our example of the dog and the mouse, we scaled lengths by a factor of 10 and we scaled volumes by a factor 10^3. We might ask whether the dog's skin would have 10^2 times the area of the mouse's. It would probably be pretty close.

But we have to be careful with scaling arguments. Here is an apparent paradox: A human adult is about 3 times as tall as a one-year-old child. Does he weigh $3^3 = 27$ times as much?

27×20 pounds = ?

Oh, no. But look at the toddler. How much thicker all his parts are relative to their length than the adult's. Does that sound like the wrong way around? Remember the dog and the mouse when we compared species of different sizes, it was the bigger ones that had the thicker limbs. Evidently Nature has other variables at her disposal beside size. The child has a larger fraction of fat than the adult, for one thing. This will come up again when we talk about keeping warm, in Chapter 8.



The same boy at ages 1¼, 2½, 6, 7, 11, 14, and 18, showing typical changes in proportions. A baby would be cold if it were as long and thin.

EXERCISE 2 Suppose that a doll is one-third as tall as the child that plays with it. To make a dress for the doll should require ______ (what fraction?) as much cloth as a similar dress for the child.

EXERCISE 3 If the doll in Exercise 2 is made of solid rubber, its density (mass per unit volume) is roughly the same as ours. What fraction of the child's weight should the doll weigh? (See Problems 11 and 12, Elephant and Dinosaur.)

3.2 THE STRENGTH OF BONE

How strong are our bones? In this section we develop a vocabulary suitable for such questions.

First we have to decide in what terms we want our "how strong" question answered. If we want to know how much weight a particular bone can support before it breaks, the answer could be in pounds or newtons or dynes, any unit of *force*. That number would characterize a particular bone, not bone as a substance. If we want a quantity characteristic of the material the bone is made of, we somehow have to divide out the physical dimensions of the bone. Again we invoke the argument that the strength of a bone is proportional to the cross-sectional area of the bone. So the quantity we are after is the maximum force per unit cross-sectional area. That quantity refers to bone, not to a particular bone. Force per unit area is called **stress:**

$$\text{stress} = \frac{\text{force}}{\text{area}}$$

If we are talking about a leg holding up the weight of the body, we are talking about compressive stress.

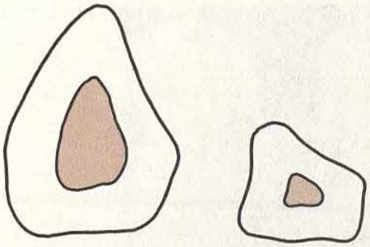


FIGURE 3.3 Human tibia and fibula (bones of the lower leg), cross section, actual size.

Bone has a rupture strength of 10^9 dynes per square centimeter (= 10^8 newtons/m^2). That is the maximum compressive stress before the material yields. It is also called the *yield stress*. In Exercise 4 you are asked to convert that to pounds per square inch, and you get an impressively large answer. The tibia, the front bone in the lower leg, has a cross-sectional area of about 3 cm^2 at its thinnest place, not counting the marrow. (See Fig. 3.3) So it could support a load of

$$\begin{aligned} 10^9 \frac{\text{dynes}}{\text{cm}^2} \times 3\ \text{cm}^2 &= 3 \times 10^9 \text{ dynes} \\ &= 3 \times 10^4 \text{ newtons} \\ &= 7 \times 10^3 \text{ pounds} \end{aligned}$$

That is well over 30 times your weight, and this is just one leg. At this point you may breathe easier, remembering that you have jumped from high places. In landing you suffered accelerations many times *g* without breaking any bones. But not too many times *g*. That is why you bent your knees as you hit the ground.[2]

EXERCISE 4 Recall that 1 kilogram weighs 9.8 newtons, or 2.2 pounds; 1 newton = 10^5 dynes. Express 10^9 dynes/cm^2 in pounds per square inch (psi). For comparison, note that the stress on the sides of an automobile tire is around 30 psi.

Elasticity

Our question about the strength of bone has been answered purely in terms of the longitudinal compression the bone can withstand before breaking. That answer has ignored the fact that bones bend. It might have been more realistic to begin our discussion with the elastic properties of bone. After being bent, the bone returns to its original shape, like a spring. It does that provided it is not bent so much that it breaks. Now we start to address the question of how stiffness is measured. Should we ask how much bending force gives how much bend? The stiffness of a particular bone depends on its length and on the detailed shape of its cross section. Nature's weight-bearing designs of bone cross sections (Fig. 3.3) give us a lot of resistance to bending for little weight of bone. They have their analogues in the I-beams used in steel-frame buildings.

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[2] See Sec. 3.4, Falling from Great Heights.

Again, we would like to define a constant that characterizes the stiffness of the material rather than the stiffness of a particular bone. Figure 3.4 shows that, as the bone bends, some of the material is put under tension, some under compression. The dotted line represents a **neutral surface,** where there is no stress. Inside the neutral surface, that is, toward the center of curvature, the stress is compressive. Outside the neutral surface, the stress is tensile. It turns out that a single **elastic modulus** describes rather well the resistance of a material to either tension or compression. But because bending is rather complicated, we shall define this elastic modulus for a simple tensile stress.

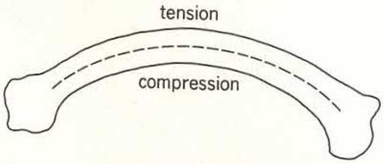


FIGURE 3.4 Bending a bone (exaggerated). Below the neutral surface (dotted) the bent bone is under compressive stress. Above the neutral surface, the stress is tensile.

Suppose that we were to pull on a wire having a uniform cross section. Take a 1-meter length of steel wire, for example, 1 mm^2 in cross section. A stretching force of 98 newtons (= 10 kilogram weight = 22 pounds) would stretch it by 0.5 millimeters, or 0.05%. The same load would stretch a 2-meter length of wire by 1 mm, that is, twice as much. The fractional elongation, which we call the longitudinal **strain** (Fig. 3.5), is still five parts in 10,000. How does the strain depend on the cross-sectional area? If we double the cross-sectional area, we have to double the load to get the same strain. So the strain depends on the force per unit area, on the **tensile stress.** Half the stress would give roughly half the strain. The strain is proportional to the stress for small strains. The constant of proportionality is called an **elastic constant:**

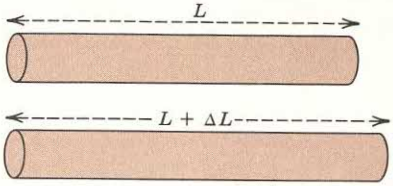


FIGURE 3.5 Strain = $\Delta L/L$.

$$\textbf{Young's modulus} = \frac{\text{stress}}{\text{strain}} = \frac{\text{tensile force per unit area}}{\text{elongation per unit length}}$$

Put in the numbers for that wire:

$$\text{stress} = \frac{98 \text{ newtons}}{10^{-6} \text{ m}^2}$$

$$\text{strain} = \frac{5 \times 10^{-4} \text{ m}}{1 \text{ m}}$$

$$\text{Young's modulus for steel} = \frac{98 \times 10^6 \text{ newtons/m}^2}{5 \times 10^{-4}} = 2 \times 10^{11} \text{ newtons/m}^2$$

Young's modulus for bone has a value roughly one-twentieth that of steel. Looking at Table 3.1, note that the yield stress of steel is only 3 times that of bone.

The maximum strain a material can stand before yielding is given by dividing the yield stress by Young's modulus:

$$\text{yield strain} = \frac{\text{yield stress}}{\text{Young's modulus}}$$

TABLE 3.1 Elastic Stiffness and Strength

	Young's Modulus dynes/cm^2	Rupture Strength = Yield Stress dynes/cm^2	Density gram/cm^3
Aluminum	7×10^{11}	1×10^9	2.7
Bone	10×10^{11}	1×10^9	1.7 to 2
Steel, stainless	20×10^{11}	2.8×10^9	7.8
Tungsten	35×10^{11}	4.2×10^9	19.3
Sugar maple	1.3×10^{11}	1.1×10^9	0.7
White oak	1.4×10^{11}	1.2×10^9	0.8
Rubber	2×10^7	10^8	1

Units conversion:
$1 \text{ newton/m}^2 = 10 \text{ dynes/cm}^2$
$1 \text{ kg/m}^3 = 10^{-3} \text{ grams/cm}^3$
$14.5 \text{ pounds per square inch} = 10^5 \text{ newtons/m}^2$

Again consulting Table 3.1, you note that the yield strains of bone and of steel are quite comparable, around one part in a hundred. If you are thinking about ski accidents, you might be worried about an ankle held firmly in a boot that does not release from a ski binding. You might ask: How far off center can the full weight of a 165-pound skier be before its pull would break a tibia? The answer is about 1 foot. (See Fig. 3.6.)

In recent years, measurements of stiffness and of **yield stress** of bones in live animals have been compared with measurements on wet bones dissected out. The living bones showed much greater values. Evidently, the muscles help the bones resist bending. This has been of interest in the study of ski injuries and the design of safety bindings for skis. The leg can withstand much greater stress when that stress is applied slowly than when it is applied so suddenly that the muscles do not have time to respond. Modern ski bindings take this into account. They are set to release earlier for force applied suddenly than for force applied gradually.

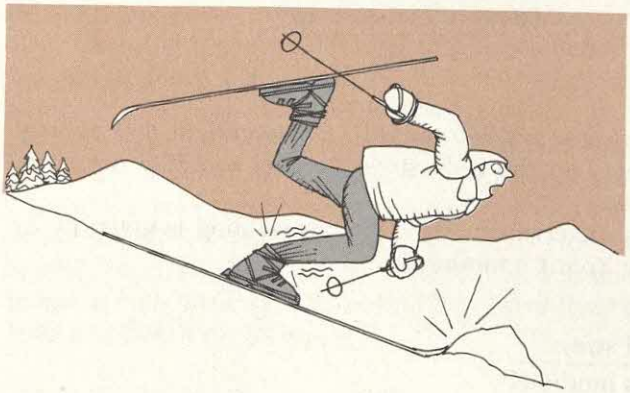


FIGURE 3.6 How far off center can the full weight of a skier be before its pull would break a tibia?

How well did nature do in her choice of a building material for our skeletons? Table 3.1 compares bone with some other materials. Steel has a rupture strength about 3 times that of bone. But it would weigh about 4 times as much. Pound for pound, bone is stronger than steel. How about wood? Wood has about the same rupture strength as bone and is typically less than half as dense. Sure enough, maples and oaks can grow much taller than we can. Of course, they don't have joints, for which bone is ideally suited.

EXERCISE 5 If a wire 1 meter long is elongated by 1 millimeter, the tensile strain is ________.

EXERCISE 6 If a wire 1 mm in diameter is loaded with a 10^3-newton weight, the tensile stress is ________ newtons/m^2.

EXERCISE 7 If a 10-cm-long bone is compressed by 0.1 mm, the compressive strain is ________. Answer: 1 part in a thousand.

3.3 SPECIAL TOPIC: Falling from Great Heights

A flier bails out of his burning aircraft at 23,000 feet. His parachute fails to open. He lands on the edge of a ravine covered with snow. Three-and-a-half months later he is back at work.

Incredible? We shall use the story to

1. discuss an acceleration that is not uniform, and
2. apply the kinematics of uniform acceleration to the practical problem of how to land safely.

Yes, the story is true. When you bail out of an airplane, you fall with a downward acceleration of 22 miles per hour per second ($= g = 9.8$ m/sec^2) for the first second or so. But the faster the air rushes past you, the more resistance it offers. As you speed up, your acceleration decreases. Your weight pulls you down; the increasing wind resistance pushes you up. The net force on you becomes less and less. Gradually, you approach a speed at which the two opposing forces are equal. At that speed the net force on you is zero and you do not accelerate past it. It is called your **terminal velocity.** For an adult human being it is around 120 miles per hour (= 54 m/sec). With a parachute, it would be less than one-tenth of that. Now we shall calculate the upward acceleration you suffer on landing. How many g's?

Obviously, that depends on how hard you land. If you land on a hard surface, the most favorable impact is a "whole-body" landing. Parachutists are taught to land so they touch first with the feet, then fall sideways so that gradually the whole body absorbs the impact. This maximizes the time for absorbing the momentum, so it minimizes the force. Also, it distributes

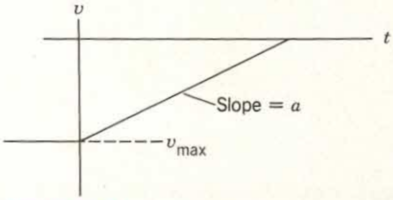


FIGURE 3.7 Velocity of your center of mass versus time, after your feet hit the ground.

the force over a large area. This reduces the compressive stress (force per unit area) on any one part of the body.

The distance your body's center of mass travels during the deceleration is essentially equal to half your height. That assumes your center of mass is near your middle.

We take that "stopping distance" s to be 1 meter and calculate the upward acceleration during the "stopping time" t. We assume that the acceleration is uniform during this time. Let $v_{\max}$ be the speed on impact, the 54 m/sec. Look at the triangle in Fig. 3.7.

$$\begin{aligned}\text{stopping distance } s &= \tfrac{1}{2}\, v_{\max}\, t \\ &= \text{area of the triangle} \\ \text{stopping time } t &= \frac{s}{\frac{1}{2}v_{\max}} = \frac{1\text{ m}}{\frac{1}{2}(54\text{ m/sec})} = 0.037\text{ sec} \\ \text{acceleration}\quad a &= \frac{v_{\max}}{t} = \frac{54\text{ m/sec}}{0.037\text{ sec}} = 1458\text{ m/sec}^2 \\ &= 148\ g\text{'s}\end{aligned}$$

This means that the body has to support 149 times your normal weight. Repeat: Total force of the ground on your body = 148 $mg + mg = 149\ mg$. If you weighed 165 pounds, that would be a force of more than 12 tons. You would be crushed. You could not survive such a hard landing.

But our analysis shows us that, for given landing speed $v_{\max}$, the acceleration is inversely proportional to the stopping distance s. Eliminating t between the last two equations above, we get

$$a = \frac{v_{\max}}{t} = \frac{\frac{1}{2}v_{\max}^2}{s}$$

Changing the stopping distance s from 1 meter to 4 meters gives one-quarter the acceleration (37 g's) and a force of 38 times the body weight, lasting 4 times as long. Remember the snowy ravine? Now the story of the flier seems less miraculous. Recall that the strength of bone is such that each tibia can support about 30 times the body weight.

At this point you might want to recall the experiments done preparatory to the U.S. space program to test what accelerations human subjects could withstand. A large centrifuge was used, and accelerations went up to 20 g's. The subjects lost consciousness but suffered no injuries. To be sure, they were strapped to soft support couches that distributed the force over a maximum area of the body. In Problem 23 we calculate the average compressive stress at 20 g's. The numbers are quite modest.

We now look at a less spectacular fall but with a hard landing. Say you jump from a height of 2 meters onto a concrete floor and don't bend your knees. The landing velocity $v_{\max}$ is now a mere 6.3 m/sec. With stiff legs, the deceleration has to occur in a stopping distance of perhaps 1 centimeter. This requires an upward acceleration of 1960 m/sec^2, or 200 g's. That means the ground exerts an upward force 201 times your normal weight. You can see why stiff-legged landings often result in broken ankles.

EXERCISE 8 Verify the numbers in this last paragraph.

3.4 TIME AND FREQUENCY SCALING

An architect may build a scale model in order to help visualize a design. An engineer may build a scale model in order to make physical measurements. At the beginning of this chapter we talked about the way weights and volumes scale with size: They go as the cube of the linear dimensions. Areas go as the square of the linear dimensions. How do the time constants, the characteristic frequencies of the model, scale with size? Again, we shall be concerned with the effects of gravity.

Time to Fall

It takes an object 1 second to fall 4.9 meters from rest. In 2 seconds, an object falls 19.6 meters, that is, 4 times as far. Here we are neglecting wind resistance and assuming a constant acceleration of $g = 9.8\ \text{m/sec}^2$. Looking at Fig. 3.8, we see that the distance fallen is represented by the area of a triangle on the velocity-versus-time graph. (See Chapter 1, Kinematics.) If we write

$$v = gt$$

the area of a triangle of altitude v and base t is

$$\begin{aligned}\text{distance fallen} = y &= \tfrac{1}{2}vt \\ &= \tfrac{1}{2}gt^2\end{aligned}$$

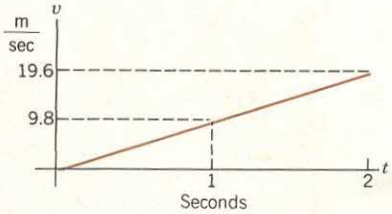


FIGURE 3.8 Falling from rest, velocity versus time. The distance fallen is the area of the triangle.

Evidently, the distance fallen is proportional to the square of the time. If we turn that around, we can say that the time it takes to fall is proportional to the square root of the distance fallen:

$$t = \left(\frac{2y}{g}\right)^{\frac{1}{2}}$$

EXERCISE 9 Verify the distances fallen in 1 sec and 2 sec.

EXERCISE 10 How long does it take to fall 9.8 m? (HINT: $\sqrt{2} = 1.414$.)

EXERCISE 11 How long does it take to fall $3 \times 4.9\ \text{m} = 14.7\ \text{m}$? (HINT: $\sqrt{3} = 1.732$.)

We now have the answer to our question about how time scales in free fall. The time to fall scales as the one-half power of the linear dimensions.

Pendulum: Dimensional Analysis

A weight swinging on a rod or string is referred to as a pendulum. A "point mass" swinging on a "massless rod" is a physicist's idealization. We call it a *simple pendulum*. That means a pendulum is simple if the support is light and if the weight is compact enough so it can be considered as concentrated at one point—its center of mass.

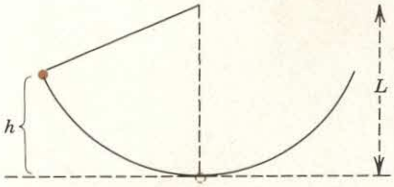


FIGURE 3.9 The pendulum swings in a circular arc of radius L. The maximum height is h.

Let the weight be displaced from its equilibrium position far enough to be raised up to a height h (see Fig. 3.9). How does the time it takes to swing back to the equilibrium point depend on h? Does it go as the square root of h? Careful! This is not free fall. We cannot just take the results of the previous paragraph and apply them blindly. But an experiment is easy here. A key on a string is fine. A tape dispenser will do. Try timing swings, even very roughly. Since the pendulum overshoots the equilibrium point and comes back almost to the same height, the "time to fall" is just one-quarter of the **period,** the time for one complete oscillation. Try timing 5 or 10 small swings to improve the accuracy. Then double the release height h and repeat.

What did you find? When you doubled the height of the swing, did the period increase by 41%? This is what the free-fall result would have predicted: The time to fall should be multiplied by $\sqrt{2} = 1.41$. What you probably found is a barely measurable increase in the time. Indeed, for small swings, the period turns out to be independent of the oscillation amplitude.

Are you surprised, or did you "know it all the time"? The mathematics to prove this result is a bit too difficult for this book. But now we should ask, what are the important variables that determine the period of a pendulum? Surely, the length L of the string matters. The strength of the earth's gravity is not a variable in the sense that you can change it. But you know that g is going to enter the formula. How about the mass M of the pendulum? If you don't know, put it in the list. Arbitrarily, we end the list with three variables. Now we demonstrate the method called **dimensional analysis** for obtaining approximate formulas. If we were to leave out an important variable, the method would probably fail. Or it might give unphysical results. Here is how it works.

We ask how our three variables, L, M, and g, might be used in the formula for the period T. They all have different physical dimensions. The period T can be measured in seconds, length L in meters, mass M in kilograms, acceleration g in meters per second per second. It would not make any sense to add or subtract quantities having different dimensions. For example, adding meters and seconds has no meaning. But multiplying and dividing them does make sense. For example, dividing a length (meters) by a time (seconds) gives a speed (meters per second). If we use only multiplication and division to combine our three variables, the most general formula would look something like this:

$$T = (\text{some number}) \times L^A M^B g^C$$

The exponents A, B, and C are the unknown powers to which each of the variables is raised. The method of dimensional analysis is used to determine these exponents. The "number" in the parenthesis does not have any dimensions. It is a pure number, and the method does not determine it. For obtaining approximate results, the hope is that it is not a number like 10^6 but somewhere close to 1.

EXAMPLE To find the volume (meters^3) of a pyramid (Fig. 3.10), you can probably guess that you have to multiply the area of the base (m^2) times the altitude (m). But how can you guess that the product has to be multiplied by $\frac{1}{3}$ to get the volume?

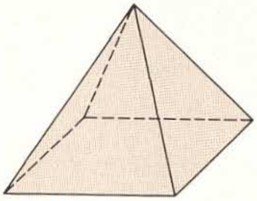


FIGURE 3.10 Pyramid. How do you find its volume?

If we forget about the magnitudes and keep only the units, the pendulum formula becomes purely dimensional:

$$\text{sec} = (\text{meter})^A \times (\text{kg})^B \times \left(\frac{\text{meter}}{\text{sec}^2}\right)^C$$

Our one equation then yields three: one equation in seconds, one in kilograms, one in meters. For example, seconds appears to the first power on the left, to the $-2C$ power on the right:

$$1 = -2C$$

Therefore, $C = -\frac{1}{2}$. Kilograms do not appear at all on the left, giving $B = 0$. That leaves

$$\text{sec} = (\text{meter})^A \times \left(\frac{\text{meter}}{\text{sec}^2}\right)^{-1/2}$$

Meters do not appear on the left, implying that $A - \frac{1}{2} = 0$, or $A = \frac{1}{2}$.

$A = \frac{1}{2}$. That means the length enters to the one-half power, which is the result we were after. The formula now says

$$T = (\text{number}) \times L^{1/2} M^0 g^{-1/2} = (\text{number}) \sqrt{\frac{L}{g}}$$

The mass M does not enter the formula. We need not have included it in the list in the first place. A heavy pendulum has the same period as a light pendulum of the same length. A stronger g would give a shorter period. Usually, that is not something we have control over. But accurate pendulum timing to investigate variations in g from place to place is useful in geological research and prospecting.

Dimensional analysis is no help in determining the "number" out front. A proper theoretical treatment gives the formula

$$T = 2\pi\sqrt{\frac{L}{g}}$$

for the period of a simple pendulum. Remember, the 2π is exactly correct only for small oscillations.

EXERCISE 12 Increasing the length L by 1% increases the period T by _______%. (HINT: $\sqrt{1.01} = 1.005$.)

EXERCISE 13 Increasing the length L by 0.1% increases the period T by _______%.

EXERCISE 14 Decreasing the length L by 1% decreases the period T by _______%. (HINT: $\sqrt{0.99} = 0.995$.)

EXERCISE 15 A pendulum swinging with an angular amplitude of 10° from the vertical has a period about 0.2% longer than the small-amplitude period given by the formula. Argue that dimensional analysis is powerless in dealing with a dimensionless variable like angle or with two variables having the same dimensions, like h and L. (On the other hand, see Problem 29, Pendulum with Finite-Amplitude Swing.)

SUMMARY

Weight is proportional to the cube of the linear dimensions. Strength is proportional to the square of the limb thickness.

The bones of larger animals are thicker in proportion to their length.

Weight-induced stress scales linearly with size.

The yield stress of our bones in compression is much greater than the demands our weight might make on them.

Tensile strain is defined as elongation per unit length. Compressive strain is defined as shortening per unit length.

Young's modulus measures the elastic stiffness of a material under longitudinal tension or compression. It is defined as the quotient of stress : strain.

Bending involves tensile stress toward the outside of the bend, compressive stress toward the inside of the bend.

Wind resistance increases with speed. This puts a limit on the speed with which an object falls in air.

For given impact speed, the stopping acceleration is inversely proportional to the stopping distance.

For given stopping distance, the stopping acceleration is proportional to the square of the impact speed.

The time to fall at constant acceleration is proportional to the square root of the distance fallen.

In making physical measurements on the undamped motion of a scale model, time scales as the square root of the linear dimensions.

The period of a pendulum is independent of its mass, insensitive to its amplitude, but proportional to the square root of its length.

The period of a pendulum is proportional to the square root of $1/g$.

KEY TERMS

Scaling principles Area is proportional to the square of the linear dimensions, volume to their cube.

Stress Force per unit area. One distinguishes compressive stress, tensile stress, and shear stress (not discussed in the text).

Strain The measure of deformation. Displacement per unit length.

Dyne A unit of force equal to 10^{-5} newtons. 1 dyne = 1 gm cm/sec^2. A force of 1 dyne accelerates a mass of 1 gm by 1 cm/sec^2.

Tibia The front bone in the lower leg.

Fibula The rear (smaller) bone in the lower leg.

Elasticity The ability of a material to return to its original shape after small deformation.

Elastic modulus Also elastic constant. A measure of elastic stiffness, defined as the quotient of stress/strain. For small deformations, the stress and strain are proportional. The common elastic moduli are Young's modulus, bulk modulus, and shear modulus. This chapter has defined only Young's modulus.

Young's modulus The proportionality constant relating compressive stress to compressive strain. Or, the constant relating tensile stress to fractional elongation. For most materials, these two moduli are almost equal.

Tensile Refers to tension or pulling. The opposite of *compressive*.

Neutral surface When a beam is bent, some of it is under compression, some under tension. The boundary between these regions, a surface of zero stress, is the neutral surface. For a rectangular beam, it is in the middle. Also called neutral axis.

Yield stress The limit of elasticity. That stress beyond which the material no longer returns to its original shape upon removal of the stress. The *yielding* is called *plastic* deformation.

Terminal velocity Fastest speed reached when falling in air. For an adult human being, around 120 miles/hour.

Period Time for one complete oscillation. Reciprocal of the frequency.

Dimensional analysis Use of the physical dimensions (units) of the variables to find their exponents when combined in a formula. A shortcut to proper physical theory.

Alopecia The development of a hairless spot. This occurs on the back of the head in babies allowed to lie on their backs too long without being turned.

REVIEW QUESTIONS

- A giant 3 times my height would weigh ________ times my weight if built to my proportions.
- If his legs were 3 times as big around, his leg muscles would have ________ times as many muscle fibers as mine.
- Consequently, the stress in his legs in order to hold him up, that is, the force per unit cross-sectional area, would be ________ times that in mine.
- In order to have the same stress in bones and muscles, an animal weighing 27 times as much should have legs with 27 times the cross-sectional area, or ________ times the diameter.
- The yield stress of a material can be expressed in: pounds / pounds per square inch / percent
- A bone with a yield stress of 10^4 pounds per square inch and effective cross-sectional area of 0.5 in.^2 can support a compressive load of ________ pounds.
- Argue that when you bend a yard stick, about half of its volume is under compression, half under tension.
- If a wooden meter stick is stretched to 1.001 m, the elastic strain in it is ________%.
- This requires a stretching force of about: 1 pound / 20 pounds / 1 ton
- An elastic modulus can be measured in: pounds / pounds per square inch / percent
- For wood and bone, the values of the elastic moduli are about 10^2 times the values of the corresponding yield stresses. This implies that the yield strains are roughly ________%.
- Look at Table 3.1. Which has greater strength per unit weight, steel or tungsten?
- Which has the greater terminal velocity in free fall, a Ping-Pong ball or a billiard ball? A mouse or a human being?
- Argue that stopping acceleration should be proportional to the square of the impact speed.
- Argue that stopping acceleration should be inversely proportional to the stopping distance.
- Argue that the time to fall a given distance (free fall) should be proportional to the square root of that distance.
- Argue that increasing the length of a pendulum by 2%: increases / decreases the period by 1%.
- The frequency of a simple pendulum 1 foot long is about: 10^{-1} Hz / 1 Hz / 10^1 Hz

PROBLEMS

✶ **1. Scaling Weight and Volume** A human being 1% taller than I am and built to my proportions would have a volume $(1.01)^3$ times mine, that is, ______% greater than mine. Being built of the same materials, his body would have the same density as mine. Therefore, his weight would be ______% greater than mine.

✶✶ **2. Scaling in Two Dimensions** A state map might be drawn on a scale 1:1,000,000. That means 10 km is represented by 1 cm on the map. How many such maps would it take to "paper" the state?

✶ **3. Rope** A rope 1 cm in diameter contains 24 strands of fiber.

(a) What is the cross-sectional area of the rope? ______ cm^2 (HINT: πr^2.)

(b) How much of this area is taken up by each strand? ______ cm^2

(c) How many strands would there be in a rope 2 cm in diameter?

(d) What diameter rope would 48 strands make?

(e) If the 1-cm rope holds 400 pounds, how much tension can the 2-cm rope stand?

(f) What should be the diameter of a rope that can hold 800 pounds?

✶ **4. Cubes** The 3 cm × 3 cm × 3 cm cube in our example has a bottom area of 9 cm^2.

(a) What is the bottom area of a 4 cm × 4 cm × 4 cm cube?

(b) What is the ratio of the bottom areas of the two cubes? large:small

(c) The weight of the 4 × 4 × 4 cube is ______ times that of the 3 × 3 × 3 cube.

(d) The stress (= weight per unit area) on the bottom of the 4 × 4 × 4 cube is ______ times that on the 3 × 3 × 3 cube.

✶ **5. Bones + Skin** If you were to grow so that every linear dimension in your body increased by 1%, your shape would not change: A 10-cm-long bone would become a 10.1-cm-long bone; a 10 cm × 10 cm area would become a 10.1 cm × 10.1 cm area, and so on.

(a) Your surface area (skin) would increase by ______%. (HINT: Area changes as the square of the linear dimensions, and $(1.01)^2$ = ______.)

(b) Your volume would increase by ______%, and so would your weight.

(c) The stresses on your bones and muscles having to do with bearing weight would increase / decrease by ______%. (HINT: Weight is multiplied by ______ and area by ______; hence the weight per unit area by ______.)

(d) Repeat the calculations of parts (a) through (c) for a 20% increase in linear dimensions. (HINT: If you increase something by 20% twice, that does not give a 40% increase, but a 44% increase. OK? 1.2 × 1.2 = ______.)

✶✶ **6. Fur Coats** If it takes four sheep to make a fur coat, how many rabbits would it take to make a coat of the same size? Data: The average rabbit weighs about 3 pounds. The average sheep weighs about 60 pounds. Of course, rabbits and sheep are not built in the same proportions, but you can solve the problem assuming that they are.

✶✶ **7. Alien Life Forms** On a planet with an acceleration of gravity g only half the value it has here, how much bigger would you expect their land animals to be than ours? Assume their bone and muscle to be constituted of the same materials as ours. Support your answer with cogent argument. You might begin as follows:

Assume that the size of their largest animals is dictated largely by their ability to support their weight. The maximum stress (force per unit cross-sectional area) in their bones and muscles should be the same as ours.

If you multiply all the linear dimensions of a given species by a factor f, the volume and hence the mass is multiplied by a factor f^3, provided that the shape stays the same. But the weight is only multiplied by $\frac{1}{2}f^3$, because their g is half of ours. The cross section of their limbs is multiplied by f^2. That means the stress in those limbs is multiplied by ______. (Complete the argument.)

✶ **8. Scaling—Wire** How does the maximum weight a wire can support depend on it cross-sectional area? On its length?

✶✶ **9. Density** One cubic centimeter of oak wood has a mass of 0.8 grams. (See Table 3.1)

(a) 1 m^3 has a mass of ______ kg.

(b) 1 kg has a volume of ______ cm^3.

(c) The ratio of the densities of oak and maple is 0.8:0.7. This means that 1 kg of maple occupies ______% more / less volume than 1 kg of oak.

✶ **10. Steel Cable** How much weight can a steel cable 1 cm^2 in cross section bear without breaking? Express in tons. Conversions: 1 ton = 2000 pounds; 1 pound = 4.445 newtons.

NOTE: If the yield stress is exceeded, the cable deforms plastically and gets thinner. This increases the stress for given load, so the cable eventually breaks.

✶✶ **11. Elephant** In the text it is stated that the human tibia can support over 30 times our weight before it ruptures. Of course, nothing is said about the joints, but let's argue this one just from the strength of bone. An elephant weighs about 80 times as much as a human being, and its tibia is about 3 times as big around as ours.
(a) Show how it is that the elephant can still support its weight on two legs.
(b) How many times its weight can it support using all four legs? (HINT: Each tibia is ________ times as strong as ours.)

NOTE: Have you ever seen an elephant jump?

12. Dinosaur A large dinosaur may have weighed 400 times as much as an adult human. Its tibia was about 3 times as big around as ours. Standing still on all four legs, the compressive stress would have been about _____ times the stress in ours when we stand still on two legs. Do you really think such an animal could have walked on land?

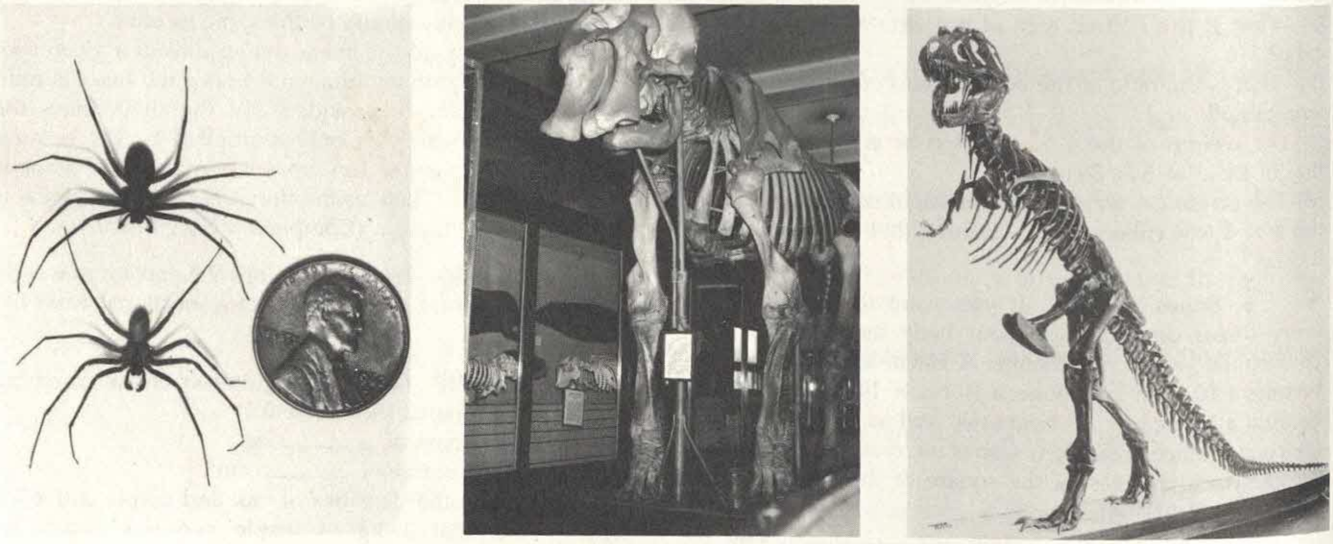
NOTE: It is conjectured that they lived in swamps.

✶ **13. Young's Modulus**
(a) How much would a 196-newton weight stretch a 1-meter length of steel wire 1 mm^2 in cross section?
(b) What would be the strain in the wire?
(c) Divide the stress by the strain. Answer in newtons/m^2.

✶ **14. Yield Stress** Would you expect that a steel wire 1 meter long, 1 mm^2 in cross section, could support a weight of 10^2 newtons? (= 22.5 pounds) 10^3 newtons? See Table 3.1.

✶ **15. Compression** A steel disk 1 cm^2 in cross section and 1 mm thick is loaded with a 9800-newton weight, that is, the weight of 1 metric ton.
(a) Find the compressive stress in the disk.
(b) Find the compressive strain. Assume that the Young's modulus for compression is the same as for tension.
(c) How much thinner does the disk become by being loaded? [Answer: 5×10^{-7} m]

✶✶ **16. Bending a Beam, Bending the Tibia** A useful measure of how much a straight rod is bent is its curvature. Imagine a small segment of the rod fit onto a circle. The radius R of this circle is called the radius of curvature.



A B C

Spider, elephant, dinosaur (Poisonous brown spider, African elephant, Tyrannosaurus rex). Can an animal with 400 times your weight possibly support it all?

The reciprocal $1/R$ is called the curvature. The tensile stress is maximum at the outside surface of the rod. Call d the distance of this surface from the neutral surface. The maximum tensile strain is given by d/R. Thus a cylindrical rod of radius r bent into a circular arc of radius R has a maximum tensile strain of r/R.

(a) Look up the Young's modulus and the yield stress of steel in Table 3.1. Their quotient is the yield strain. So a steel rod can stand a strain of about ________% before it reaches its *elastic limit*. Beyond that, it does not come back to its original shape. Then the deformation is no longer *elastic*, but is called *plastic*. Some steels break just beyond the elastic limit. Bone does too.

(b) A steel rod 1 mm in radius can be bent into a circular arc of radius 1 meter (curvature = 1 m^{-1}) and still be elastic. Argue that a steel rod 2 cm in radius would yield if it were bent this far.

(c) The tibia has an effective d of about 1.5 cm at its thinnest point. Use Table 3.1 to find the maximum curvature the tibia can have without breaking. Express the result in m^{-1}.

★★ **17. Rubber** The elastic properties of polymers are rather different from those of crystalline solids. Nevertheless, it makes sense to talk about a Young's modulus and about a yield stress.

(a) Use Table 3.1 to find the force needed to stretch a rubber band 1 mm^2 in cross section by 1%.

(b) In your experience, can rubber stand elastic deformations of 100%? That would mean doubling its length in tension, halving it in compression.

(c) Do the relative magnitudes of the Young's modulus and yield stress support your answer to part (b)? Explain.

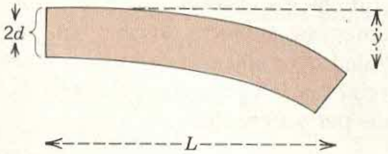
NOTE: For strains more than a few percent, the relation between stress and strain becomes quite nonlinear for rubber and other polymers.

★★ **18. Cantilever Beam** For a rectangular beam of length L, width w, and thickness $2d$, with Young's modulus E, as shown in the figure, the displacement y of the end due to bending under a load F applied at the end is given by the formula

$$y = \frac{1}{2}\frac{L^3F}{Ed^3w}$$

The other end is assumed clamped.

(a) If the load F is doubled, the displacement y is multiplied times ________. The formula is valid for small displacements only. We speak of a linear spring, or "linear theory."



PROBLEM 18. Rectangular beam bending under a load applied at the end. The left end is assumed clamped.

(b) If L is doubled, the displacement y is multiplied times ________.

(c) If L is increased by 1%, the displacement y is increased by ________%.

(d) If d is increased by 1%, the thickness of the beam is increased by ________% and the displacement y is decreased by ________%.

(e) Apply the formula to an oak plank 2 meters long, 30 cm wide, 2.5 cm thick, with a load of 200 newtons at the end. Use Table 3.1.

(f) Was it a good approximation to neglect the weight of the plank? Try to make a correction.

★★ **19.** $s = \frac{1}{2}v^2/a$

(a) For a given stopping distance s, if the impact speed v is doubled, the acceleration a is multiplied times ________.

(b) For given acceleration a, if the stopping distance s is doubled, the impact speed v has been multiplied times ________.

(c) For given stopping distance s, if the impact speed v is increased by 1%, the acceleration a is increased by ________%.

(d) For given acceleration a, if the stopping distance s is increased by 1%, the impact speed has been increased by ________%.

★★ **20.** $s = \frac{1}{2}v^2/a$ Suppose that you desire to keep the acceleration less than 20 g = 196 m/sec^2, for safety reasons.

(a) What is the largest allowable impact velocity, given a stopping distance s of 2 meters? (HINT: You can do this one in your head.)

(b) Repeat for a stopping distance s = 0.5 m.

★ **21.** $s = \frac{1}{2}v^2/a$

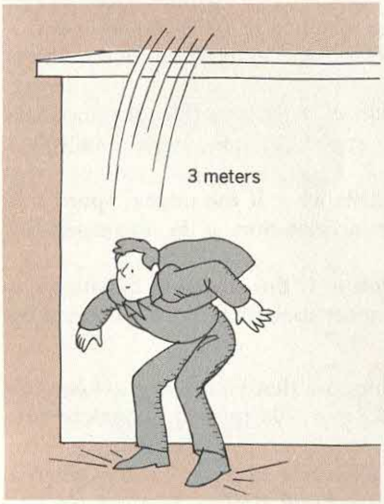
(a) Sketch a graph of s versus v for given acceleration a.

(b) Sketch a graph of acceleration a versus stopping distance s for given v.

★★ **22. Soft Landing** Suppose that your stopping distance is 50 cm. That is, the distance your center of mass moves down after your feet touch the ground is 0.5 m. What is the fastest impact speed that will keep your average stopping acceleration less than 5 g's.

✶ **23. Accelerating at 20 *g*'s. Stress?** The support couch exerts force 21 times the subject's normal weight. Let that force be distributed over 600 square inches. Suppose that the normal weight is 165 pounds. Calculate the average stress, in pounds per square inch.

✶✶ **24. Falling from a Height** You want to jump down from a 3-meter-high platform. Suppose that your legs can bend enough so that your center of mass has 0.6 meters of vertical descent between the instant your feet touch the ground and the instant your center of mass comes to rest. Assume that the force of the ground on your feet during the "landing time" is constant.
(a) Sketch a graph of the velocity of your center of mass versus time for the whole 3.6 meters of vertical descent. (HINT: Is it a straight line?)
(b) Calculate the acceleration during the landing and express it both in g's and in m/sec^2.
(c) The upward force of the ground on your feet is ______ times your normal weight.



PROBLEM 24. Jumping down from a 3-meter-high platform. By bending your feet, your knees, your waist, you give your center of mass 60 cm to stop after your feet first hit the ground.

✶✶ **25. Wind Resistance** An approximate formula for wind resistance at speed v is

$$\text{wind force} = bv^2.$$

(a) If a body weighing 735 newtons falls at a contant speed (terminal velocity) of 54 m/sec, the proportionality constant b is equal to ______ kg/m.
(b) What is the wind resistance at 27 m/sec?
(c) What is the downward acceleration of a body falling at 27 m/sec?

NOTE: This formula holds for speed exceeding 1 millimeter per second. This is to be contrasted with the situation in Problem 1.54, Falling without Acceleration. There, for very slow speeds, the air resistance is proportional to v, not to v^2.

✶✶ **26. Lunar Gravity** On the moon gravity is one-sixth as strong as on earth. How does the time to fall a given distance scale between earth and moon? Answer in the form: "The time to fall a distance y on the moon is ______ times the time to fall the same distance on earth."

✶✶ **27. Period of Pendulum Independent of Mass** Without using dimensional analysis, present an argument to show that the period of a pendulum should be independent of its mass.

✶✶ **28. Period and Frequency** A pendulum having a period of half a second has a frequency of two oscillations per second. The technical term is 2 Hertz (Hz).
(a) To double the frequency, the period has to be halved. This requires that the length of the pendulum be multiplied times ______.
(b) A carpenter's tape measure used as a pendulum makes quite a satisfactory metronome for setting the beat in music. Calculate the length in inches for 0.5 and 1 Hz, that is, 60 and 120 half-swings per minute.

NOTE: "The Star-Spangled Banner" might be marked 80.

✶✶ **29. Pendulum with Finite-Amplitude Swing** In the text, the period of a pendulum oscillating with a small amplitude, that is, swinging away from the vertical by only a small angle α, is given as $2\pi\ell^{1/2}/g^{1/2}$. For larger angles, theory predicts a slight lengthening of the period with increasing amplitude α:

$$T = (2\pi\ell^{1/2}/g^{1/2})(1 + \tfrac{1}{4}\sin^2\tfrac{1}{2}\alpha + \tfrac{9}{64}\sin^4\tfrac{1}{2}\alpha + \ldots)$$

The dots represent terms of higher order, higher powers of $\sin\frac{1}{2}\alpha$; those are going to be negligible until the angular amplitude α gets close to 90°.
(a) Verify that, for $\alpha = 60°$, the period is 7% longer than for "zero" (i.e., small) amplitude.
(b) How much longer (%) than the zero-amplitude period does a 10° swing take? Does the sin^4-term matter? [Answer: 0.19%]

✶✶ **30. How Fast Can I Run?** When small animals run, their legs are in rapid oscillation. The legs of large animals have longer periods. In other words, having legs twice as long does not enable the animal to run twice as fast. A comfortable jogging speed is one for which the oscillation frequency is close to the natural pendulum frequency T of the leg. Here is a scaling argument:

step size $\propto$ leg length L

speed = step size × frequency $\propto \frac{L}{T}$

But

$T \propto L^{1/2}$ for a pendulum

so

speed $\propto L^{1/2}$

(a) According to this argument, an animal having legs twice as long should have a running speed ________ times as fast.

NOTE: We would expect Nature to obey our scaling law roughly at best. Remember, species differ in ways other than size.

(b) A runner might cover about 3 meters in one period, that is, two paces. The natural swinging frequency of the leg is about 1 second. How much faster than 3 m/sec can *you* run a mile?

NOTE: The 4-minute mile is now an international standard in track competition.

✱✱ **31. Scaling Time** One scene in the movie "Change of Scale" shows a bridge being blown up. Suppose that the bridge filmed was a scale model one-hundredth life size. That means a 50-meter bridge would be 50 cm in the model. To make the explosion look natural, the movie has to be played at ________ (what fraction?) of the speed at which it was filmed.

✱✱ **32. Bending the Tree Limb by Dimensional Analysis** When you hang on the end of a tree limb, the distance y that it sags under your weight W depends on its length L, its cross-sectional area A (assumed uniform), and the Young's modulus E of the wood. From experiment you know that y is proportional to W and to L^3. Use dimensional analysis to show that

(a) y is inversely proportional to E.

(b) y is proportional to A^{-2}.

✱✱ **33. Frequency of Guitar Strings by Dimensional Analysis** You know that the frequency of vibration of the strings of guitars and other musical instruments increases when the string tension is increased. Also, the more massive strings have lower frequencies. Shortening the string length raises the frequency.

(a) Use dimensional analysis to find how the frequency might depend on string tension F, mass M, and length L.

(b) When you shorten the vibrating string by pressing it against the finger board, you also change M, but you leave M/L, the mass per unit length, unchanged. Rewrite the result of part (a) to show how the frequency depends on tension, mass per unit length, and length.

☐ QUESTION FOR FURTHER THOUGHT

α. **Bed Sores** Bed sores may occur where prolonged local pressure against a set of blood vessels impedes the circulation. A common place is right at the coccyx, a protruding bone at the base of the spine. Here the skin bears a disproportionate share of the weight of a bed patient lying on his or her back. Why do such sores afflict adults but not babies?

An obvious answer is that the baby is so much lighter than the adult, so there is less force squeezing on the blood vessels. True. But isn't that force distributed over a much smaller area in the baby? And isn't it *stress* (= force per unit area) that counts? We might try to compare the average stress exerted by the bed sheet on the body in order to make comparisons.

Compare a 10-pound baby and a 165-pound adult: The baby's linear dimensions are scaled down about 1:5 relative to the adult, so the comparable surface area is only one twenty-fifth that of the adult. How does the average stress on the baby's skin compare with that of the adult's? The ratio of the stresses

$$\frac{\text{baby}}{\text{adult}} = \frac{10 \text{ pounds}/165 \text{ pounds}}{\frac{1}{25}} = 1.5$$

The stress on the baby's skin is about 1.5 times *greater* than on the adult's! Evidently, average stress is not what counts for bed sores, so that we shall have to look elsewhere. But we can use the insight we have gained from our simple exercise in proportions to see why babies that are not turned often enough can develop hairless spots **(alopecia)** on the back of the head: A much larger fraction of the total body weight resides in the infant's head than in the adult's.

Getting back to bed sores, it looks as though the baby's blood vessels can stand local pressure better than the adult's. How is it possible that a stress sufficient to collapse an adult's blood vessel can be withstood by the corresponding blood vessel in the baby?

Present your ideas for the resolution of this paradox in a brief paragraph.

A hummingbird feeding.

4

ENERGY AND POWER

SOME QUESTIONS THIS CHAPTER WILL ANSWER

Someone says work is force times distance. When would such a stripped-down definition be misleading?

The foot-pound is a unit of work. Is the joule a unit of work?

How would you find the mechanical advantage of a pulley system?

State the work-energy theorem.

When is the concept of potential energy useful? When isn't it?

What is an elastic collision?

For what kind of collisions is kinetic energy conserved?

For what force does *mgh* represent potential energy?

If I tell you the vertical takeoff speed of a jumper, can you tell me how high the jump will be? Do you have to know the weight of the jumper?

Estimate how much power it takes for you to run up a flight of stairs in 4 seconds. Answer in watts.

What is lift-over-drag ratio? Why is it important in determining the power requirement of an aircraft engine?

Why can big birds in general not fly as well as little birds?

4.1 WORK

You are given the task of getting 1000 pounds of sand from the first floor to the second floor, 12 feet up. Should you carry the 100-pound bags up one at a time or two at a time? (See Fig. 4.1.)

The question really has to do with personal preference. The lifting *force* you need for two bags is twice the force you need for one. So it seems to require more *effort* to move them two at a time. On the other hand, you might finish the job in less time that way. Which way would you develop more fatigue? That may depend on how strong you are. There is one measure of the task that is called **work.** One bag at a time or two at a time, the same amount of work is done either way. You can express the amount of work quantitatively: To raise 1000 pounds up 12 feet requires 12,000 foot-pounds of work. It does not matter how you subdivide it. Ten loads at 1200 foot-pounds each, or five loads at 2400 foot-pounds each, the work comes out the same.

This kind of bookkeeping on the quantity we call *work* will form the subject of this chapter. Here is a crude definition of the technical term:

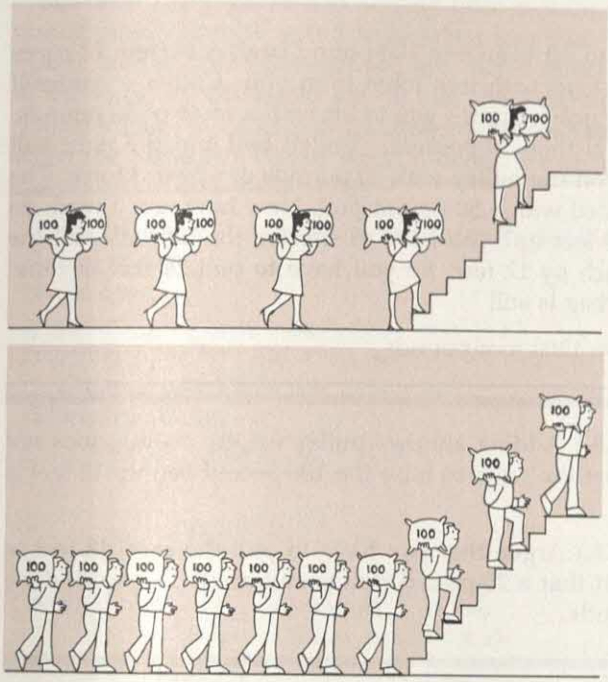


FIGURE 4.1 Should you carry the 100-pound bags up one at a time or two at a time?

DEFINITION *Work* is the product of force times the distance that force acts through.

Each 100-pound bag requires 100 pounds of force to lift it. That hundred pounds of upward force acts through 12 feet of vertical ascent to do 1200 foot-pounds of work per bag. Add it up for 10 bags, and so on.

EXERCISE 1 The force needed to lift 1 kg is 9.8 newtons. The work done in lifting 1 kg up 1 meter is ______ newton meters.

DEFINITION 1 newton meter = 1 kg m^2/sec^2 = 1 joule
The **joule** is the SI (Système International) unit of work.

Conversion 1 foot-pound = 1.358 joules

EXERCISE 2 Verify that conversion factor.

Block and Tackle

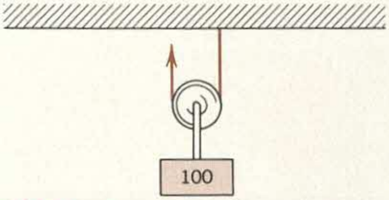


FIGURE 4.2 Block and tackle. A 50-pound pull can lift a 100-pound load.

What if you don't want to lift even one 100-pound bag? A system of ropes and pulleys can lift the load with less force from you. Convince yourself that the setup of Fig. 4.2 only requires you to lift with a force of 50 pounds. The tension in the rope is then 50 pounds. The left half and the right half of the rope each pull up on the pulley with 50 pounds of upward force. The 100-pound bag can be lifted with a 50-pound pull. Now how much work do you do to lift the bag 12 feet up? You have to shorten the left half and the right half of the rope each by 12 feet. So you have to pull 24 feet of rope. The work done on each bag is still

50 pounds × 24 feet = 1200 foot-pounds

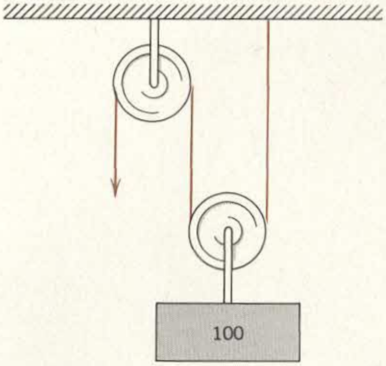


FIGURE 4.3

EXERCISE 3 (See Fig. 4.3.) Adding another pulley on the ceiling does not change things. Argue that the work to raise the 100-pound bag up 12 feet is still 1200 foot-pounds.

EXERCISE 4 (See Fig. 4.4.) Argue that you have to pull the rope 48 feet to raise the bag 12 feet, but that a 25-pound pull is sufficient. The work done is still ______ foot-pounds.

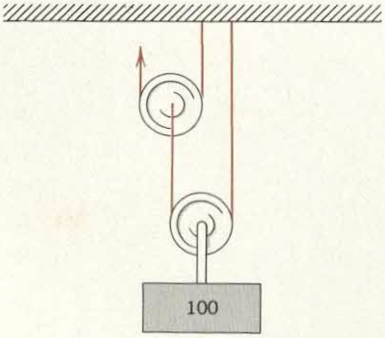


FIGURE 4.4

The **block-and-tackle** problems have not been introduced here in order to teach you how to lift bags of sand. The point is to make intuitive the concept of work—force times distance—and to accustom you to *energy bookkeeping*. That term will be made clear in the next section.

4.2 THE WORK-ENERGY THEOREM

EXAMPLE Let a constant force of 1 newton push on a 1-kg air-track glider and accelerate it through 2 meters. The force then does an amount of work equal to

1 newton × 2 meters = 2 joules

What happened to the work? It has gone into changing the speed of the glider. How much? That depends on how fast the glider was going initially. Suppose it was at rest. The acceleration

$$a = \frac{1 \text{ newton}}{1 \text{ kg}} = 1 \, \frac{\text{m}}{\text{sec}^2}$$

would last until the glider had traveled the 2 meters. How fast is it going at the end? You know how to solve this kind of problem pretty well by now. We picked easy numbers on purpose. After 1 second it is going at 1 m/sec and has covered 0.5 m. After 2 seconds it has gone the whole 2 meters and is going at 2 m/sec.

But what if the glider had already been moving forward before the force was applied? Then it would have taken less than 2 seconds to cover the 2 meters, and it would have gained less than 2 m/sec in speed from the action of the force. What should we do our bookkeeping on? Is there a quantity characterizing the motion of the glider that does change by the same amount when the same amount of work is done? We tackle that question algebraically:

Constant force F on mass m, acts through distance s
Initial speed v_i
Final speed v_f
Acceleration a lasts time t, so $v_f - v_i = at$
Average speed $= \frac{1}{2}(v_f + v_i)$, so
distance $s = \frac{1}{2}(v_f + v_i)t$
Eliminate the time:

$$s = \frac{\frac{1}{2}(v_f + v_i)\,(v_f - v_i)}{a}$$

$$= \frac{\frac{1}{2}v_f^2 - \frac{1}{2}v_i^2}{a}$$

$$\begin{aligned}\text{work done} &= Fs \qquad \text{(Now use } F = ma\text{)}\\ &= mas \qquad \text{(Now put in for } s\text{)}\\ &= \tfrac{1}{2}mv_f^2 - \tfrac{1}{2}mv_i^2\end{aligned}$$

The quantity $\frac{1}{2}mv^2$ is called the **kinetic energy.** This is the quantity that allows bookkeeping. The theorem that we have proved is

WORK-ENERGY THEOREM The work done on the mass is equal to the change in its kinetic energy.

Now come back to our example, the 1-kg glider with 2 joules of work done on it. What increased by 2 joules regardless of the initial speed? The kinetic energy.

EXERCISE 5 Let $m = 1$ kg, $s = 2$ m, and $v_i = 1.5$ m/sec. Then the initial kinetic energy is

$\frac{1}{2}mv_i^2 = 1.125$ joules [Verify]

The final kinetic energy is

$\frac{1}{2}mv_f^2 = 1.125\text{ J} + 2\text{ J} =$ ________

Verify that the final velocity v_f is indeed the result of an acceleration of 1 m/sec^2 over 2 meters.

Why is kinetic energy, $\frac{1}{2}mv^2$, an interesting concept? Essentially, it is *because* it allows us to do another kind of bookkeeping. Kinetic energy is a quantity that is larger for a fast object than for a slow one, larger for a massive object than for a light one. In Chapter 2 we defined *momentum,* also a measure of "how much motion." Which is the better measure of motion, kinetic energy or momentum? Two hundred years ago, that was a controversial question. Both lend themselves to bookkeeping, but different kinds of bookkeeping. Both tell something about how hard it is to start a certain motion, or how hard it is stop it. The kinetic energy of a bullet, for example, is a measure of how far it will penetrate into a block of wood. It is also a measure of how much it heats up the block of wood. The logical connection between kinetic energy and temperature will be made in Chapter 6, Pressure of Air and Water.

Potential Energy

To lift 1 kilogram takes a force of 9.8 newtons. To lift 1 kilogram up 1 meter takes an amount of work equal to 9.8 joules. What if, having been raised 1 meter, 1 kg is dropped? What happens to the work we did on it? [Answer: It goes into kinetic energy.] After falling 1 meter, the 1-kg mass will have a kinetic energy of 9.8 joules. That is what the work-energy theorem says. Gravity provides a downward force of 9.8 newtons. That force acts through a distance of 1 meter. Work done = 9.8 joules. If we had dropped it from double the height, the kinetic energy gained in falling would have been doubled. Had we dropped double the mass, we also would have doubled the kinetic energy. Remember the sand bags. The energy bookkeeping is straightforward. The work done in lifting a mass up against gravity all goes into kinetic energy when the mass is dropped back down to its original level.

It is useful to have a word that describes "how the work is stored" while the mass is waiting to be dropped. But it is not work that is stored. Neither is it kinetic energy. The useful word is **potential energy.** When the 1-kg mass is raised up 1 meter, 9.8 joules of work is done on it. We say its potential energy is increased by 9.8 joules. When it falls down 1 meter, that potential energy is lost and "converted" into kinetic energy. This is a convenient way to talk. It implies that the sum of potential energy and kinetic energy is what enters the bookkeeping.

Here is a quick review of our energy bookkeeping:

To lift mass m against gravity requires a force mg

To raise mass m up to height h requires work mgh

Change in potential energy = Δ (P.E.) = mgh

When mass m falls down from height h,

Δ (P.E.) = $-mgh$

Change in kinetic energy = $\Delta\ (\frac{1}{2}mv^2) = -\Delta(\text{P.E.})$

Repeat: $\Delta\ (\frac{1}{2}mv^2) = mgh$

Notice that we never said what the value of the potential energy is. We talked only about changes in P. E. In other words, the level at which the P. E. of an object is zero was not defined. We shall continue to live with that ambiguity. If you don't like it, you can define your own zero. After all, the zero is arbitrary. How about sea level? That way negative values of P.E. won't seem strange. The point is, regardless of where you take the zero, the statement

$$\Delta\,(\tfrac{1}{2}mv^2) = -\Delta\,(\text{P.E.})$$

is equivalent to the statement

kinetic energy + potential energy = constant

You will read about the "Law of Conservation of Energy." People may have in mind other forms of potential energy besides gravitational. For example, in Chapter 13 we shall take up electrostatic potential energy. But this kind of bookkeeping is what the terminology is about.

EXERCISE 5 My mass is 75 kg. Consequently, I weigh 735 newtons. When I step up a 0.15-m step, I increase my P.E. by ______ joules. If I jump down 0.15 m, my landing speed is ______m/sec. That makes my kinetic energy on landing $\frac{1}{2}mv^2 =$ ______ joules.

EXERCISE 6 To climb the full 3 meters between floors requires work 735 newtons × 3 m = ______ joules.

Note that it does not matter how steep the stairs are. In climbing stairs, I move forward as well as up. But my weight pulls vertically down on me. So the work I do against gravity depends only on my vertical displacement. The definition of work has to be refined when the force and the displacement are not in the same direction.

DEFINITION Work is the product of force times the component of the displacement parallel to that force.

So the work done against gravity would be quite the same if I went up a ramp.

EXERCISE 7 A car might have a mass of 10^3 kg. When you drive up a hill 10^2 meters high, by how much is the potential energy of the car increased?

EXERCISE 8 A fly might have a mass of 50 milligrams ($= 50 \times 10^{-3}\,g = 50 \times 10^{-6}$ kg). How much work is required to lift it up 1 meter? Does the fly increase its potential energy by that much in climbing 1 meter higher on the wall? In climbing 1 meter higher in the air?

4.3 ELASTIC COLLISIONS

In Chapter 2 we defined what is meant by an **elastic collision** between two objects: Their relative speed after the collision is the same as their relative speed before:

$$|v_2 = v_1| = |v_2^* - v_1^*|$$

We now show that an elastic collision conserves the total kinetic energy of the two colliding objects.

All that is needed for the derivation is the conservation of momentum:

$$m_1 v_1 + m_2 v_2 = m_1 v_1^* + m_2 v_2^*$$

Rewrite the condition for elastic collision in the form

$$v_2 - v_1 = -(v_2^* - v_1^*)$$

or

$$v_1 + v_1^* = v_2 + v_2^*$$

Rewrite the momentum conservation equation:

$$m_1 (v_1^* - v_1) = -m_2 (v_2^* - v_2)$$

The two equations can now be combined so as to relate the kinetic energy before to the kinetic energy after the collision. Just multiply the equations together:

$$m_1 (v_1^{*2} - v_1^2) = -m_2 (v_2^{*2} - v_2^2)$$

In kinetic-energy language:

$$\tfrac{1}{2}m_1 v_1^2 + \tfrac{1}{2}m_2 v_2^2 = \tfrac{1}{2}m_1 v_1^{*2} + \tfrac{1}{2}m_2 v_2^{*2}$$

Repeat: The total kinetic energy is the same after the collision as before.

On the air track, elastic collisions are the ones between gliders fitted with spring bumpers. During the collision, while the bumper spring is being compressed, work is done against the spring. We say that "elastic potential energy" is stored in the spring. As the spring expands and does work on the gliders, that elastic P.E. is converted back into the kinetic energy of the gliders. What is special about elastic collisions is that no energy is lost in the two conversions, kinetic-to-potential and potential-to-kinetic. Why is that special? Doesn't the work-energy theorem guarantee that the energy books will balance?

Nonconservation of Mechanical Energy

You know about collisions that are not elastic. For example, if the bumper spring is replaced with putty, the two gliders stick together. In that case, the kinetic energy after the collision will be less than before (see Exercise 9).

What about the work-energy theorem? How can we save the energy bookkeeping? The answer is that some of the work done on the putty goes into internal motions of the putty. The molecules of the putty end up with the kinetic energy needed to balance the books. It is kinetic energy of random motion, and we call it heat. The putty gets warm.

Energy bookkeeping involving heat will be taken up in Sec. 8.3, under the heading "First Law of Thermodynamics." Here, you may want a guideline to tell when the potential-energy concept is applicable. When it is not, mechanical energy can be "lost" by doing hidden internal work. The rule is that the potential-energy concept is limited to central forces that depend only on the positions of objects, not on velocities or on the time. That says that work done by frictional forces, by electric motors, by heat engines, or by the wind cannot be described in terms of potential-energy changes. A central force is a force directed along the line joining two interacting objects.

EXERCISE 9 A 1-kg glider moving at 1 m/sec bumps into an identical glider at rest. The putty between them makes them stick together, and they move off together.

(a) Conservation of momentum implies that their speed after the collision is ______ m/sec.
(b) Kinetic energy before collision = 0.5 joules. (Verify.)
(c) Kinetic energy after collision = ______ joules.
(d) What fraction of the original kinetic energy was lost?

EXERCISE 10 Same as Exercise 9, except that the stationary glider's mass is 2 kg.

4.4 SPECIAL TOPIC: How High Can I Jump? About as High as a Flea

Our muscles are capable of doing work. In what form is energy stored in them, waiting to be converted into work? Is it potential energy? The word is *chemical energy*. The scaling argument that follows assumes that the amount of energy a muscle can store is proportional to how much muscle there is. It shows why it is reasonable that a flea and a man should be able to jump to about the same height (see Fig. 4.5). ■

FIGURE 4.5 How high can I jump? About as high as a flea.

Our giant (Sec. 3.1) will not do well in the competition for who can jump highest. He weighs too much in relation to his strength. Can you imagine an elephant jumping? From a crouched position, I can't even jump up half my height off the ground. A flea can jump about 200 times its height. The actual height of the jump is about the same. Here is the argument why a big animal and a small animal have roughly the same limitation here.

We are talking about a jump from rest. As your legs straighten, you leave the ground with an upward center-of-mass velocity v and kinetic energy $\frac{1}{2}mv^2$. After take-off, your center of mass continues to rise to a height h above its take-off altitude. At this point all that kinetic energy has been converted to potential energy:

$$mgh = \tfrac{1}{2}mv^2$$

Of course, your mass m is a needless variable here. It cancels out. But your kinetic energy was the result of your leg muscles doing work to straighten your legs and accelerate that mass m. It is a reasonable assumption that the amount of chemical energy rapidly available in the leg muscles is proportional to the mass of muscle. If leg muscle forms the same fraction of total mass in man as in a flea, then the energy available for jumping should be proportional to total mass. Consequently the height h should also be the same.

There is actually an error in that argument. You crouch down, bending your knees. During the upward acceleration of your center of mass your legs straighten and raise your center of mass a distance s before your feet lose contact with the ground. The amount of work done by the muscles is not just the kinetic energy $\frac{1}{2}mv^2$ that they give your body. There is also the potential-energy gain mgs due to raising your center of mass up to its take-off altitude. That term was left out of the scaling argument of the previous paragraph. We now put in numbers to see how important it is.

For a grown-up, the straightening distance s might be 0.6 meters. If the jump height h is also 0.6 m, then the work done by the leg muscles in the jump is

$$\begin{aligned}\text{work done by legs} &= mgs + \tfrac{1}{2}mv^2\\ &= mgs + mgh\\ &= mg\,(0.6\text{ m} + 0.6\text{ m})\end{aligned}$$

We made a 100% error in using 0.6 meter in place of 1.2 meter in our scaling argument. For the flea, the distance s the legs stretch is at most a couple of millimeters. Writing mgh in place of $mgs + mgh$ was a very good approximation there. In that case, why can't the flea jump twice as high as we can? Perhaps it can. You are asked for your ideas in the Question for Further Thought at the end of the chapter.

EXERCISE 11 In the argument above, we replaced the kinetic energy $\frac{1}{2}mv^2$ at take-off by mgh. Why is this correct?

4.5 POWER

The rate of doing work is called **power.** The SI unit of power is the **watt:**

$$1 \text{ watt} = \frac{1 \text{ joule}}{\text{second}}$$

A 1-watt motor puts out 1 joule of work per second. If something does work at the rate of 1 watt for 1 hour, it does 1 watt-hour of work:

$$1 \text{ watt-hour} = 3600 \text{ watt sec} = 3600 \text{ joules}$$

1 **kilowatt-hour,** the unit favored by the electric utility companies (1 kwh = 3.6×10^6 joules), is the work done in lifting about 5000 human beings 1 meter off the ground. Alternatively, it is the work done in lifting one human being 5000 meters.

EXERCISE 12 **Verify.**

A warning may be useful about possible confusion arising from the popular usage of words. The electric utility company sells you "power." Presumably, it charges you for how much power you use. Your bill tells you how many kwh you used. In other words, you are charged for watt-hours, not for watts—for energy, not for power.

The next paragraph goes through an estimate of how much power the human body can deliver. The amount is impressive. Peak power is about 1 kilowatt. The Special Topic that follows presents just enough theory of flight to show why human-powered flight is not practical: We just don't have enough endurance.

■ CLIMBING STAIRS

When I step up a 12-inch step, I am doing 1 foot-pound of work on every pound of my body. That means I am doing 165 foot-pounds of work on my 165 pounds. Converting to metric, every kilogram gets raised 0.305 meters, so my 75 kg requires

$$\text{work} = 75 \text{ kg} \times 9.8 \frac{\text{newtons}}{\text{kg}} \times 0.305 \text{ m} = 224 \text{ joules}$$

to be raised up 1 foot. In my house it is about 8 ft. 6 in. between floors (2.6 m). That means I do 1.9×10^3 joules of work against gravity every time I walk upstairs. Let's round that off to 2000 joules. How long does that amount of work take? If I take the stairs two at a time and run as fast as I can, it takes me about 2 seconds. That means I am putting out 1000 joules per second:

$$10^3 \frac{\text{joules}}{\text{sec}} = 10^3 \text{ watts} = 1 \text{ kilowatt}$$

And that neglects the work my muscles do that just heats them up! Total power is more than twice the output power.

I can't keep up that pace for more than a few seconds. I run out of breath. When I walk in the mountains, I can comfortably gain around 400 meters of altitude in an hour. That is a power output of

$$75 \text{ kg} \times 9.8 \frac{\text{newtons}}{\text{kg}} \times \frac{400 \text{ m}}{3600 \text{ sec}} = 82 \text{ watts}$$

That is about an order of magnitude down from my peak exertion. Here is a beautiful example of Nature's way of designing for emergencies. I have an enormous short-run capability, many times greater than my circulatory system can handle in the long run.

It will be useful to review how the power was calculated in each case. We multiplied the lifting force mg times the vertical distance s to be gained. That is the work done against gravity. This is divided by the time t it takes to do that work, to obtain the power:

$$\text{power} = \frac{\text{work}}{\text{time}} = \frac{mgs}{t} = mg \cdot \frac{s}{t}$$

Power is the product of a force times a velocity. It is the lifting force times the rate of vertical displacement s/t. We call s/t the vertical component of the velocity v. Accordingly, we write

$$\text{power} = Fv_{\parallel}$$

where $v_{\parallel}$ is the component of the velocity v parallel to the direction of the force F (Fig. 4.6.). In climbing stairs this is only about 70% of Fv, since the velocity v is directed along the staircase, not vertically up. (See Appendix A.)

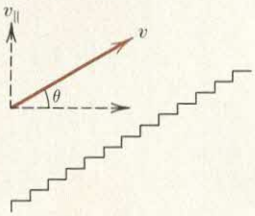
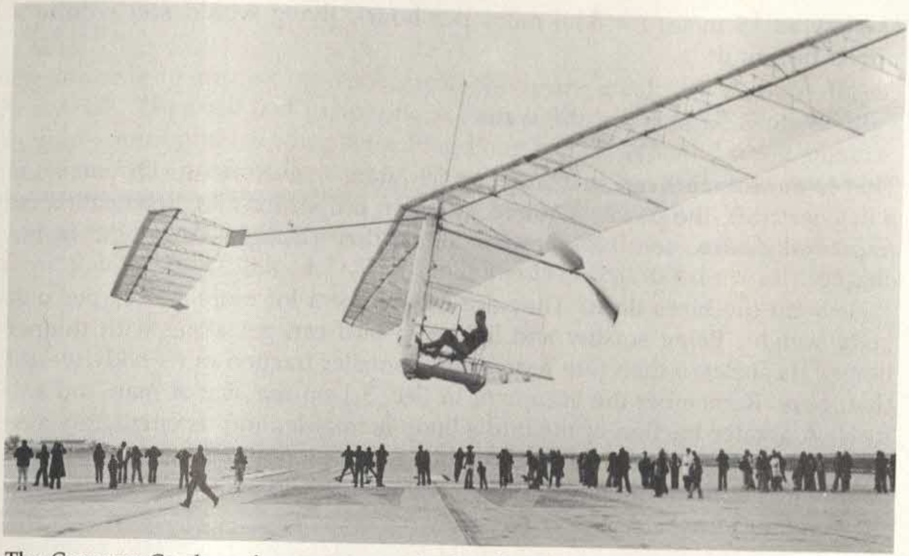


FIGURE 4.6 Climbing stairs. The power required is your weight times the vertical component of your velocity.

EXERCISE 13 Verify that figure of 82 watts for my power output in mountain climbing.

EXERCISE 14 How much power is required to accelerate a 1-ton car from 19 m/sec to 21 m/sec in 0.5 sec? Use $m = 10^3$ kg. Neglect friction.

EXERCISE 15 A 10^3-kg car accelerates from rest to 10 m/sec in 4 seconds. Calculate the average power needed to supply the additional kinetic energy. Again neglect friction. There is no implication that the acceleration is constant.



The *Gossamer Condor,* a human-powered aircraft.

4.6 FLYING

If we could only fly like the birds! Recently, the old dream of muscle-powered flight has been realized with a pedal-powered aircraft. The pilot was a superb athlete. Most of us could not sustain quite enough power output to power flight. The numbers are right near the margin of the possible. This section attempts to estimate the minimum power needed.

In the previous section, when we calculated the power needed in climbing stairs, we neglected air resistance. This was legitimate because the frictional "drag" force of the air tending to slow you down is so small. It is much less than the lifting force the stairs have to supply, which is equal to your weight. When you fly through the air, the **drag** force acting in the direction opposite to your velocity relative to the air can no longer be neglected. The aerodynamicist talks about the "**lift**-over-drag ratio" of a wing. This is the ratio of the upward force to the horizontal retarding force of the air moving past the wing in horizontal flight.

What you have to know about the physics of flight is that every wing design has a **stalling speed.** Below this speed, the wing no longer gives appreciable lift. Using our terminology, below the stalling speed, the L/D ratio becomes very small. At somewhat higher speeds, the L/D ratio can take on values as high as 25. A lift-over-drag ratio of 10 is more common. With L/D = 10, a weight of 1000 newtons could be held aloft with a forward thrust of 100 newtons. The forward thrust would be supplied, say, by a propeller. The figure 1000 newtons represents the weight of a human adult fitted with very light wings. Now even with that optimistic L/D ratio of 25, a lift of 1000 newtons requires a thrust of 40 newtons. Given a stalling speed as slow as 15 m/sec (= 33.6 miles per hour), flying would still require a power output of

$$40 \text{ newtons} \times 15 \frac{\text{m}}{\text{sec}} = 600 \text{ watts}$$

That is more than I can sustain for much more than a minute. If you want a heavier craft, the power required grows in proportion. For speculation on improved design, see the Question for Further Thought at the end of the chapter.

How do the birds do it? They are capable of a lot more power per unit body weight. Being smaller and lighter, a bird can get along with thinner bones. Its skeleton therefore represents a smaller fraction of its body weight than ours. Remember the argument in Sec. 3.1 on the size of man and animals. A greater fraction of the bird's body is muscle, and its circulatory system is designed for a higher metabolic rate per pound than man's. The larger birds either fly briefly and then rest, or they do a lot of gliding, letting the wind supply the power. ■

4.7 USEFUL MATHEMATICS: VECTORS

So far we have talked only about motion in a straight line. All collisions were head-on. A velocity was either + or − because on a straight line the only directions are forward and backward. Now we want to talk about sideward, about things moving at an angle to each other. We need to free ourselves from the constraint to one dimension and go to the real three-dimensional world. In three dimensions, a velocity will no longer be represented by one number, but by three numbers: the x, y, and z components of the velocity **vector.** (See Appendix A.)

The next section goes back to freely falling objects. It treats the kinematics of throwing.

■ 4.8 SPECIAL TOPIC: How Far Can I Throw?

When the ball leaves your hand, it has motion in a well-defined direction. We talk about its *initial velocity*. If you throw it toward the east and a little bit up, our convention would say that the initial velocity has an x-component and a z-component. If the acceleration is vertically down (gravity), the subsequent velocity has only x- and z-components, no y-component. The mathematician would say that the motion is confined to the xz plane. It is motion in two dimensions. ■

Before doing any arithmetic or algebra, it is best to make some simplifying assumptions. One is to neglect the spin of the ball and represent the ball's position as a function of time by a point in the xz plane,

$$x = x(t)$$
$$z = z(t)$$

The other is to neglect air resistance. Those are good assumptions for a heavy ball. They are bad assumptions for a light ball. You know that they would be terribly misleading for a Ping-Pong ball. They could never predict the curve ball in baseball pitching. They get worse the faster the speed of the wind past the ball. They do make the kinematics simple.

Suppose that the initial velocity is 20 m/sec, directed at an angle of 30° above the horizontal (Fig. 4.7). The components of that initial velocity vector are

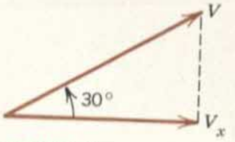


FIGURE 4.7

$$v_x = 17.32\ \frac{\text{m}}{\text{sec}}$$
$$v_y = 0$$
$$v_z = 10\ \frac{\text{m}}{\text{sec}}$$

EXERCISE 16 Verify that $\sqrt{v_x^2 + v_z^2} = 20$ m/sec, and that $v_z/v_x = \tan 30°$. (HINT: Tangent = opposite/adjacent.) The acceleration of 9.8 m/sec² is directed vertically downward. The components of the acceleration vector are

$$a_x = 0$$
$$a_y = 0$$
$$a_z = -9.8\ \frac{\text{m}}{\text{sec}^2} = -g$$

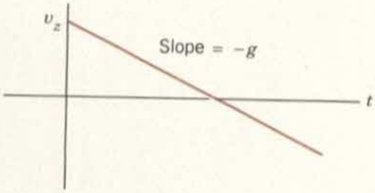


FIGURE 4.8

Finding the trajectory of the ball represents the solution of two problems: a uniform motion in the x-direction and a motion at constant acceleration in the z-direction. For example, we might ask where the ball is 2 seconds after being released. The x-component of the velocity remains at a constant 17.32 m/sec. So the x-displacement is just

$$x(t) = 17.32\ \frac{\text{m}}{\text{sec}} \times 2\ \text{sec} = 34.64\ \text{m}$$

The z-component of the velocity is not constant (Fig. 4.8). At $t = 2$ sec, it is

$$v_z(t) = \frac{10\ \text{m}}{\text{sec}} - \left(9.8\ \frac{\text{m}}{\text{sec}^2}\right) \times (2\ \text{sec}) = -9.6\ \frac{\text{m}}{\text{sec}}$$

The ball rose and has already started to lose altitude. The average v_z over the 2 seconds is +0.2 m/sec. The ball ends up only 0.4 m higher than it was when released.

EXERCISE 17 Let the initial velocity vector have components (v_{x0}, v_{z0}). Derive the formulas

$$\begin{aligned} v_x(t) &= v_{x0} \\ v_z(t) &= v_{z0} - gt \\ x(t) &= v_{x0}t \\ z(t) &= v_{z0}t - \tfrac{1}{2}gt^2 \end{aligned}$$

A geometric derivation is suggested: Sketch graphs of $v(t)$ versus t and interpret the areas of the little triangles, as in Sec. 1.1, Kinematics.

EXERCISE 18 When does the ball reach its highest point? Call that time t_{top}. At t_{top} the ball has stopped rising: $v_z(t_{\text{top}}) = 0$.

$$v_{z0} - gt = 0$$

$$t_{\text{top}} = \boxed{}$$

Note that t_{top} is independent of v_{x0} and depends only on v_{z0}.

Now to come back to the question asked in the section title: How far can I throw? The range of a projectile is defined as the horizontal distance it travels before coming back to its original height. You can see that the range is the value of x at time $t = 2\ t_{\text{top}}$:

$$\text{range} = v_{x0} \times 2t_{\text{top}} = \frac{2v_{x0}v_{z0}}{g}$$

That is the result of the symmetry between the up and down portions of the trajectory. Problem 29 shows that the best angle for throwing far is 45°. In other words, if the initial speed $v_0 = \sqrt{v_{x0}^2 + v_{z0}^2}$ is given, a 45° launching angle maximizes the range. That means $v_{x0} = v_{z0} = \sqrt{\tfrac{1}{2}}\ v_o$. A baseball pitcher can throw close to $v_o = 40$ m/sec. That gives his throw the following range:

$$\text{range} = \frac{v_0^2}{g} = \left(40\ \frac{\text{m}}{\text{sec}}\right)^2 \Big/ \left(9.8\ \frac{\text{m}}{\text{sec}^2}\right)$$

$$= 163 \text{ m} = 536 \text{ feet}$$

That is a long throw! Of course, you remember that we neglected air resistance. Anyway, if you can't compete in that league, remember that a 1% increase in your speed v_0 gives you a 2% increase in maximum range. [HINT: Range is proportional to v_0^2, and $(1 + 0.01)^2 = 1.02$.]

EXERCISE 19 How much kinetic energy does that pitcher impart to a 150-gram ball with a 40-m/sec pitch?

NOTE: Don't underestimate his muscular effort. He has to impart some kinetic energy to his arm too, which has a mass of several kilograms.

SUMMARY

To raise a weight of 9.8 newtons up to a height of 1 meter requires an amount of work equal to 9.8 joules.

A mechanical device that allows raising a weight with *half* the force requires that force to act through *twice* the distance to raise the weight to the same height. The work done is the same.

According to the work-energy theorem, work done on a mass is converted into kinetic energy. This is the basis of energy bookkeeping and the idea that energy is neither created nor destroyed (energy conservation).

Elastic collisions are lossless; that is, the total kinetic energy is the same after as before.

In collisions that are not elastic, kinetic energy of the colliding masses is converted into energy of internal motion or deformation.

When a muscle does work, it uses up chemical energy.

Man and flea can jump about equally high. This is roughly predicted by simple scaling laws, assuming that the amount of work a muscle can do rapidly is proportional to its mass.

Power is the rate of doing work = $Fv_{\parallel}$. 1 watt = 1 joule/sec.

The peak rate of human power output is more than 10 times the rate we can sustain for longer times.

Human-powered flight is impractical because we cannot sustain the needed power output to overcome air resistance.

Mass is a scalar. Displacement, velocity, acceleration, force, and momentum are vectors.

Under the action of gravity, the horizontal and vertical components of displacement obey independent equations.

NOTE: This is only true in the absence of frictional forces.

KEY TERMS

Work A force does work when the object on which the force acts is displaced in the direction of the force. The work done is the product of the force times the component of the displacement in the direction parallel to the force. Alternatively, the work is the product of the displacement times the component of the force in the direction parallel to the displacement. If the force is not constant over the displacement, the average force must be used. Then the definition requires a sum of products using small displacements.

Block-and-tackle One or more pulleys and ropes so arranged as to amplify the force available for pulling. With one rope and one pulley, the available pull can be doubled.

Work-energy theorem The work done on a mass is equal to the change in its kinetic energy.

Kinetic energy Energy of motion = $\frac{1}{2}mv^2$, where m = mass and v = speed.

Potential energy (P.E.) The energy an object has by virtue of its position. The work done in getting an object to its position from some reference position. For gravitational potential energy simply write mgh, where mg is the weight and h is height above some reference altitude. A potential energy function can only be defined for forces that are functions of position alone.

Joule Unit of energy. The work done by 1 newton acting through 1 meter. 1 joule = 1 newton × 1 meter.

Elastic collision A collision between two objects whose relative speed after is the same as before. Elastic collisions conserve kinetic energy, that is, they are lossless.

Power Rate of doing work = $Fv_{\parallel}$.

Watt Unit of power. 1 watt = 1 joule/sec.

Kilowatt-hour (KWH) The amount of energy supplied if work is done at the rate of 1000 joules per second for 1 hour. 1 kwh = 3.6×10^6 joules.

Lift The force of the wind on the wing perpendicular to the wind direction.

Drag The force of the wind on the wing parallel to the wind direction.

Stalling speed The slowest speed at which an airplane will fly.

Vector In three dimensions, a vector is a set of three numbers called components that transform in the same way as a displacement under a change of reference frame. Accordingly, a vector is something having a magnitude and a direction. Examples: Displacement, velocity, acceleration, force, and momentum.

Scalar A quantity that is invariant under a change of reference frame. Accordingly, a scalar has a magnitude and does not have a direction, nor is the magnitude a function of direction. Examples: mass and energy.

Component of a vector The x-component of a vector is the projection of that vector on the x-axis.

Magnitude of a vector Also called absolute value. If a vector has components (V_x, V_y, V_z), its magnitude is $(V_x^2 + V_y^2 + V_z^2)^{1/2}$.

Simple machine A device for transforming forces. The six basic types of simple machine are the lever, the wedge, the wheel and axle, the inclined plane, the pulley, and the screw.

Mechanical advantage The mechanical advantage of a simple machine is the ratio of the output force to the input force.

REVIEW QUESTIONS

- How much work is done when a 98-newton weight is raised 2 meters?

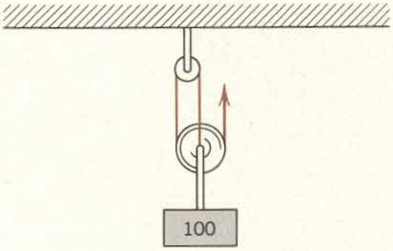
NOTE: That is a 10-kg weight.

- Is all that work converted into potential energy of the 10 kg?
- If the weight is dropped and falls to its original level, how many joules of kinetic energy does it have just before it hits the ground?

- Why does the kinetic energy not depend on whether it drops straight down or slides down a frictionless ramp?
- What happens to the kinetic energy when the weight hits the soft ground?
- Show how a rope and pulley can be used to lift a heavy weight with a force equal to half that weight.
- If the speed of an object is doubled, its kinetic energy is multiplied times ________.
- If the speed of an object is increased by 1%, its kinetic energy is increased by ________%.
- In order to increase the kinetic energy by 1%, the speed has to be increased by ________%.
- If two equal masses have an elastic head-on collision, they simply interchange velocities. True or False? Prove your answer.
- If two equal masses have a totally inelastic collision so that they stick together, anywhere up to 100% of their total kinetic energy may be lost, that is, may go into internal motion. True or False? (HINT: How about a symmetric collision?)
- If you jump straight up, the kinetic energy you lose as you gain altitude is converted into potential energy in such a way that you lose an equal amount of kinetic energy in every inch your center of mass rises. True or False? Explain.
- At first we argued that the height you can jump from a crouched position is proportional to the energy your leg muscles provide. What was wrong with that argument?
- In jumping, you raise your weight *mg* up to an effective height *s* before your feet leave the ground. This takes a time *t*. What is your average power output during this time?
- I weigh 165 pounds. If I walk up a flight of stairs having a 10-foot vertical rise, I do 1650 foot-pounds of work against gravity. Why does the slope of the stairs not affect the amount of work done against gravity?
- If I run down again, what happens to the potential energy I gained in the climb?
- Argue that, if I run up the stairs very fast, I actually do a bit more work. (HINT: What happens to my kinetic energy?)
- The amount of work done by an airplane engine in powering the plane for a mile is proportional to the lift-over-drag ratio of the wings. True or False? Explain.
- For a given lift-over-drag ratio, the power needed to fly a level course is proportional to the speed. True or False? Explain.
- A jet engine capable of providing a thrust of 10^4 newtons at a speed of 2×10^2 m/sec is advertised as delivering 2000 watts of power. Why do you think there is a misprint in the advertisement? What might the writer have intended?
- If the initial speed is doubled, the range of the throw is multiplied times ________.

PROBLEMS

✶ **1. Block and Tackle** Argue that in order to raise the load up 12 feet, you have to pull the rope 36 feet. How much tension is needed in the rope to lift a 100-pound load?



PROBLEM 1. Block and tackle. Mechanical advantage is 3:1.

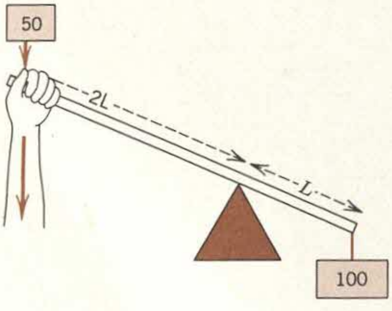
✶✶ **2. Mechanical Advantage** A pulley is one of the six classes of **"simple machine."** The others are the lever, the wheel and axle, the wedge, the inclined plane, and the screw. All of these are devices for transforming forces.

In the pulley system of Fig. 4.3, a force of 50 pounds can raise a weight of 100 pounds. We say that the system has a mechanical advantage of 2. The mechanical advantage is defined as the ratio of the output force of the machine to the input force to the machine. If we neglect frictional losses in the machine, the ratio of the distances through which those forces must act in doing work is the *reciprocal* of the mechanical advantage.

(a) In Fig. 4.3, in order to lift the weight 1 foot, the rope has to be pulled ________ feet.

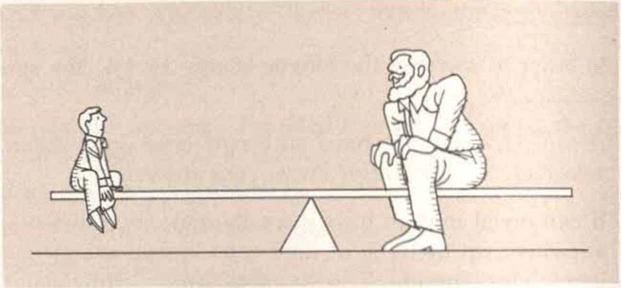
(b) In Fig. 4.4, in order to lift the weight 1 foot, the rope has to be pulled ________ feet. The mechanical advantage is ________.

(c) The figure here shows a lever that allows a 100-pound weight to be lifted with a 50-pound force. The mechanical advantage of this lever is ________. To raise the weight 1 foot, the force applied vertically downward at the long end has to act through ________ feet.



PROBLEM 2. Mechanical advantage: Lever allows a 100-pound weight to be lifted with a 50-pound force.

(d) The teeter-totter is another "lever" arrangement (see the second figure). Describe in words. How can the light child "lift" the heavy grown-up?

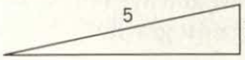


PROBLEM 2. The bigger one has to sit closer to the middle in order to balance.

✶ **3. Inclined Plane** A ramp with a grade of 1 in 5 has a mechanical advantage of 5. In pulling the load 5 meters up the ramp, it is raised 1 meter. A load weighing 1000 newtons can be pulled up a frictionless ramp by a force of 200 newtons.

(a) How much work is done by the 200-newton force in pulling the load 5 meters?

(b) How much work is done in raising the 1000-newton weight by 1 meter?



PROBLEM 3.

✶ **4. Pushing the Car** Suppose that it requires a force of 100 newtons (= 22 pounds) to keep a car rolling slowly on a level road. This is just the force to overcome friction. How much work is done in pushing it a mile (= 1.6×10^3 m)? ________ joules.

✶ **5. Pushing the Pencil** Suppose that it requires a force of 1 dyne (= 10^{-5} newtons) to move the pencil across the paper in writing. How much work is done in writing the solution to this problem—about 0.1 meter of pencil lines? ________ joules. Express the answer also in ergs. Conversion: 1 erg = 10^{-7} joules = 1 gram cm^2/sec^2.

** **6. Coefficient of Friction** (a) How much work is done by a force of 6 newtons pushing a blackboard eraser across 1 meter of blackboard?

NOTE: That 6 newtons is a force in the direction of the motion. It is the tangential component of the total force pushing the eraser. The work goes into warming up the board and the eraser and making a sh-shing sound.

(b) To erase properly, I have to push the eraser into the board with a force of about 8 newtons. But that 8 newtons does no work. The board does not deform appreciably under the 8-newton push, and the work of deforming the eraser is small. The ratio of the tangential frictional force impeding the motion to the normal force is called the *coefficient of friction*. In this case it is 0.75. Verify.
(c) Coulomb's Law of Friction asserts that the coefficient of friction is independent of the normal force. In practice, this is only an approximate law. Assuming it to be true for our blackboard, what would be the frictional force if I double the normal force to 16 newtons?
(d) How much work would then be done in moving the eraser 1 meter?
(e) What is the net force with 16 newtons normal force and 12 newtons tangential force? [HINT: Pythagoras! In part (b), the net force is 10 newtons.]

* **7. Car at Highway Speed** A reference book says that it takes 9 kilowatts to overcome tire and air friction when running a car at 44 miles/hour (= 20 m/sec). Calculate the horizontal force of the road on the tires.

** **8. Friction** A 5-kg chair is dragged slowly across the floor by a horizontal pull of 30 newtons. Some of the energy expended goes into the air as a scraping sound. Some of it goes into warming the floor.
(a) How much work is done in dragging it 2 meters at constant speed?
(b) If the pull is increased to 50 newtons, the chair speeds up as it is pulled. Assume that it starts from rest and the frictional force opposing its motion remains 30 newtons. Calculate its acceleration.
(c) How long does it take to go 2 meters? How fast is it moving at the end?
(d) How much work does the 50 newtons do on the chair?
(e) How much of that work goes into kinetic energy of the chair? Into heat?
(f) What is the average power supplied to the chair over the 2 m?
(g) What is the instantaneous power supplied to the chair as it passes the 1-meter mark?

** **9. Climbing Stairs, Climbing Mountains** A staircase with the riser equal to the depth of the step makes a 45° angle with the vertical.
(a) Let ϕ be the angle the climber's velocity v makes with the vertical. Show that power $= mgv \cos \phi$. For "All the trig you need," see Appendix B.
(b) For a 45° staircase, show that this power is about 70% of mgv.
(c) A pleasant trail up a mountain might have a 1-in-5 grade. That is another way of saying $\cos \phi = \frac{1}{5}$. If I average 400 meters of vertical ascent per hour, how fast am I walking on the trail? ________ m/sec.

* **10. Climbing Mountains to Work Off Your Lunch** A glass of milk can deliver roughly 100 Calories of chemical energy in being metabolized ($= 4.2 \times 10^5$ joules). How high do you need to climb to "use up" that much energy? You can be consoled to learn that your straightforward answer is about twice too big because your body is at best 50% efficient in converting chemical energy into work.

** **11. Potential Energy of a Linear Spring** For small deformations, stress and strain are proportional. (See Sec. 3.2) To stretch a spring by an amount x from its equilibrium length requires a force kx. The "spring constant" k is proportional to the Young's modulus of the material and depends on the geometry of the spring. How much work does this stretch require? The guess

$$\text{work} = \text{force} \times \text{distance} = kx \cdot x$$

is *wrong*. The force is not constant during the displacement. The average force over the displacement x is only $\frac{1}{2}kx$. The correct answer is

$$\text{work} = \text{average force} \times \text{distance} = \tfrac{1}{2}kx \cdot x$$

Consequently, the elastic potential energy stored in the spring is

$$\text{P.E.} = \tfrac{1}{2}kx^2$$

(a) To stretch the 1-meter steel wire of Section 3.2 by 1 millimeter requires a force of 98 newtons. The average force over that displacement is 49 newtons. Hence the potential energy stored in the stretched wire is ________ joules.
(b) Verify that the spring constant is $k = 0.98 \times 10^5$ newtons/m.

✶ **12. Work in Jumping** How much work must I do on my 75 kg to jump 0.6 m off the ground? Assume that I start by crouching, that is, lowering my center of mass 0.6 m below its "takeoff" altitude. [Answer: 882 joules]

✶ **13. Vertical Jump**
(a) How high can you jump with a vertical take-off velocity of 2 m/sec?
(b) How long does it take to achieve full altitude?
(c) How long does it take to fall back to the ground, that is, back to the take-off altitude?
(d) How fast is your center-of-mass traveling when you hit the ground?

✶ **14. Vertical Jump**
(a) To jump to a height of 1 meter, what must be the take-off speed?
(b) To stay in the air for 1 second, what must be the take-off speed? How high does that take you?

✶✶ **15. Jumping down** You plan to jump down from a 6-meter-high platform. This can be dangerous! Assume that your legs can bend enough to give your center of mass 0.6 meters of vertical descent between the instant your feet touch the ground and the instant the center of mass comes to rest.
(a) Plot the velocity of your center of mass versus time for the whole jump. Assume constant acceleration during the landing, that is, after your feet touch the ground.
(b) Calculate the landing acceleration. Express the answer in m/sec^2 and in g's.
(c) The force on your feet during the landing is _______ times your normal weight.
(d) Assume a body mass of 75 kg. How much potential energy will you gain in climbing up to the 6-m platform?
(e) How much kinetic energy will you have the instant your feet touch the ground?

✶✶ **16. Jumping: Man Versus Flea** Suppose man and flea can both jump 0.6 m high from rest. Assume that the upward acceleration a over the leg-straightening distance s is constant.
(a) If $s = 0.6$ m for man, calculate a. Express it in g's. (HINT: $mas = mgh$ from conservation of energy.)
(b) Calculate the force on his legs. Answer in the form "_______ times his weight."
(c) If $s = 2$ mm for the flea, calculate the acceleration a and the force on the flea's legs during the acceleration.

For man, an upward acceleration of 1 g already puts a sizable strain on the leg muscles. It is like carrying another person your own weight on your shoulders. An acceleration of 300 g, even if our legs could deliver it, would put intolerable stress on our internal organs. Is the flea stronger than we are? It is much stronger per unit weight. See Sec. 3.1, Size of Man and Animals.

Problems 17–20 are Easy Problems on power.

✶ **17 Walking Upstairs** How much power does a 75-kg woman expend in walking upstairs if she takes 10 seconds to walk up one floor and it is 3 meters between floors? Neglect all frictional losses. _______ watts.

✶ **18 Jogging** Suppose that jogging on flat ground at 3 meters per second requires about 100 watts of power from the legs. (This is a rough estimate.) All that energy goes into friction, since no potential energy is gained and kinetic energy is essentially constant. (It is, if we neglect the "bouncing" up and down.)
(a) What is the average forward component of the force of the ground on the runner?
(b) What is the average upward component of the force of the ground on the runner?
(c) Why does the upward component do no work?

✶ **19 Jogging Uphill** A 75-kg jogger runs up a 1-in-12 grade. That means the roadway makes an angle with the horizontal whose sine is $\frac{1}{12}$, or $\sin\theta = 0.0833$. His speed is 3 m/sec. Assume that 100 watts of the power delivered by his legs goes into friction.
(a) How much goes into raising his center of gravity?
(b) How much power do his legs put out?
(c) How much altitude does he gain in 10 seconds? Compare with Problem 17.

✶ **20. Accelerated Running** How much average power does this runner require to accelerate from zero to 6 m/sec in 2 seconds? Assume that 100 watts are required on the average to overcome friction, and he is on flat ground. The horizontal force on him is partly the force needed to overcome friction, partly that needed to accelerate him. Do the bookkeeping on these forces.

At the end of the 2 seconds, how much energy has gone into kinetic energy? How much into friction?

✶ **21. Joules, Watts, and Calories** When you climb a staircase or a mountain, not all the chemical energy you "burn" in your muscles goes into potential energy, that is, into raising your weight. Some goes into heating your body and the ground under your feet. In solving this problem, neglect this "lost" energy. in other words, approximate the conversion of food energy into useful work as 100% efficient.

(a) How many joules are required to raise 75 kg (= 165 pounds) up 100 feet (= 30.5 m)?
(b) Convert that to calories (1 calorie = 4.18 joules) and remember that the Calorie of the nutritionist is one kilocalorie (= 10^3 lowercase calories).
(c) A lump of sugar gives up about 18 Calories when metabolized. How high does that amount of energy take a 165-pound climber?

NOTE: The efficiency of conversion is actually well below 50%.

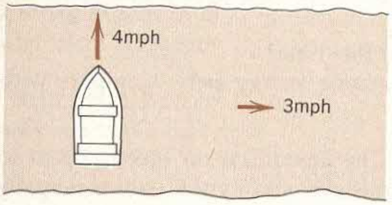
✶✶ **22. Waterfall Warms River** A waterfall is 10 meters high.
(a) What is the kinetic energy of 1 liter (= 1 kg) of water at the bottom of the waterfall? Neglect the energy of horizontal motion.
(b) One calorie is defined as the heat energy needed to raise the temperature of 1 gram of water by 1 degree Celsius. How much does the 10-meter fall raise the temperature of the water?
(c) If 10 tons (= 10^4 liters) of water go over the falls each second, how much power is wasted? ________ watts.

NOTE: The 10-meter drop could have been harnessed to drive a generator. That is the meaning of the word "wasted."

✶✶ **23. Airplane Power Requirement** A certain light airplane has a mass of 500 kg including the pilot (total weight = 4900 newtons). At its cruising speed of 50 m/sec (= 112 mph) its lift-over-drag ratio is 10.
(a) How much power does it require for level flight at a constant 50 m/sec?
(b) How much power does it require to increase its altitude at the rate of 10 m/sec while flying at 50 m/sec, that is, to fly uphill at a 1-in-5 grade? (HINT: Don't forget the drag. Assume L/D unchanged.)

✶✶✶ **24. Power at Constant Acceleration** At uniform acceleration, it takes just as long to accelerate (i) from 9 m/sec to 11 m/sec as (ii) from 19 m/sec to 21 m/sec.
(a) Find the average power output needed to accelerate a mass of 1000 kg at an acceleration of 1 m/sec^2 during stages (i) and (ii).
(b) What can you say about how the power requirement for constant acceleration depends on speed? (HINT: How do the kinetic-energy increases compare?)
(c) [More difficult] Write down a formula for power as a function of mass m, acceleration a, and speed v.

✶✶ **25. Components of a Vector** The speed of the boat in still water is 4 miles/hour. It heads its bow straight across the river, which is 1 mile wide (see figure).
(a) Without knowing the speed of the river, calculate how long the crossing will take.
(b) Without knowing the speed of the river, can you say in what direction the boat actually travels? Yes or No?
(c) The river flows east at 3 miles/hour. Argue that the speed of the boat is 5 miles/hour. Sketch!
(d) The eastward projection of that 5 miles/hour velocity vector is 3 miles/hour. Someone who sees the boat in silhouette from the front (i.e., from the north) sees only that it moves to the left at 3 miles/hour. Let θ be the angle made by the velocity vector with the shore (with an eastward vector). Show that $\cos\theta = \frac{3}{5}$. (Trig? See Appendix B.)
(e) Someone who sees the boat in silhouette from the side (from the east) sees only that it moves to the right at 4 miles/hour. What angle does the velocity vector make with a northward vector? Answer in the form $\cos\theta =$ ________.

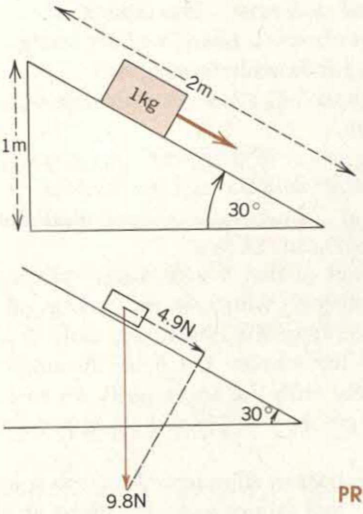


PROBLEM 25.

✶✶ **26. Addition of Velocities** The boat (still-water speed 4 miles/hour) heads down the river (speed 3 miles/hour).
(a) How long does it take to go 1 mile?
(b) How long does it take to go 1 mile down and back?
(c) Compare with the time to go 1 mile across the river and back.

NOTE: A similar problem will be assigned in Chapter 20, in connection with the Michelson–Morley experiment (Problem 20.22).

✶✶ **27. Inclined Plane** A 1-kg block slides down a frictionless plane inclined at 30° to the horizontal. This means that in sliding down 2 m along the plane, it loses 1 m of altitude (see figure).
(a) According to energy bookkeeping, it loses 9.8 joules of potential energy. Therefore, its kinetic energy at the bottom must be ________ joules.
(b) Its weight is 9.8 newtons. The component of this force along the plane is 9.8 newtons × sin 30° = 4.9 newtons.



PROBLEM 27.

According to Newton's Second Law, its acceleration is ______ m/sec^2.

(c) With that acceleration, how long does it take to go the 2 m?

(d) What is its speed at this time?

(e) Now calculate its kinetic energy $\frac{1}{2}mv^2$. Compare with part (a).

★★ **28. Throwing** The limitations on speed appear to be imposed by strength (force) and by skill, not essentially by the lack of available energy.

(a) Why can't you throw a bowling ball as fast as a baseball?

(b) Why can't you throw a Ping-Pong ball as far as a baseball?

★★ **29. Maximize the Range** For an initial velocity with horizontal and vertical components (v_{x0}, v_{z0}), the ball reaches the top of its trajectory at time $t_{\text{top}} = v_{z0}/g$. You have already gone through that algebra. In that time it covers horizontal distance $x = v_{x0}t_{\text{top}} = v_{x0}v_{z0}/g$. The range—the distance the ball has gone by the time it gets back to its original height—is just twice this distance:

$$\text{range} = \frac{2v_{x0}v_{z0}}{g}$$

Suppose that we are given v_o and asked to find the launching single θ that maximizes the range. We can write $v_{x0} = v_0 \cos\theta$ and $v_{z0} = v_0 \sin\theta$ so that

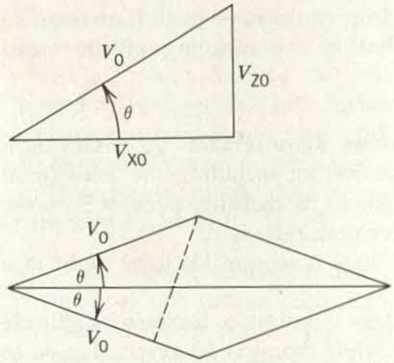
$$\text{range} = \left(\frac{2v_0^2}{g}\right) \sin\theta \cos\theta$$

Now use your calculator—just trial and error—to find the θ that maximizes sinθ cosθ. It is easiest if you make a table. (HINT: The maximum value is 0.5 so that the maximum range is v_0^2/g. If your calculator does not have sin and cos functions, here is another way. Let $s = v_{0z}/v_0$. Then, from Pythagoras, you can write $v_0^2 = v_{0x}^2 + v_{0z}^2$, so $V_{0x} = V_0 \times (1 - s^2)^{1/2}$. This gives range $= (2\,v_0^2/g)\,s\,(1 - s^2)^{1/2}$. Now you can use your calculator by trial and error to maximize that function of s. No fair calculus—but it does save time!)

★★ **30 Range as a Function of Launching Angle** This is not a problem in trigonometry, but it does use the sine function. Using the definition of the sine of an angle, convince yourself that $\sin\theta = v_{z0}/v_0$. The task here is to rewrite the formula

$$\text{range} = \frac{2v_{x0}v_{z0}}{g}$$

in terms of the initial speed v_0 and the initial angle θ. We use a geometric trick. The area of the right triangle with sides v_{x0} and v_{z0} is one-half the product of the sides (see the first figure).



PROBLEM 30.

Four such triangles will make a rhombus (an equilateral parallelogram (see the second figure). The two labeled sides make an angle of 2θ, so that the area of the rhombus is base × altitude $= v_0 \times v_0 \sin 2\theta = v_0^2 \sin 2\theta$. This is where we need trigonometry, to write the altitude (the dotted line) as $v_0 \sin 2\theta$, using only the definition of the sine of the angle as the ratio of the side opposite the angle 2θ (that dotted line) to the hypotenuse (v_0).

Now our task is accomplished: In the range formula we substitute the area of the rhombus ($= v_0^2 \sin 2\theta$) for 4 times the area of the triangle ($= 2v_{x0}v_{z0}$) to obtain

$$\text{range} = \frac{v_0^2 \sin 2\theta}{g}$$

We should check to see that it agrees with the maximum range v_0^2/g of Sec. 4.8. The largest value taken on by the sine function is 1; that is the sine of 90°. (Verify by referring to the definition in the box.) Accordingly, the range should be maximum for

$$\begin{aligned} \sin 2\theta &= 1 \\ 2\theta &= 90^\circ \\ \theta &= 45^\circ \end{aligned}$$

which was our previous result.

(a) Prove that the range is the same at 46° as at 44° launching angle.
What other launching angle gives the same range as 30°?

✶✶✶ **31 Throwing a Ball Off a Cliff** To throw the ball as far as you can on to a flat plain, do you still try for a 45° launching angle? Present argument in words. Then find how far the ball goes when thrown off a 50-meter cliff at 25 m/sec for launching angles of 40° and 45°.

☐ QUESTIONS FOR FURTHER THOUGHT

α. **Flea Versus Man** When a flea jumps, some of the energy it has on take-off is lost to the air because of wind resistance. For man's jump from rest, even though the take-off speed is comparable, the frictional loss to the air is negligible. Why? Another way to argue that is to point out that the terminal velocity for the flea is much lower than for man.

Argue that this gives man an advantage in the who-can-jump-higher competition. Do you think this outweighs the flea's advantage in strength-to-weight ratio? Do you think the fraction of the body weight devoted to leg muscle is in fact comparable? What are other advantages of flea versus man in this competition?

I measured the height h of my jump by comparing how high my hand can reach standing on tip-toe with my reach jumping. How would you make measurements on fleas? Who will win?

β. **Why Stall on Landing** Why does the pilot try to have the airplane at stalling speed just an instant before the wheels touch the ground?

γ. **Muscle-Powered Flight** Suppose that a wing design achieved a lift-over-drag ratio of 50:1, with a stalling speed of 10 m/sec. Comment on the practicality of using such a wing for human-powered flight. How much would craft-plus-pilot weigh? What minimum power output would be required for level flight? Can big wings be made strong enough and yet light enough? How fast would you be able to climb? How about wind? Does it represent a hazard or a possible source of power?

Satellite photograph showing clouds blown by counterclockwise wind patterns.

5

ROTATION

SOME QUESTIONS THIS CHAPTER WILL ANSWER

When the spinning skater pulls her arms in, why does she then spin faster?

Why does the stone whirling on the end of a string move twice as fast if the string is pulled in to half its original length?

Explain why whirling the stone twice as fast at constant string length quadruples the string tension.

Explain why the Gulf Stream flows counterclockwise.

Explain why we have two tides per day.

5.1 ANGULAR MOMENTUM

Paddling in the stern of the canoe, you happen to turn the craft sharply to the right in order to avoid an obstacle. The canoe rotates clockwise. But unless you make another stroke to stop the rotation, it just keeps turning. That is an example of conservation of angular momentum.

Here is another one, from ice skating: With your hands far from your body, go into a slow spin. When you pull your hands in close to your body, you automatically spin faster. Ice is wonderful for demonstrating the laws of mechanics! What is not so wonderful is when your car skids in rounding a curve. That is also an example of conservation of angular momentum.

A stone being twirled on the end of a string is an example that allows some easy quantitative experiments (see Fig. 5.1). When the string is shortened, the stone moves faster. The law is simple if we idealize the situation:

1. Neglect gravity and friction, that is, consider the stone to be moving on a frictionless plane.
2. Neglect the mass of the string and consider the stone as a "point mass."

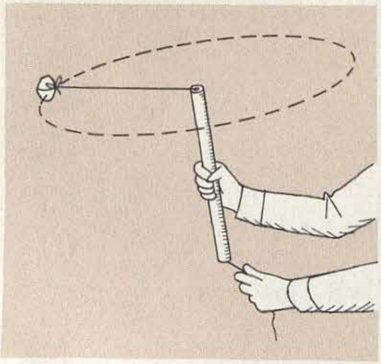


FIGURE 5.1 A stone being twirled on the end of a string. When the string is shortened—by pulling it through the tube—the stone spins faster.

For slow changes in string length, the law is an inverse proportionality: When the string length is halved, the speed of the stone is doubled. The speed v is inversely proportional to the radius r of its path:

$$rv = \text{constant}$$

If the change in string length is not slow, the law has to be formulated more carefully. The velocity vector v then has a component parallel to the string. The law has to be written

$$rv_{\perp} = \text{constant}$$

The component of the velocity perpendicular to the string is inversely proportional to the string length.

Multiplying this expression by the mass m of the stone gives us the **angular momentum** of the stone:

DEFINITION $mrv_\perp$ = angular momentum

Notice that this quantity does not have the same dimensions as momentum. Momentum is measured in kg m/sec. Angular momentum is kg m^2/sec.

You may wonder, under what conditions is angular momentum conserved? For the stone, the answer is that the angular momentum is unchanged if the only forces on it are parallel to the string.

Some insight can be gained by writing the angular momentum in terms of the rate of rotation. Suppose that you are twirling the stone at the rate of 2 revolutions per second. That 2/sec is the rotation frequency ν, and is also written 2 Hertz (abbreviation 2 Hz). The stone takes $\frac{1}{2}$ a second to make its circle once. That $\frac{1}{2}$ second is called the *period* T of the rotation.

DEFINITION $T = 1/\nu$ = period.

The speed of the stone is

$$v = \frac{2\pi r}{T}$$

the circumference of its circular path divided by the time it takes to go around once. If we want to make the formula look pretty, we write it in the form

$$v = \omega r$$

The symbol ω (= omega) is called the **angular velocity** and is defined as

DEFINITION $\omega = 2\pi/T$

$= 2\pi\nu$ = angular velocity

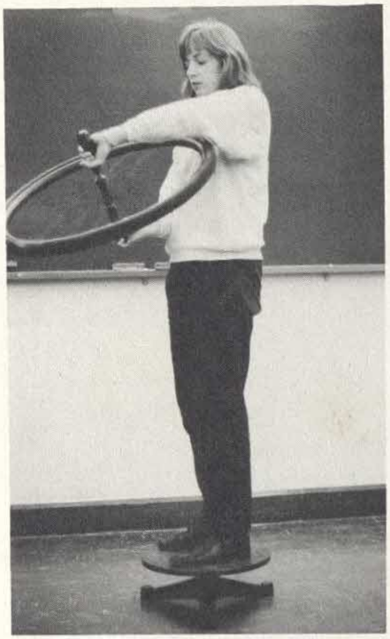
It is measured in radians per second. You may recall that the **radian** is a natural measure of angle, since there are 2π radians in one revolution (= circumference/radius).

Using our new notation, we write the angular momentum as

angular momentum $= mrv = mr^2\omega$

When you see the r^2 in this formula, your experiments with the stone on the string become more vivid. If the string is pulled in so its length r is halved, the speed v of the stone is doubled, to be sure. But its angular velocity is quadrupled. The rotation rate goes as the inverse square of the string length for angular momentum to be constant.

The tetherball shows the law beautifully. As the rope winds around the pole, pulling the ball in closer, the ball moves faster and faster. A good classroom demonstration uses a little merry-go-round or bar stool mounted on good bearings so it rotates freely. Take a couple of weights in your hands and start spinning. Then pull your hands close to your body. The spin is



much faster, just as with the skater. You can really get dizzy. When you push your hands out again, the spin slows to its original rate.

A bicycle wheel mounted on a handle makes a dramatic demonstration of angular-momentum conservation. Stand or sit on your rotating platform holding the handle vertically. With your other hand, start the wheel spinning in a horizontal plane. As you spin the wheel clockwise, you find yourself turning counterclockwise. If you stop the rotation of the wheel, your own rotation stops too.

The bicycle wheel is handy for quantitative experiments too. Its angular momentum is easy to calculate from its angular velocity. Because most of the mass is in the rim and tire, essentially at the same distance r from the hub, the angular momentum is very close to $mr^2\omega$. To be more precise, the m here is the mass of rim and tire, and the r is the "effective" radius. The mass in the hub contributes very little to the angular momentum because it is so close to the center of rotation.

Moment of Inertia

To calculate the angular momentum of a rotating object with the mass distributed at various distances from the center is a bit more complicated. Multiply each mass point m_i by its own effective r_i^2 and add up all those $m_ir_i^2$ products. The resulting sum is called the **moment of inertia.** You might want to think in terms of atoms. The ith atom has mass m_i and is a distance r_i from the rotation axis.

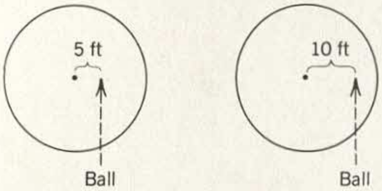
DEFINITION moment of inertia $= \sum_i m_ir_i^2$

Using that terminology, we can write

$$\text{angular momentum} = (\text{moment of inertia}) \times \omega$$

for any rigid object. A big wheel has a larger moment of inertia than a little wheel, even if their masses are the same.

Describing the spinning skater in those terms, we can say that pulling her arms in close to her body decreases the moment of inertia. Conservation of angular momentum implies that this increases the angular velocity.



(a) (b)

FIGURE 5.2 Catching a ball on a merry-go-round: (a) 5 feet from the center; (b) 10 feet from the center.

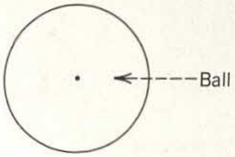


FIGURE 5.3 Catching a ball on a merry-go-round when the ball comes from a radial direction, that is, aimed toward the center of the merry-go-round.

EXERCISE 1 You are standing on a stationary merry-go-round, 5 feet from the center. Someone throws you a ball, as shown in Fig. 5.2*a*. If the bearings are frictionless, when you catch it you will start spinning counterclockwise. Now suppose that you were standing 10 feet from the center and caught the same ball thrown just as hard (Fig. 5.2*b*). Argue that your angular velocity would be twice as great. Assume that the mass of the merry-go-round is so much greater than your own mass that moving further out toward the edge did not change the moment of inertia of the whole system appreciably.

If the ball comes from the radial direction (Fig. 5.3), argue that catching it does not impart any angular velocity to the merry-go-round. It does not matter how far from the center you are standing. Evidently what matters is the component $v_\perp$ of the ball's velocity perpendicular to the line drawn between you and the rotation axis (center) of the merry-go-round.

5.2 CENTRIFUGAL FORCE, CENTRIPETAL FORCE

What force is it that keeps the string stretched when the stone is whirled around in a circle?

The obvious answer is that you are pulling on the string to keep the stone from flying off in a straight line. That happens to be the correct answer, too. It may not be completely satisfying. When you ride on a merry-go-round, you feel yourself being pulled toward the edge. The faster the rotation, the stronger the pull. Also, the further you are from the center, the stronger the pull. The force pulling you away from the center of rotation is just like the force that stretches the string when you whirl the stone. Isn't that what we call **centrifugal force?**

Certainly. But now we have to be very careful not to confuse the logic. If we talk about a force *away* from the center, a centri*fugal* force, that is equal and opposite to the force *toward* the center that keeps the stone and the merry-go-rounder from flying off. That force is called a centri*petal* force. An object going round in circles is changing its velocity vector at every instant. Even though its speed is constant, it is always being accelerated. Its acceleration vector is directed toward the center of the circle: centripetal acceleration. The **centripetal** force and the centripetal acceleration obey Newton's Second Law, $\mathbf{F} = m\mathbf{a}$. The force toward the center is proportional to the acceleration toward the center. But an observer riding along with that object would not observe its centripetal acceleration, its constant change of direction. Of course, the observer could observe the tension in the string. So he would ascribe that tension to centrifugal force on the object rather than to an acceleration he does not observe. In the language of Chapter 2, centrifugal force is a *fictitious force.* It is a force arising from making an observation in an accelerated reference frame. The fictitious centrifugal force is equal and opposite to the centripetal force, which is a real force.

How the centripetal force depends on the angular velocity and on the length of the string is derived in the box: For a given rotation rate, doubling the string length doubles the force. For given string length, doubling the rotation rate quadruples the force. Here is the formula: At angular velocity ω and distance r from the center of rotation,

$$\text{centripetal acceleration} = \omega^2 r$$

Accordingly, the centripetal force on a mass m is

$$\text{centripetal force} = m\omega^2 r.$$

The following exercises express this formula in different ways.

PROOF OF v^2/R FORMULA

CENTRIPETAL ACCELERATION How does it depend on speed and turning radius? [Answer: $a = v^2/R$]

Proof

We are talking about uniform motion at speed v around a circular path of radius R. Figure 5.4 shows a small arc of this left-turn path, corresponding to a travel time t, making the arc of length vt. At the beginning of this arc the velocity is in the x-direction; after time t, the path has moved sideways by a displacement y away from that initial x-axis. Since the speed is unchanged, the acceleration vector is in the y-direction, perpendicular to the velocity vector at the beginning. Writing a for this (centripetal) acceleration, the sideward displacement y during the time t can be written

$$y = \tfrac{1}{2}at^2$$

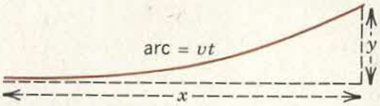


FIGURE 5.4 Motion in a circular arc at speed v. The x-direction is the original direction of motion at the left-hand end of the arc.

To write a formula for a in terms of v and R, we shall construct another expression for y involving the arc length vt and the radius R, and then equate the two expressions for y. The geometry is shown in Fig. 5.5. For the large right triangle, with R as hypotenuse, write (Pythagoras)

$$x^2 + (R - y)^2 = R^2$$

which becomes

$$x^2 + R^2 - 2Ry + y^2 = R^2$$

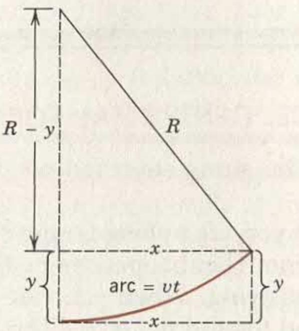


FIGURE 5.5 Proof of the v^2/R formula.

The R^2 terms cancel, leaving

$$2Ry = x^2 + y^2$$

If the time t is short enough, the arc vt can be approximated[1] by a straight line—the hypotenuse of a right triangle, for which

$$x^2 + y^2 = (vt)^2$$

holds. Eliminating the $x^2 + y^2$ between the last two equations gives

$$2Ry = (vt)^2 \quad \text{or} \quad y = \tfrac{1}{2}\frac{(vt)^2}{R}$$

This is the promised geometrical formula for y, known as the "sagitta formula."[2] Setting this equal to $\frac{1}{2}at^2$ gives t^2 on both sides of the equation:

$$\tfrac{1}{2}at^2 = \tfrac{1}{2}\frac{v^2t^2}{R}$$

or, finally,

$$a = \frac{v^2}{R}$$

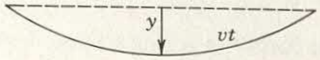


FIGURE 5.6

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[1] Passing to the limit of small times t makes this argument rigorous.

[2] If you complete the picture of the bow and arrow in Fig. 5.6, the displacement y is seen to be the length of the arrow—the **sagitta**.

EXERCISE 2 An object going around in circles of radius r at speed v covers one circumference $2\pi r$ in a time T, one period:

$$vT = 2\pi r$$

The angular velocity is $\omega = 2\pi/T = v/r$. Prove that centripetal acceleration

$\omega^2 r$ can also be written $\omega^2 r = \dfrac{v^2}{r} = \left(\dfrac{2\pi}{T}\right)^2 r$.

EXERCISE 3 If the merry-go-round doubles its angular velocity ω, but you move in toward the center so you halve the value of r, what happens to your centripetal acceleration? It is multiplied times ______. Note that your speed v stays the same.

EXERCISE 4 You are running around in circles. Suppose that your speed increases by 1% and the radius of the circle increases by 1%. What happens to your centripetal acceleration? It increases / decreases by ________%.

EXERCISE 5 Curves in the road are banked to keep the cars from skidding toward the outside. Use a sketch to prove that if the speed is so fast that the centripetal acceleration is equal to g, then the road should be banked at 45°. (Problem 20 is also about banking curves. See photo there.)

EXERCISE 6 A 1-kg stone is whirled around on the end of a 1-meter string at the rate of 1 revolution per second. Show that the string tension is 39.5 newtons. (Problem 19 shows that adding gravity does not change this result.)

5.7 SPECIAL TOPIC Why We Get Dizzy on a Merry-Go-Round

If you want to stand up on a merry-go-round or on a bus rounding a curve, it helps to hang on. There seems to be a force pulling you toward the edge of the merry-go-round, toward the outside of the curve on the bus. We call it centrifugal force. But if you start to walk toward the edge of the merry-go-round, you feel another kind of force: For a counterclockwise rotation (most merry-go-rounds?) you feel yourself veering toward the right. If you walk toward the center of the merry-go-round, again you feel a force toward the right. This force, having to do with your velocity *relative to the rotating platform* and directed at right angles to it, we call **Coriolis force.** (We will discuss Coriolis force in the next section.) Is it the centrifugal force that makes us dizzy or the Coriolis force? It turns out that both play a part.

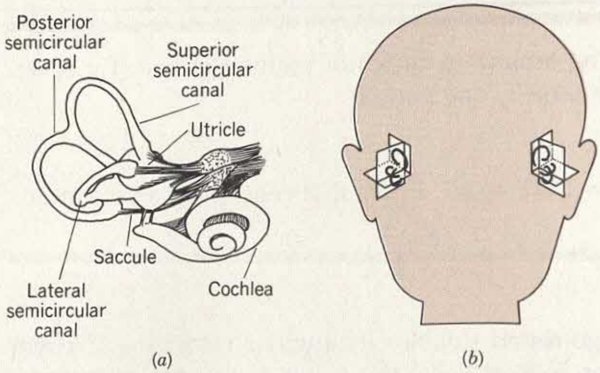
Posterior semicircular canal
Superior semicircular canal
Utricle
Saccule
Lateral semicircular canal
Cochlea
(a)
(b)

FIGURE 5.7 The balancing organ of the ear. Note the three semicircular canals.

Actually, some people feel dizzy just watching a movie taken on a rotating platform. Our eyes have a lot to do with vertigo. But if you close your eyes while you are being spun around, you can still get that queasy feeling. It is not so bad if you sit or stand quite still; that points the finger at the Coriolis force. Here are the anatomical clues: The balancing organ of the ear consists of a **labyrinth** or **vestibular apparatus** containing a *utricle* and *saccule* and three *semicircular canals,* in each ear (see Fig. 5.7). The utricle and saccule have some pebbles **(otoliths)** whose weight presses on the surrounding sensory cells to give a continuous signal about "which way is down." The three **semicircular canals,** which lie in approximately mutually perpendicular planes, are partially filled with a viscous fluid *(endolymph),* and each contains a pendular organ called a *cupula*. If the direction of the g-force changes, for example, if you tilt your head, there is fluid flow in the canals. The flow displaces the cupula; sensory cells, in turn, respond to the direction and amplitude of the displacement. Apparently, the utricle and saccule give information about slow changes in the "down" direction; the cupula and surrounding fluid respond to fast changes.

If your head is accelerated, that fluid starts to flow. For example, being accelerated to the left makes fluid flow toward the right, just as if you had tilted your head toward the right. That is what is happening when the bus is turning left: There is an acceleration toward the left (toward the center of the turn), and that is why you have to hang on. It requires a force to accelerate your body toward the left **($\mathbf{F} = m\mathbf{a}$).** We call that acceleration *centripetal acceleration* ("pulling toward the center"), and the force a centripetal force.

In the box labeled v^2/R we worked out how the centripetal acceleration depends on the speed of the bus and the sharpness of the curve. But even without the quantitative argument, we can see what rotation does to the balancing organ. In that left turn, the right ear is going faster than the left ear and experiences more centripetal acceleration than the left. "Down" appears to be in different directions for the two ears. Confusing, indeed! If you are at the center of the merry-go-round, the left ear is accelerated toward the right, the right ear toward the left.

The remarkable thing is not that this kind of thing makes us dizzy, but

"CENTRIPETAL FORCE/CENTRIFUGAL FORCE."

Centripetal force is a real force, unlike the equal and opposite centrifugal force, which just feels like a force because your frame of reference is rotating.

Back in Chapter 2 we warned you not to expect laws of physics like the conservation of momentum to hold in accelerated reference frames. The bus is certainly one of those, and in order to talk in our customary language, including statements like "$F = ma$," we introduce fictitious forces like the centrifugal force to "cancel out" the centripetal force and keep us in one place—on the bus, that is.

that we quickly get used to it; the sensation of rotation wears off in about half a minute. The cupula is elastic and returns slowly to its equilibrium position. But angular acceleration, a change in the rate of rotation, again produces the sense of rotation. When the merry-go-round stops, you feel the world spinning the other way. Indeed, the fluid flows back, displacing the cupula in the opposite direction. And again, it takes about 30 seconds for that sensation to wear off as the cupula returns to equilibrium.

SUGGESTION If you have not already done so, read the box with the proof of the formula $a = v^2/R$ now.

Moving the Head

We have been considering the head motionless in the rotating frame of reference (merry-go-round or bus). That way, the balancing organ experiences only one of the two types of "fictitious forces" arising from the rotation of the reference frame, only centrifugal force. If you move your head, Coriolis force also contributes to modifying what you experience as the "down" direction.

Without a quantitative derivation, one can nevertheless get an intuitive grasp of the accelerations that we feel as Coriolis forces. Points on the merry-go-round closer to the edge are moving faster than points close to the axis. If you walk along a radius toward the edge, your tangential velocity (to the left, for counterclockwise rotation) due to the rotation has to increase; that is, you are given a tangential acceleration to the left by virtue of your radial velocity toward the edge. It is called a Coriolis acceleration. Evidently, it is proportional to the rate of rotation and also proportional to the radial walking speed. As you walk radially outward, the floor of the merry-go-round exerts a force on you toward the left. If the floor were slippery, you would slide toward the right. You experience the Coriolis force toward the right.

Note that the Coriolis acceleration is always opposite to the direction of the Coriolis force, just as the centripetal acceleration is always opposite to the centrifugal force. Both forces are what we have called fictitious forces. Both accelerations are due to being in an accelerated frame of reference.

A force proportional to a velocity is a concept we have not met so far in this book. If you turn the velocity around, you turn the force around. Indeed, if you walk toward the center of the merry-go-round, the force is the other way. Because you have also turned around, it is still a force to your right. If you walk tangentially, say counterclockwise or *with* the merry-go-round, that is equivalent to increasing your rate of rotation, and consequently the centripetal acceleration. Again, you feel yourself pushed to the right (toward the edge) as a result of your velocity relative to the merry-go-round. A marble started rolling on a merry-go-round can roll in a complete circle, constantly turning to the right.[3] And on this great merry-go-round, the earth, spinning around once every 24 hours, the winds and the ocean currents are driven in large circular patterns by Coriolis forces, turning to the right in the northern hemisphere (counterclockwise merry-go-round), to the left in the southern hemisphere (clockwise merry-go-round). Figure 5.8, drawn for the northern hemisphere, shows why turning right gives a counterclockwise rotation around a low-pressure region.

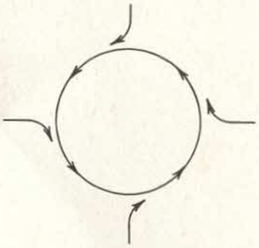

FIGURE 5.8 In the northern hemisphere, winds blowing into a low-pressure region are turned to the right by the Coriolis force. Viewed from the center, the air mass turns counterclockwise.

What about head motions? If you are facing away from the center of the merry-go-round and start turning your head to the left, your left ear moves toward the axis, your right ear away from the axis. Your left ear experiences Coriolis acceleration toward your right side; your right ear Coriolis acceleration toward your left side. Again, there is confusion about "which way is down." These Coriolis accelerations appear to be the chief cause of motion sickness. Many a mother instructs her car-sick child to "sit real still and it will go away." In the car, of course, curves are not the only cause of *g*-forces in directions other than the vertical. Speeding up and slowing down give sensations similar to going up and down hills, that is, tilting backward and forward. (See the sketch with Problem 1.61.) ■

■ 5.8 SPECIAL TOPIC: Rotating Space Stations

At the time of writing of this book, the problem of providing astronauts with a homelike environment for long stays in space has not been solved. How can one simulate gravity, a homelike weight of 9.8 newtons per kilogram of body mass, in a space station? The obvious solution, to have the craft accelerate forward at a steady 9.8 meters per second per second, would require a constant source of fuel, to give the craft ever-increasing kinetic energy. That's out. But a steady rotation, say of a cylindrical craft about its axis, would make it possible to walk on the inside surface of the cylinder. In Exercise 7 we calculate how much rotation it takes to give a centripetal acceleration (= v^2/R) equal to 9.8 m/sec^2, to give the astronaut his accustomed earthly weight. We now ask the question, does this astronaut get dizzy?

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[3] The film "Frames of Reference" has a dramatic demonstration of this phenomenon.

We might ask the question another way: Does he have any way of knowing that his craft is rotating, aside from looking out at the stars? Does this **artificial gravity** feel like real gravity? The answer is No. Not once he starts to walk around the cylinder! If he walks *with* the rotation, that increases his rate of rotation and hence his centripetal acceleration. The floor has to provide the extra "upward" centripetal force, so he feels heavier. If he walks the other way, he gets lighter. In fact, if he walks (runs?) as fast as the cylindrical floor is turning, he has no weight left at all. A somewhat unusual environment after all! Note that walking in a direction parallel to the axis of rotation does not change his weight. But if he tries stooping down, he will tend to fall over. On getting up, he will feel pushed the other way.

These forces, arising from motion relative to his rotating reference frame, are Coriolis forces. They are proportional to the rate of rotation of the cylinder and also to the velocity of the relative motion. As we saw, they depend on the direction of that relative velocity. The Coriolis force is always perpendicular to that direction; any component of the relative velocity parallel to the axis of rotation does not contribute.

What are the design considerations for rotating space stations to simulate gravity? Remember that the effective g is proportional to the square of the rate of rotation. For a given rotation frequency, it is also proportional to the radius of the craft (= $\omega^2 R$) The Coriolis forces are directly proportional to rotation frequency. If we want to minimize these forces for our astronauts, we should make the cylinder large. A budget-conscious designer may just tell them to move slowly. ■

EXERCISE 7 A small, round space station has a radius of 10 meters. What angular frequency ω would give an "effective g" of 9.8 m/sec^2 at the outside edge? How many sunsets would the astronaut see per minute?

■ 5.9 SPECIAL TOPIC: Are We Lighter at Noon than at Midnight?

It sounds reasonable. At noon, the sun is above our heads. It is a massive body, so it exerts a gravitational pull on us. At noon, that pull is up, pulling me up off my bathroom scale. Wouldn't that decrease my weight? At midnight, the sun is on the other side of the earth; its gravity pulls me down, into the bathroom scale I am using to measure my weight. Now you may point out that the sun is awfully far away, so this might be a very small effect. But are we sure we have the *sign* of the effect right? Can we conclude that the scale reads less at noon than at midnight?

No, things are not that simple. Remember, the sun pulls on the earth too. That pull is also up at noon, down at midnight. We know that the earth responds to this gravitational pull by accelerating toward the sun. After all, that is what keeps the earth in orbit. I know that the earth on which my bathroom scale stands is accelerating upward at noon. So the earth is pushing upward on the bathroom scale, pushing it up toward me. That would tend to *increase* the scale reading. The effect of the earth's acceleration toward the sun is to make me heavier at noon, lighter at midnight. So now we have to ask, which dominates? Which affects my apparent weight more at noon, the sun pulling up on me, making me lighter, or the sun pulling up on the earth, making me heavier? For the same question at midnight, just reverse all the signs.

To try to answer the question, we remember $F = ma$, and restate the question in terms of accelerations: Which acceleration would be bigger, my acceleration toward the sun because of its gravitational pull on me, or the earth's acceleration toward the sun because of the sun's pull on it?

You can easily convince yourself that what counts here is which is closer to the sun. You don't have to know about the inverse-square falloff of the gravitational force. All you have to know is that it does fall off with distance and is proportional to mass. It also helps to know about center of mass. When we talked about the center of mass back in Chapter 2, we pointed out that it is equivalent to the center of gravity. At noon in June, with the sun overhead, my center of mass is indeed closer to the sun than the earth's center of mass. Figure 5.9 is a cartoon, of course; it is not to scale. The center of the earth is about 6400 kilometers further from the sun than I am. So at noon, the sun's pull on me "talks harder" than the sun's pull on the earth in determining the reading of my bathroom scale. I am lighter at noon than I would be if the sun were not overhead.

Now think about midnight. This time the center of the earth is closer to the sun than I am. The sun's pull on the earth per kilogram of earth is greater than the sun's pull on me per kilogram of me. The earth's falling into the sun (down) talks harder than my falling into the sun. So again, with the bathroom scale falling away from me, I am lighter than I would be if the sun were not there. Instead of saying "pull per kilogram" I could, of course, use the term "gravitational field."

If all this seems complicated, remember that we are performing a weighing in an accelerated reference frame. The earth is accelerating toward the sun in its (almost circular) orbital motion. The magnitude of that acceleration is the same at all hours of the day or night. The sun's pull on me is not, because I am closer to the sun at noon than at midnight. The net effect of the sun is to make me lighter both when it is above me and when it is below me.

Now look what happens when the sun is neither above nor below, but off to the side, say at 6 A.M. and 6 P.M. Again, to make the picture simple, put me at the equator in March or September. At 6 o'clock, the pull of the sun on the earth and the pull of the sun on me are almost parallel, but not

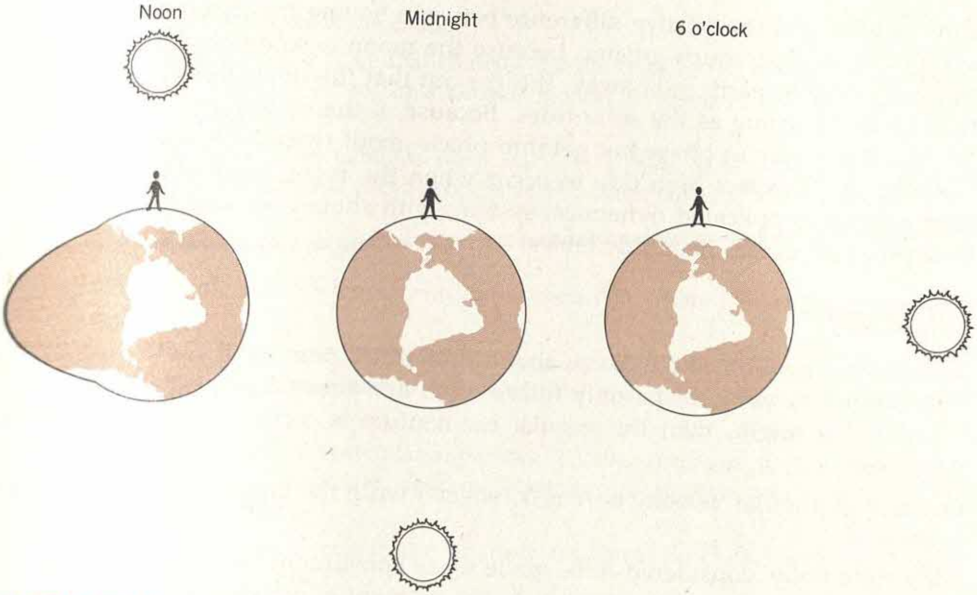

FIGURE 5.9 "Are we lighter at noon than at midnight?" At noon, the sun is above our heads. At midnight, the sun is on the other side of the earth. At 6 o'clock, the sun is off to the side.

quite. Looking at Fig. 5.9, you see that the sun's pull on the earth is straight to the right, while the pull on me is to the right and a little bit down. So the effect of the sun is to make me heavier than I would be if it were not there. This time, the effect is not the result of distances being a little different but of directions being a little different. In looking back over the argument, note that we have used the word "down" in a highly egocentric way. It is down relative to me and my bathroom scale.

What may now be clear to you is that my weight reading is periodic and has a 12-hour period. That implies a symmetry between the situations at noon and a midnight, which may worry you. But remember that the earth's radius is a tiny fraction of the earth–sun distance. An inverse-square law looks very linear over that small a range.

Tides

Why have we worried about such a small effect? A difference of 6.4×10^3 km out of 1.5×10^8 km is not going to show up on any bathroom scale. The answer is that this story is really about the **tides.** Have you ever wondered why there are two high tides and two low tides per day? Now you know! It is the small variation in the weight of the water. But you have probably read that the tides are due to the moon. Isn't the gravitational field of the moon here on earth much smaller than the gravitational field of the sun? Indeed. But the relative difference between having the moon above us and below us is so much greater, because the moon is so much closer than the sun—only 60 earth radii away. It turns out that the **lunar tides** are more than twice as strong as the **solar tides.** Because of the motion of the moon, the two don't stay in phase but get into phase about twice a month.

Please don't expect high tide to occur when the moon is overhead. The oceans are a complicated dynamical system, with shore lines and variations in depth. Navigators need tide tables! ■

SUMMARY

Angular momentum of a mass m about a reference point (call it the origin) is defined as $mrv_{\perp}$. If the only forces on m are directed toward (or away from) that origin, then the angular momentum is conserved during the motion.

In terms of angular velocity $\omega = v_{\perp}/r$, we can write the angular momentum as $mr^2\omega$.

For a rigid body, considered to be made up of constituent masses m_i located a distance r_i from some rotation axis, the moment of inertia about that axis is defined as the sum $I = \Sigma m_i r_i^2$. The angular momentum about that axis is simply $I\omega$.

For uniform motion in a circle, the centripetal acceleration is given by $v^2/r = \omega^2 r = (2\pi/T)^2 r$.

That acceleration gives rise to the fictitious force we call centrifugal force when we describe motion in a rotating reference frame. If an object is also moving relative to that reference frame, there is another fictitious force perpendicular to its direction of motion: the Coriolis force.

The difference between the gravitational acceleration due to the sun (and moon) of the whole earth and of objects at its surface (e.g., the sea) gives rise to two tidal cycles per day.

KEY TERMS

Angular momentum of a rigid body about a specified axis of rotation is the sum of the angular momenta $m_i r_i^2 \omega$ of the mass points making up the body. Here r_i is the distance of the mass m_i from the rotation axis.

Angular velocity denoted $\omega = v_{\perp}/r$. Measured in radians per second.

Radian The natural measure of angle, defined as the ratio of arc length to radius. One revolution (= 360°) is therefore 2π radians, making 1 radian = 57.3°.

Angular momentum of a rigid body about a specified axis of rotation is the sum of the angular momenta $m_i r_i^2 \omega$ of the mass points making up the body. Here r_i is the distance of the mass m_i from the rotation axis.

Moment of inertia Denoted I, and defined about a specified axis of rotation, as the sum over all the mass points m_i of the quantity $m_i r_i^2$.

Centripetal Toward the center.

Centrifugal Away from the center.

Centripetal force The force that changes the direction of the velocity when an object goes around a curve. In a circular motion at speed v, this is a force mv^2/r toward the center of the circle.

Centrifugal force The fictitious force $m\omega^2 r$ on a mass m measured in a rotating frame of reference that takes account of the acceleration it has by virtue of the rotation only.

Sagitta Latin word for arrow.

Sagitta formula Arrow length = $(\frac{1}{8})(\text{bow length})^2/R$, where R = radius of curvature of the bow.

Coriolis force The fictitious force on a mass measured in a rotating frame of reference which is proportional to the velocity of that mass relative to the rotating frame.

Labyrinth Also called vestibular apparatus. Tubular structure in each ear, containing a utricle and a saccule, used for obtaining information about the tilt of the head. See Fig. 5.7.

Otoliths Pebblelike objects in the labyrinth that press on the sensory cells to indicate tilt direction. See Fig 5.7.

Vestibular apparatus Also called labyrinth.

Semicircular canals Three tubular structures in each ear. The flow of a viscous fluid in these canals caused by tilting the head gives information about fast changes in the inclination of the head.

Torque Makes an object tend to start turning or, more generally, change its angular velocity. Definition: Torque is the moment of a force or set of forces about a specified point or axis. To calculate the moment of a force about an axis, multiply the magnitude of the force times the perpendicular distance of the line of action of that force from the axis. That perpendicular distance is called the moment arm.

Moment arm See Torque.

Orbit The path of a planet around the sun. More generally, the path of a particle in a force field. The planets and comets have elliptical orbits with the sun at one focus.

Escape velocity The fastest speed a satellite can have and still be in a closed orbit. Any faster speed causes it to escape from the gravitational field, never to return. At the earth's surface, the escape velocity is 11.2 km/second.

Artificial gravity Fictitious "downward" force produced by rotation. The term is found more in science-fiction accounts of life in a space station than in instruction books for laboratory centrifuges.

Tides Cyclical motions of parts of the earth's surface (water or earth) having roughly a 12-hour period.

Solar tides Tides caused by the gravitational pull of the sun.

Lunar tides Tides caused bythe gravitational pull of the moon.

Spring tide Tide of large amplitude caused by solar and lunar tides being in phase. (Not mentioned in text.)

Neap tide Tide of small amplitude caused by solar and lunar tides being out of phase. (Not mentioned in text.)

REVIEW QUESTIONS

- Argue that the MKS unit of angular momentum is kg m^2 sec^{-1}.
- Often it is enough to write mrv to define angular momentum. When do you have to be careful and write $mrv_\perp$?
- Explain what is meant by a moment of inertia.
- Argue that a particle moving in a circular path at constant speed is being accelerated toward the center of the circle.
- Argue that the centripetal acceleration v^2/r can also be written $\omega^2 r$.
- You are hanging on to a vertical rod on the merry-go-round. If the angular velocity increases by 1%, by how many percent does the tension in your arm increase?
- In a cylindrical space station 100 meters in radius, what rotation frequency will make the astronauts feel at home? Home means $g = 9.8$ m/sec^2.
- Argue that, if they stand up too fast, they will tend to fall over. Those sideward forces are called C ____________________ forces.
- Argue that, when the sun is overhead, you are lighter than you would be if the sun were on the horizon. Then argue that when the sun is on the other side of the earth, you are also lighter than you would be if the sun were on the horizon.

PROBLEMS

✶✶ **1 Angular Momentum** When the spinning skater pulls in her arms, her moment of inertia is decreased by 20%. Argue that her angular velocity increases by 25%.

✶ **2 Scaling** If the angular velocity of an object is increased by 1%, its angular momentum is increased by ________%. The assumption here is that the moment of ____________________ is constant.

QUICK QUESTION: How can the moment of inertia of an object be increased while keeping its mass constant?

✶ **3 Scaling** A rigid object is built to be the same shape as the model, with all the linear dimensions multiplied times 10. This means that the mass of each component is multiplied times 10^3. The moment of inertia is multiplied times ________.

✶ **4. Volcanoes Change the Length of the Day** The eruption of Maunaloa volcano moved some of the mass of the earth (some lava) to higher ground, that is, farther from the earth's center. According to the Law of Conservation of Angular Momentum, this increased / decreased the length of the day by a small amount just because it increased/decreased the earth's moment of inertia.

✶✶ **5. Canoe** The paddler is in the stern (the back) of the canoe, which is moving straight ahead. Someone off to his right (excuse me, off to the starboard) throws him a ball. After he catches it, the canoe will rotate clockwise / counterclockwise. Doubling the speed of the ball would multiply the rotation rate of the canoe times ________. Doubling the mass of the ball without changing its speed would multiply the rotation rate of the canoe times ________. Assume that the mass of the ball is much less than the mass of the canoe. Why was this assumption necessary?

✶ **6. Angular Momentum of the Earth in Its Orbit**

Data: Earth–sun distance = 1.5×10^{11} m = 1 A.U.[4]
Mass of the earth = 6.0×10^{24} kg
Period = 1 year

Calculate its angular momentum in kg m^2 sec^{-1}.

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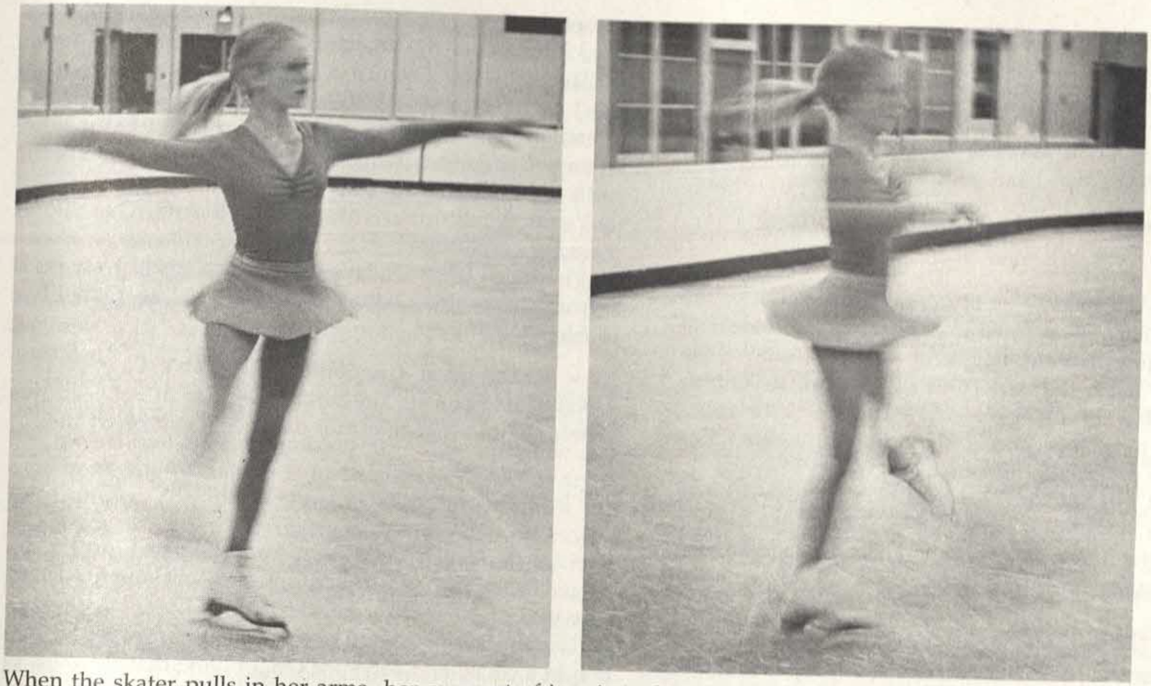
[4] A.U. = astronomical unit.



When the skater pulls in her arms, her moment of inertia is decreased. This causes an increase in her angular velocity, conserving her angular momentum.

Volcano erupting: Mount Hekla in Iceland.

★★ 7. Speed of an Earth Satellite The free-fall acceleration of objects near the earth's surface is what we have called g (= 9.8 m/sec^2). The further an object gets from the surface, the weaker the gravitational pull of the earth. The law that governs the fall-off of "g" is an inverse-square law:

$$g = \frac{\text{constant}}{r^2}$$

where r is the distance of the object from the center of the earth. The "constant" is equal to (9.8 m/sec^2) × (radius of the earth)^2. Note that the height of the object above the surface of the earth is r − (radius of the earth). That inverse-square law is called Newton's Law of Gravitation, and you will meet it in numerous places throughout this book (see, for example, Problem 13.46).

If we want our satellite to travel in a circular orbit of radius r, how fast does it have to move? Let's use the v^2/r formula for the acceleration of an object going around a circular path at constant speed v:

$$\frac{v^2}{r} = \frac{\text{constant}}{r^2}$$

(a) How fast would a satellite have to travel in order to stay in orbit just above the atmosphere, that is, $r = 6.4 \times 10^6$ m. Note that the radius of the earth is only about 100 km less than that.
(b) How long would it take to go around the earth once?

NOTE: Since there is so little air up there, there is virtually no friction to rob the object of its energy. A satellite can stay in orbit for years.

★★ 8. Kepler's Third Law In Problem 7 you found the relation between the speed v of a satellite and the radius r of its circular orbit: v^2/r = constant/r^2. That can be written more economically as

$$v^2 = \frac{\text{constant}}{r}$$

(a) Recalling that the period T is the time to go a whole circumference, we have $2\pi r = vT$. Use this equation to eliminate v above to obtain

$$T^2 = (\text{constant})r^3$$

The constant is different, of course. This relation, that the square of the period is proportional to the cube of the radius, is true for satellites of the sun as well. It was discovered by Kepler based on astronomical observations of the planets by Tycho Brahe—almost 100 years before Newton used it to deduce the inverse-square law for gravitation.
(b) The earth–sun distance is 1.5×10^{11} meters, and a year is 3.1×10^7 seconds (verify). Calculate the centripetal acceleration of the earth in its orbit. Now you know why, when you drop something, it falls toward the center of the earth rather than toward the center of the sun.
(c) Venus is only 1.08×10^{11} m from the sun. Calculate its period of revolution.
(d) The moon takes 28 days to circle the earth. How far is it from the center of the earth? (HINT: Use part (b) of Problem 7.)

★★ 9. Universal Gravitation Newton's Law of Universal Gravitation for the force of attraction between two spheres with center-to-center distance R can be written

$$F = \frac{Gm_1m_2}{R^2}$$

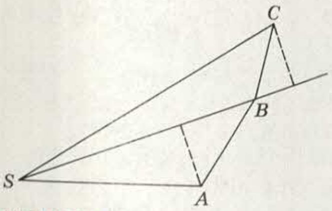
The sun is (R=) 1.5×10^{11} m from the earth, on the average. That distance is called 1 A.U. (Astronomical Unit). One year = 3.1×10^7 seconds. (Verify.)
(a) Use the law of motion, $F = ma$, with $a = v^2/R$ for motion in a circle, and solve for Gm_{sun}. Note that the mass of the earth cancels out.
(b) Venus takes 225 days to circle the sun versus 365 days for Earth. Calculate the sun–Venus distance in astronomical units. (HINT: You need neither the mass of the sun nor the universal gravitational constant G in order to solve this.)

★★★ 10. When You Slow Down a Satellite, It Speeds Up Since it takes work to lift a satellite up away from the earth, the potential energy increases with height. If it meets resistance to its orbital motion by gas molecules "lost in space," it gives up some of its kinetic energy to them, and starts to spiral in. But Problem 9 tells us that at a smaller r it has a larger v. This says that doing work on the gas results in a larger kinetic energy.
(a) Show that energy bookkeeping is not violated. Where does the extra energy come from?
(b) Show that a 1% decrease in r results in a 1% increase in kinetic energy.
(c) Here is a crucial hint: The potential-energy decrease is twice the kinetic-energy increase. Show that the amount of work done on the gas (i.e., the kinetic energy given to the gas) is equal to the kinetic energy gained by the satellite.

NOTE: More about this hint in Chapter 13, Electrostatics.

★★★ **11. Kepler's Second Law** This law states that in the motion of the planets around the sun, the line connecting the planet and the sun sweeps out equal areas in equal times. Kepler had found (Kepler's First Law) that the planetary orbits are ellipses with the sun at a focus. Equal areas in equal times means they move faster when they are close, slower when they are far away from the sun. Since the orbits are all so close to circular, this was truly a remarkable generalization and tells you that Kepler had good astronomical data to work with! Newton later proved the equal-areas law just from the assumption that the force was "central," that is, pulled the planet directly toward the sun. Here is the proof.

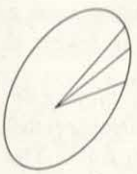
Think of the impulse imparted to the planet by the sun's pull as delivered in discrete bursts. Acutally, you know that momentum is being transferred continuously; the discrete-burst approximation is just a trick to make the argument simple. Consider a short time Δt during which the planet has velocity v_1. In that time, it travels a distance $v_1\,\Delta t$, shown as *AB* on the figure. At the end of that time interval, it gets one of its impulses toward the sun. During the next time interval Δt it travels at velocity v_2, covering the distance $v_2\Delta t$ shown as *BC* on the figure. What we are trying to prove is that the area of the triangle *ABS* is equal to the area of the triangle *BCS*.



PROBLEM 11. Proof of the "law of areas." The impulse toward the sun *S* is considered to be delivered at point *B*. The triangles *SAB* and *SBC* have equal areas.

To complete the proof, we need some ideas about vector addition. The velocity vector v_2 differs from the velocity vector v_1 by a vector directed along the "line of centers" *BS*. So the components of the two velocity vectors perpendicular to *BS* are equal. That says the two dotted lines drawn perpendicular to *BS* are equal, the one drawn from *A*, the other drawn from *C*. Now finish the proof.

Of course, the time intervals Δt can be as short as you like, so the discrete impulses can match as closely as desired the continuous transfer of momentum to the planet by the attraction of the sun. This process of "passing to the limit" of small Δt is probably familiar to you from mathematics courses. This mathematical technique was invented by Newton in connection with his work on the orbits of the planets and the Law of Gravitation.



PROBLEM 11. Kepler's Second Law. As the planet gets further from the sun, it slows down, so those triangles get longer and thinner. The triangles correspond to equal time intervals; the law says they have equal areas.

★★★ **12. Conservation of Angular Momentum from Kepler's Second Law** Use Newton's proof (see Problem 5.11) to prove the conservation of angular momentum for a ball on a string.

★★ **13. Satellite Meeting Resistance from Gas Molecules**

(a) Show that the angular momentum of the satellite spiraling in closer to the earth is decreasing. Notice that there is no violation of angular-momentum conservation here. The gas molecules carry off the satellite's angular momentum. Does it seem strange that the angular momentum is decreasing even though the angular velocity is increasing?
(b) Show that for an inverse-square law of gravitation, a 1% decrease in r results in a 0.5% increase in v.
(c) By how many percent does this make the angular momentum decrease?

★★ **14. Escape Velocity** The potential energy of an earth satellite is given by the formula

$$mgR\left(1 - \frac{R}{r}\right)$$

Here

r = distance from the center of the earth
R = radius of the earth = 6.4×10^6 m
g = 9.8 m/sec^2 = acceleration of gravity at $r = R$, meaning sea level
m = mass of the satellite

(a) Verify that $gR = (7.9 \times 10^3 \text{ m/sec})^2$.
(b) Show that the potential energy thus defined is zero at sea level. Remember (Chapter 4) that the zero of potential energy is arbitrary; that is, it is merely a convention.

(c) Show that the work done in lifting a satellite from $r = R$ to $r = 2R$ is equal to the work done in taking it from $2R$ to infinity.
(d) At what speed does a satellite move in a circular orbit at $r = R$, that is, just above the atmosphere? (HINT: The atmosphere is not even 100 km thick. You make less than a 2% error by setting $v^2/r = 9.8$ m/sec^2.)
(e) What speed does an object have to have at sea level in order to escape the earth's gravity altogether, that is, to get out to $r =$ infinity? (HINT: Do energy bookkeeping.)
(f) Suppose that you arbitrarily changed the potential-energy formula to read simply $-mgR^2/r$, omitting the "constant" term. Now the potential energy is zero at infinity instead of at sea level. Show that the answers to parts (c), (d), and (e) are unchanged.

NOTE: This is the more usual convention. It makes all potential energies negative for gravity—a comforting thought for an attractive force.

✶ **15. Speed of Earth Satellites** In Problem 14 you found that the speed for the smallest possible orbit is 7.9 km/sec (= 18 thousand miles per hour). In part (e) you found that an object given a speed of 11 km/sec at sea level eventually escapes from the gravitational pull of the earth:
(a) Argue that an object close to the earth having a speed less than 7.9 km/sec has to crash eventually.
(b) Argue that an object close to the earth having a speed greater than 11 km/sec cannot be in a stable orbit around the earth; that is, it either has to crash or escape. (HINT: Use energy bookkeeping.)
(c) Show that the escape velocity is $2^{1/2}$ times the velocity in the smallest orbit.
(d) What minimum kinetic energy (in joules) does each kilogram of satellite have to have to go into orbit? Express also in kilowatt hours.

NOTE: At the present cost of energy, that is less than 50 cents a pound. NASA charges more.

✶ **16. Angular Momentum in an Inelastic Collision** A playground merry-go-round is rotating at an angular velocity of 3.14 radians per second.
(a) Verify that its period is 2 seconds.
(b) Its mass (= 100 kg) may be considered as concentrated in a ring 1.5 m in radius. Show that its angular momentum is 7.1×10^2 kg m^2/sec.
(c) A 25-kg child jumps on with zero tangential velocity. Calculate the angular velocity after the impact. Assume that angular momentum is conserved and that the child has zero angular momentum before it lands. The child also may be assumed to land 1.5 m from the center.

✶✶ **17. Kinetic Energy of the Merry-Go-Round** Refer to Problem 16.
(a)Verify that the initial kinetic energy of the merry-go-round is 1.11×10^3 joules. (HINT: You can just add the kinetic energies of the individual parts of the merry-go-round. Note that they are all moving at (3.14 radians/sec $\times$ 1.5 m) = 4.7 m/sec.)
(b) Calculate the kinetic energy of merry-go-round plus child after the impact. Don't be surprised that kinetic energy was lost in an inelastic impact. You should be worried about energy bookkeeping: The lost kinetic energy goes into heating up the material in the vicinity of the impact. Remember how it used to smart after you jumped on?

✶✶ **18. More about the Mass on a String**
(a) Calculate the tension in a 2-meter string with a 1-kg mass whirling at 1 period per second. (That is an angular velocity of 2π radians per second.) Remember the centripetal acceleration v^2/r. You can neglect gravity at this speed, that is, assume the string is almost horizontal and set $F = ma$.
(b) Suppose that you shorten the string slowly. You can actually do this by pulling it through a smooth tube you hold in your hand. The angular velocity increases. Show that this is required by the conservation of angular momentum $mr^2\omega$.
(c) Show that a 1% decrease in r results in a 2% increase in ω.
(d) By how many percent has the speed v increased?
(e) By how many percent has the kinetic energy increased?
(f) Show that this kinetic energy increase is equal to the work done in pulling on the string; that is, energy bookkeeping works. (HINT: Has the string tension increased or decreased? To calculate the work done (= force $\times$ distance), you will have to take an average over the 2 cm by which the string was shortened.)

✶✶✶ **19. Stone Whirling on a String**
This time we do some vector addition to show that gravity does not matter.

You are going to prove that the tension (F) in the string is independent of the acceleration of gravity g. Here is how. The stone has weight mg. The string of length r makes an angle θ with the horizontal. So the stone moves in a horizontal circle of radius $r \cos\theta$. According to the v^2/R formula, its centripetal acceleration is $\omega^2 r \cos\theta$. This means that the horizontal component of the force on the stone is $m\omega^2 r \cos\theta$. The vertical component is mg.

(a) Use Pythagoras's theorem to prove that $F^2 = (mg)^2 + (m\omega^2 r \cos\theta)^2$.
(b) Draw a sketch and use it to show that $\sin\theta = mg/F$.
(c) Now do the algebra needed to prove that $F = m\omega^2 r$, independent of g.

✶✶ **20. Banked Curves—More Vector Addition** The angle of banking should be such that there is no tendency of the car to skid toward the outside of the curve. This means the net force of the road on the car should be normal to the road surface (see the figure). The vertical component of this force is mg, the weight of the car. The horizontal component of this force is the centripetal force mv^2/r, where r is the radius of the curve.
(a) Prove that the ratio of the horizontal to the vertical components of this force gives the tangent of the angle that force makes with the vertical.
(b) Prove that making $\tan\theta = v^2/gr$ satisfies the no-skid condition. Be sure to make a labeled sketch showing the various forces.

Cars going around a well-banked curve.

✶ **21. Banked Curves** What angle should the road surface make with the horizontal so that there is no tendency to skid on a curve with a 100-meter radius negotiated at 20 m/sec (= 44 mph)? Have you ever seen a highway banked that hard? Now you know why mountain roads have more conservative speed limits on the tight curves. (HINT: See Problem 20.)

✶✶ **22. Bicycle Wheel** The mass is mostly in the tire and rim, so it is easy to calculate the moment of inertia.
(a) Calculate it for a mass of 1.5 kg and a radius of 0.68 m.
(b) What is its angular momentum if it is spinning at 3 revolutions per second?
(c) A student prepares to demonstrate conservation of angular momentum by standing on a platform arranged to rotate freely on good bearings. Her mass may be taken as 70 kg; her moment of inertia about the axis of spin might be about 3 kg m^2. She is handed the spinning bicycle wheel, axle vertical, and holds it carefully by the bearings. Nothing happens. Then she touches the tire and stops its spinning. Immediately she and her platform start to rotate in the sense in which the wheel was spinning. Calculate her angular velocity. Tell what approximations you are making. (See photo in Sec. 5.1.)

✶✶ **23. Whirling on the Bar Stool** Actually, a rotating platform is better if you have one. You are holding 5 kilograms in each hand. When your arms are extended (say 0.8 m from your centerline), those weights contribute 6.4 kg m^2 to your moment of inertia.
(a) Verify.
(b) When your arms are at your sides, say 0.3 m from your centerline, show that the weights contribute only 0.9 kg m^2. So, in extending your arms, you are increasing your moment of inertia by more than 5.5 kg m^2.
(c) Suppose that the moment of inertia hands-at-your-sides is 3 kg m^2; hands-extended it is 8.5 kg m^2. Hands-at-your-sides, you are rotating at an angular velocity of 6.3 radians/sec. What will be your angular velocity when you bring those weights all the way out?
(d) Notice that it is easier to raise the weights when you are rotating than when you are at rest. Of course, the centrifugal force helps. Does that mean that the total kinetic energy is less with the weights far out than close in?
(e) If you pull your arms in again, you really have to pull. Argue that the work you do goes into kinetic energy of rotation.

NOTE: The important point here is that energy bookkeeping is not the same as angular-momentum bookkeeping. You can lose both energy and angular momentum due to friction in the bearings, of course.

✶✶ **24. Torque and Angular Acceleration** A certain flywheel has virtually all its mass m concentrated at its outer radius R. So its moment of inertia I is mR^2. A tangential force F is almost instantaneously transmitted to all parts of the wheel; that is, all parts of the wheel share equally in it, and it is everywhere tangential. If it is mounted on frictionless bearings, it will change its rate of rotation in accord with the law of motion. Each mass point will accelerate tangentially with acceleration $a = F/m$. So the angular velocity $\omega = v/R$ will show a time rate of change $\alpha = a/R$. If we substitute $a = F/m$, this becomes

$$\alpha = \frac{F}{mR}$$

The *torque* τ is defined as the product of the force times its moment arm: $\tau = FR$. In terms of the angular acceleration this can be written

$$\tau = mR^2\alpha = I\alpha$$

The formula $\tau = I\alpha$ looks a lot like $F = ma$ and may be called its rotational analogue. In the numerical work that follows, the important thing is to keep the units straight!

(a) Take $R = 0.5$ m and $m = 10$ kg. Calculate I.

(b) Let $F = 44.5$ newtons (i.e., 10 pounds). Calculate the torque.

(c) Calculate the angular acceleration α. (HINT: It comes out in radians per second per second.)

(d) How long until the wheel, starting from rest, is rotating at two revolutions per second? (HINT: That is 12.56 radians per second.)

✶✶ **25. Kinetic Energy** Argue that the kinetic energy of rotation of a rigid object can be expressed as $\frac{1}{2}I\omega^2$. (HINT: Each mass point m_i has kinetic energy $= \frac{1}{2}m_i v_i^2$ and $v_i = \omega r_i$.)

✶ **26. Centrifuge** To make liquid suspensions settle rapidly, a simulated gravitational field is produced by rapid spinning.

(a) Calculate the angular velocity (radians per second) of a 10,000-rpm (revolutions per minute) centrifuge.

(b) A test tube 10 cm from the center experiences a gravitational field (an acceleration) of ________ newtons per kilogram.

(c) Argue that the force the liquid exerts on the tube is about ten thousand times as great while it is spinning as it is at rest. Would you use thin-walled glass tubes?

✶ **27. The Hammer Throw** The "hammer" is a track-and-field event in which a 3-kg ball is whirled around the body in a circle of 1.8-meter radius. Before it is released, it may be moving at 20 m/sec. Calculate the force on the athlete's hands at this moment.

Problem 27

✶✶ **28. Quantization of Angular Momentum—Bohr Theory of the Hydrogen Atom** In Chapter 18 we shall meet the "quantum" of angular momentum, $\hbar = 1.05 \times 10^{-34}$ joule sec. According to the Bohr theory, the usual orbit of the electron around the proton has exactly this much angular momentum. We now calculate the speed v and radius r of this circular orbit. The electrostatic inverse-square-law attraction of electron to proton (Chapter 13) is given by 2.3×10^{-28} newton m^2/r^2. Set this force equal to mass times acceleration mv^2/r, where $m = 0.911 \times 10^{-30}$ kg is the mass of the electron.

(a) Set $mvr = \hbar$, and eliminate r between the two equations to solve for v.

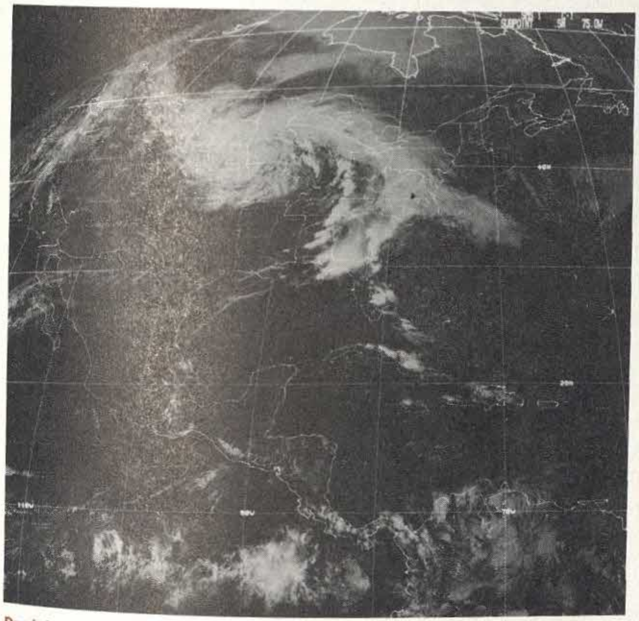
NOTE: The proton is so much more massive than the electron that it is treated as if it remains at rest.

(b) Now solve for r. [Answer: $r = 0.53 \times 10^{-10}$ m] This gives a good estimate of the size of a hydrogen atom.

✶ 29. **Why Don't We Fly Off Our Rotating Earth?** Because gravity holds us down. In fact, the earth pulls you toward its center with a force of 9.8 newtons on every kilogram in your body. But how about the centrifugal force? If you were standing at the equator, every mass m in your body would also feel a fictitious force $m\omega^2R$ acting upward. What fraction is that of the 9.8 newtons/kg acting downward? Remember, $\omega = 2\pi/24$ hours, and $R = 6.4 \times 10^6$ m. Another way to ask the question is, by how many percent will your bathroom-scale weight reading increase if you move to the north pole?

✶✶✶ 30. **Coriolis Force** In the movie "Frames of Reference," the two physics professors are getting on a platform rotating counterclockwise. They are playing with one of those well-machined dry-ice pucks on a polished table. One starts the puck moving toward the other but finds it coming back into his own hand as the platform completes half a turn. In the rotating frame of reference, the puck has gone in a closed loop: clockwise / counterclockwise? Argue cogently.

✶✶✶ 31. **Coriolis Force Drives the Weather** The earth rotates around a north–south axis, counterclockwise when viewed from the north. A north wind blows southward. In the northern hemisphere, the further south the air gets, the faster the earth moves underneath it. In the earth's frame of reference, the moving air experiences a westward force, that is, a force to the right. In the same way, air moving northward in the northern hemisphere experiences an eastward force, again a force to the right.



Problem 31 Satellite weather photograph showing counterclockwise wind pattern around a pressure "low" in the Midwest. See also photo on p. 115.

(a) Argue that air rushing into a low-pressure region in the northern hemisphere find itself moving in clockwise / counterclockwise circles around the low.

(b) Look at a national weather map in the newspaper. Locate a low-pressure area, and confirm that the winds turn around it as you predicted in part (a).

(c) Would you expect hurricanes in the southern hemisphere to twist in the sense opposite that of hurricanes in the northern hemisphere?

✶✶ 32. **A Pulsar Is a Rotating Neutron Star** An "old" star, having used up its fuel for nuclear fusion, can undergo gravitational collapse and become a neutron star. Suppose that a star with mass $M = 3 \times 10^{30}$ kg (= 1.5 solar masses) and radius 8×10^8 m has a rotation period of 30 days. It collapses to an average density of 4×10^{14} g/cm^3.

(a) Verify that its radius is now 1.2×10^4 m.

(b) The moment of inertia is given by a formula $I = fMR^2$, where f is a numerical constant close to 0.3.

NOTE: For a *uniform* sphere, f would be 0.4. What is the ratio of the old to the new moment of inertia, $I_{\text{old}}/I_{\text{new}}$?

(c) Calculate the new rotation period, assuming that angular momentum is conserved.

NOTE: A neutron star has two visible beacons, like rotating search lights, at each magnetic pole, so we can observe the flashes. The pulsar in the Crab Nebula blinks 30 times per second, corresponding to a rotation frequency of 15 Hz.

✶✶ 33. **Rotational Spectrum of H_2** The moment of inertia of a diatomic molecule about its center of mass is $I = \Sigma m_i r_i^2 = 2m(R/2)^2$. For H_2, m is the proton mass and R is the equilibrium proton-proton separation in the molecule (= 0.074 nm).

(a) Calculate I.

(b) If angular momentum is quantized in units of $\hbar = 1.05 \times 10^{-34}$ joule sec, calculate the longest rotational period possible for this molecule.

NOTE: The longest period in the observed rotational spectrum of molecular hydrogen is 2.72×10^{-13} sec.

✶ **34. Angular Momentum of the Rotating Earth**

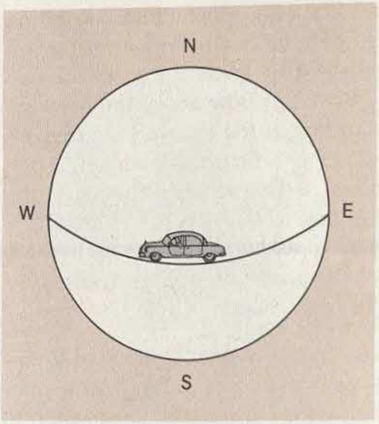
(a) The angular velocity of the earth is $\omega = 2\pi$ radians/24 hours = ________ radians/sec.

(b) Its mass is 6×10^{24} kg; its radius is 6.4×10^{6} m; and its moment of inertia is 1.0×10^{38} kg m^2. Verify that this moment of inertia agrees with the formula $0.4\ MR^2$.

(c) A 1-metric-ton (= 1000 kg) car starts to drive west on a road near the equator (see the figure). Argue that this means it acquires some angular momentum opposite to that of the earth. Conservation of angular momentum implies that the earth has to speed up a bit.

(d) Estimate the fraction by which the angular velocity of the earth is increased by bringing that car up to 20 m/sec. Express in the form "________ parts in 10^{23}."

✶✶ (e) The work done to bring the car up to that speed is 2×10^{5} joules. Verify and argue why you neglect the work done in speeding up the earth. (HINT: Remember, the car is being carried along at 465 m/sec before it starts.)



PROBLEM 34. If you drive west, the earth has to speed up a bit to conserve angular momentum.

Albuquerque balloon fiesta.

6

PRESSURE OF AIR AND WATER

SOME QUESTIONS THIS CHAPTER WILL ANSWER

Bone and muscle are more dense than water. Why is the average density of the human body a bit less dense than water?

What is Archimedes' Principle?

Ships are made of steel. Steel is much more dense than water. How can ships float?

If a certain kind of wood has specific gravity 0.5, its density is half that of water. A piece of this wood will float with more than half / half / less than half of its volume sticking out above the water level.

The buoyant force exerted by the water on the bottom surface of that piece of wood is equal to / less than / greater than its weight.

Explain the meaning of the phrase "760 millimeters of mercury is equal to 1029 centimeters of water."

Why does the nurse's blood-pressure gauge contain mercury—a highly toxic liquid?

When the weather report says "barometic pressure 29.5 inches," what is it talking about?

The density of the air is directly proportional to its pressure / temperature. This is a statement of ____________________'s Law.

The momentum transmitted per second by each molecule of an ideal gas to one wall of the container is proportional to: the average speed / the mean square speed

Explain how it is that a scuba diver coming up too fast could get "the bends."

About how far up is the oxygen content of the air only half of what it is down here?

6.1 DENSITY AND SPECIFIC GRAVITY

The founders of the metric system devised the weights and measures in such a way that one liter of water has a mass of one kilogram. Another way to say that is that the **density** of water is 1 kilogram per liter. Still another way is to say its **specific gravity** is 1. Of course, to be precise one has to specify the pressure and temperature, since water is somewhat compressible and does expand on being warmed from room temperature. The 1-liter, 1-

kilogram equivalence is exact for water at a pressure of 1 atmosphere and a temperature of 4°C (for Celsius). At higher pressures, the same kilogram of water has a somewhat larger volume. But the **compressibility** of water is so small that each additional atmosphere of pressure only increases the density by 0.005%, and thermal expansion reduces the density only 0.001% per degree C. This insensitivity of density to changes of **pressure** and temperature is characteristic of most liquids and solids and quite different from the behavior of gases. Air, for example, is 20,000 times more compressible than water, and has a 400-fold greater **thermal expansion coefficient.**

Since the human body is constituted largely of water, it is not surprising that our density is very close to 1 kg/liter. That is the same as 1 gram/cm^3, good[1] to 3 parts in 10^5. Bone is somewhat more dense (between 1.7 and 2 grams/cm^3), and so is muscle. Fat is somewhat less dense (around 0.9 gm/cm^3). That the average body density usually comes so close to that of water must have to do with our having air-filled cavities—like the lungs. Indeed, most of us will float in water when our lungs are full, and sink when we have exhaled completely: By exhaling, we decrease the total volume of the body enough to bring the average density from just below that of water to just above. For example, we might be talking about changing the specific gravity from 0.97 to 1.01.

When such small changes are important, you might argue that it would be harder to stay afloat in warm water. It is less dense than cold water, and, of course, our body temperature is regulated to stay very constant, close to 37°C, regardless of the environment. It turns out that the density differences associated with a few degrees of temperature change are too small to notice. On the other hand, things dissolved in water can make noticeable density changes. Seawater, for example, has only 30 parts per thousand of salt, but anyone who has swum in the ocean knows how much easier you float in salt water than in the swimming pool: The density of seawater is 2.5% greater than that of fresh water.

6.2 ARCHIMEDES' PRINCIPLE[2]

Who knows, maybe it was while noticing his own buoyancy change as he inhaled and exhaled in the bath that Archimedes hit on the idea: *The buoyant force exerted by the water is equal to the weight of water displaced.* The next section presents a "proof," but here we just want to admire: What a beautiful intuitive generalization it is! If an object sinks, it is because it is more dense than water. So the weight of water it displaces—the buoyant force on it—is less than its own weight and therefore not enough to hold it up. An object less dense than water only submerges so far that the weight of water displaced by the submerged volume—the buoyant force—is equal to the object's weight, and the two forces are in equilibrium.

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[1] 1 ml = 1.000027 cm^3.

[2] Archimedes, Greek mathematician, scientist, and inventor, lived in 287–212 B.C. in Syracuse, Sicily. The story is told that he was asked by King Hieron to figure out a way of testing if the king's crown was indeed made of pure gold or was alloyed with a cheaper metal like silver. Archimedes' solution was to immerse it in water and compare the rise in the water level with that caused by an equal weight of gold, that is, to compare densities. Upon hitting on this solution while bathing, the story continues, he ran naked out of the bath house to his home shouting "εὑρηκα" (pronounced Eureka), "I have found it."

Here is a specific example:

EXERCISE 1 Oak has a specific gravity of 0.7. What fraction of a block of oak is submerged when it floats, and what fraction sticks out above the water?

SOLUTION Suppose that the block of wood has a volume of 1000 cm^3. Since we know its density, we know it weighs as much as 0.7 kilograms, that is, 0.7 kg × g.[3] When it floats, the water has to buoy it up with a force of 0.7 kg × g. According to Archimedes, it displaces a volume of water having a weight of 0.7 kg × g, or 700 cm^3.

The answer, that a block having a specific gravity of 0.7 floats with 70% of its volume submerged is, of course, independent of its size. We only gave our block a volume of 1000 cm^3 to aid the intuition.

Pressure and Buoyant Force

We now look more closely at how that buoyant force comes about. Our block might be, for example, a cube 10 centimeters on a side. That would give each face an area of 100 cm^2 (= 10 cm × 10 cm). If the top surface of the cube were held submerged 1 cm below the surface, the water above it, that is, a volume of 100 $\text{cm}^2 \times 1\ \text{cm} = 100\ \text{cm}^3$, would weigh 100 grams × g. This has to be borne by the 100 cm^2 surface. Each square centimeter of surface has to support 1 gram of water. If you think about the water molecules adjacent to the cube face, they are being compressed, and the compressive stress is the same in all directions. This means that if the block were only dipped so that its lower surface were 1 cm below the water level, the upward force on the lower surface would amount to that same 100 grams × g: 1 **gram weight** for each cm^2 of surface. What we are saying is that 1 cm below the water level, the water exerts the same pressure up as down.[4]

[3] This is pronounced "0.7 kilogram weight." How convenient not to convert to newtons here! The medical literature is full of kilograms used as a unit of force rather than mass. See Sec. 1.5.

[4] It exerts the same pressure sideways too: The water molecules down there have no preferred directions; all directions in space are the same to them. This **isotropy** of compressive stress is a property of liquids and gases, and is called **Pascal's Principle.** Some solids also possess this symmetry property. Anisotropic crystals do not, nor do cellular materials like wood or bone.

If 1 cm of water will push on that surface with a force of 100 grams × g, then at a depth of 7 cm, the force would be 700 grams × g, just enough to buoy up the 700-gram block. Now you see how **Archimedes' Principle** comes about: The block sinks just low enough so that the water pressure at the bottom surface gives an upward force equal to the weight of the block (see Fig. 6.1).

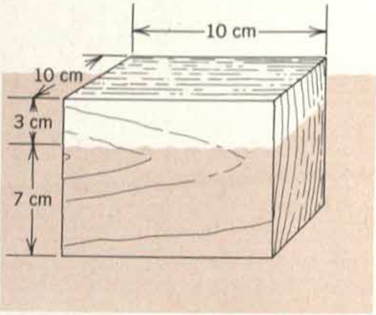

FIGURE 6.1 The 7 centimeters of the oak block that are under water displace 700 grams of water. The whole block weighs as much as the displaced water.

Pressure and "Head"

If we focus attention on 1 square centimeter of water surface, then 1 cm down below the surface there is 1 cubic centimeter of water above that 1 cm^2. That much water weighs 1 gram × g. Seven centimeters down we have a column of water weighing 7 grams × g, and so on. If we focus on an area A of surface and follow it down to a depth h, that defines a column of water having a volume hA. (The volume of a cylinder is height times cross-sectional area.) You know the density of water, but for generality call the density ρ (Greek rho). Then the mass of that vertical cylinder of water is ρhA, and its weight (= mass × g) is ρghA. Now focus on the horizontal area A that supports that much weight. The stress (= force per unit area) on that area A is the weight divided by the area.

$$\text{stress} = \frac{\rho ghA}{A}$$

or ρgh. When compressive stress is **isotropic,** it is just called *pressure*. So the pressure at depth h is given by the formula

$$\text{pressure} = \rho gh$$

EXERCISE 2 An oak dowel rod 1 cm^2 in cross-sectional area and 10 cm long is dropped into a graduated cylinder full of water. The walls constrain it to be almost vertical.

(a) How many centimeters of its length are submerged when it is in equilibrium? (HINT: Remember, in equilibrium its weight (= 7 grams × g) has to be equal to the buoyant force of the water on the 1 cm^2 area of the bottom of the rod.)

(b) If the rod is pushed down so that it is completely submerged, what is the buoyant upward force exerted on it by the water? Argue that this force does not depend on how far it is pushed, even though the force on the bottom surface does.

Proof of Archimedes' Principle

By now you may have guessed how the proof goes. Before the wood block was lowered into the water, there was water in those 700 cm^3. The weight of that water was supported by the water underneath it. It was in equilibrium. In other words, water pressure exerted an upward force equal to the weight of that 700 cm^3 of water. When that 700 cm^3 of water is replaced by 700 cm^3 of wood, the buoyant force on that volume does not change. Therefore, it is equal to the weight of the water displaced.

That proof is perfectly general and does not depend on the shape of the floating object, or the material of which the object is made, or the nature of the fluid. The only assumption is the implicit one that the force exerted by the surrounding fluid on the surface of the submerged object is the same as the force exerted across the same surface if the object is replaced by something else. The something else could be more of the same fluid. That makes the "surface" imaginary, to be sure.

EXERCISE 3 22.4 liters of air weighs 29 gram-weight. The same volume of helium gas weighs 4 gram-weight. What is the buoyant force of the air on a 22.4-liter balloon? [Answer: 29 grams × *g*] What is the most the rubber and string can weigh for the helium-filled balloon to fly? Argue that the answer is 25 grams × *g*.

*6.3 THE UNITS OF PRESSURE (SPHYGMOMANOMETER)

In the example of the oak block, it was convenient to measure force in grams × *g* or kilograms × *g*. Remember, at a depth of 7 cm, the pressure is just enough to lift the 100 cm^2 area of the bottom of the block with a buoyant force of 700 grams × *g*. You will recall that that was the weight of the block in our example. Rather than saying that 7 centimeters below the water level the pressure is 7 gram weight per square centimeter, the medical literature often just says "7 **cm of water**," or 7 cm H_2O. Using cm H_2O as a unit of pressure is particularly convenient when dealing with water or with blood, which has about the same density. Thus we would say that the blood pressure in the large vein leaving the brain is about 40 cm H_2O lower than the pressure in the right atrium of the heart. There is a "pressure head" of about 40 cm, the height of the brain above the heart. When you lie down, that pressure difference virtually disappears. Note that the corresponding pressure difference on the arterial side of the brain goes away at the same time.

Your physics professor may well be horrified at the thought that you will emerge from this course quoting pressures in "cm H_2O" instead of some good physicist's units. If pressure is force per unit area, then the modern unit of pressure should be the newton per square meter, officially called the **pascal.** For our oak block, the pressure on that lower surface 7 cm down should then be written

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*This section is optional and can be omitted without loss of continuity.

$$\begin{aligned}\text{pressure} &= \rho g h \\ &= \left(1000\,\frac{\text{kg}}{\text{m}^3}\right) \times \left(9.8\,\frac{\text{newtons}}{\text{kg}}\right) \times (0.07\text{ m}) \\ &= 686\text{ newtons/m}^2\text{, or 686 pascals}\end{aligned}$$

Since the bottom of the block has an area of $(0.1\text{ m})^2 = 0.01\text{ m}^2$, the buoyant force is

$$\begin{aligned}\rho g h A &= \left(686\,\frac{\text{newtons}}{\text{m}^2}\right) \times (0.01\text{ m}^2) \\ &= 6.86\text{ newtons}\end{aligned}$$

which is indeed the weight of a block having a volume of $(0.1\text{ m})^3$ and a specific gravity of 0.7. Verify:

$$\begin{aligned}\text{weight} &= \text{mass} \times g \\ &= \rho \times \text{volume} \times g \\ &= \left(700\,\frac{\text{kg}}{\text{m}^3}\right) \times (0.001\text{ m}^3) \times \left(9.8\,\frac{\text{newtons}}{\text{kg}}\right) \\ &= ?\end{aligned}$$

It may seem awkward to give the density as 700 kg/m^3 instead of as 0.7 gm/cm^3, but that is what we have to do if we insist on staying strictly in "MKS" units (MKS is meter-kilogram-second).

EXERCISE 4 Of course, you could write 0.7 kg/liter for the density of oak and 1 liter for the volume of the block. You have set the liter equal to the cubic decimeter, an approximation good to three parts in a hundred thousand:

$$1\text{ liter} = (10^{-1}\text{m})^3 = 10^{-3}\text{ m}^3$$

But then, if you want to find pressure using the formula "pressure = $\rho g h$," it comes out in newton m/liter. Unfamiliar? Write 1 newton m/liter = 1 newton $\text{m}/(10^{-1}\text{m})^3$. Try it:

$$\begin{aligned}\text{pressure} &= \rho g h \\ &= \left(1\,\frac{\text{kg}}{\text{liter}}\right)\left(9.8\,\frac{\text{newtons}}{\text{kg}}\right)(0.07\text{ m}) \\ &= \underline{\qquad\qquad}\ \frac{\text{newtons}}{\text{m}^2}\end{aligned}$$

Did you get the same answer as before?

Much of the biomedical literature uses CGS (= centimeter-gram-second) units rather than MKS. In CGS units, the acceleration of gravity g is written 980 cm/sec^2 or 980 dynes/gm. (You may recall that, if you write $F = ma$ in CGS units, the unit of force is 1 dyne = 1 gm cm/sec^2, which translates to 10^{-5} newtons.) In these units the pressure 7 cm below the water level is written

$$\begin{aligned} \text{pressure} &= \rho g h \\ &= \left(1\,\frac{\text{gm}}{\text{cm}^3}\right) \times \left(980\,\frac{\text{dynes}}{\text{gm}}\right) \times (7\text{ cm}) \\ &= 6860\,\frac{\text{dynes}}{\text{cm}^2} \end{aligned}$$

[Compare with 686 newtons/m^2 above.] Evidently, the dyne/cm^2 is a smaller unit than the newton/m^2:

$$1\,\frac{\text{newton}}{\text{m}^2} = 10\,\frac{\text{dynes}}{\text{cm}^2}$$

Repeat: The newton is 10^5 dynes; the m^2 is 10^4 cm^2.

Now you might think that three or four metric units for pressure would be enough, especially after remembering that American engineers have a preference for the pound per square inch for hydraulics and the ton per square foot for buildings and roads. Alas, there are more. The blood-pressure gauge **(sphygmomanometer)** in common use has mercury as its measuring liquid, so it reads pressure directly in millimeters of mercury, called **torr** or "mm Hg." The pressure in an artery of the arm might typically peak at 120 mm Hg above ambient, that is, above the air pressure in the room. To translate into our earlier units, we need to know that the specific gravity of mercury is 13.546 so that 1 mm Hg = 1.35 cm H_2O. That "systolic" blood pressure of 120 mm Hg, translated into centimeters of water,[5] would be

$$13.546 \times 12.0\text{ cm} = 162.6\text{ cm H}_2\text{O}$$

Now you know why the nurse uses mercury instead of water in her sphygmomanometer (pressure cuff). It would be a very bulky instrument indeed if it used a less dense measuring liquid.

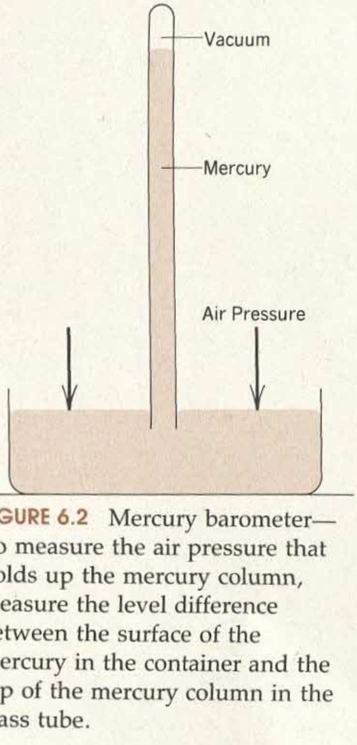

FIGURE 6.2 Mercury barometer—To measure the air pressure that holds up the mercury column, measure the level difference between the surface of the mercury in the container and the top of the mercury column in the glass tube.

The same goes for the scientific **barometer,** which is a tube filled with mercury, closed at the upper end, dipping into a mercury trough at the lower end. The pressure of the air acting on the mercury surface supports the mercury against gravity so that it appears to hang from the "vacuum" at the top of the tube.[6] See Fig 6.2. On an average day, the air pressure is close

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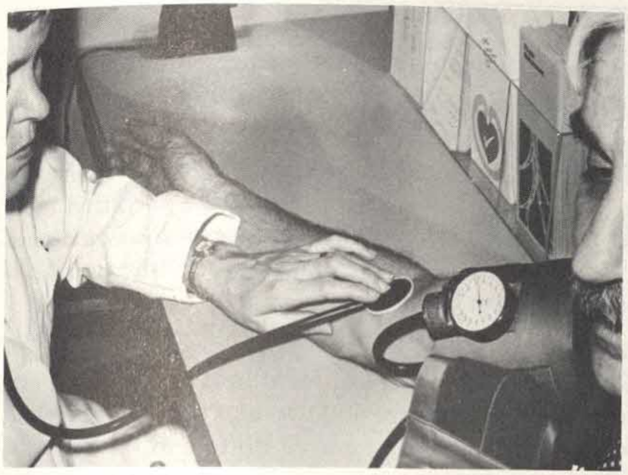
[5] To avoid confusion, the medical literature consistently uses millimeters for pressure head of mercury and centimeters for head of water.

[6] Of course, the so-called **Torricellian vacuum** actually contains mercury vapor. Since the vapor pressure of mercury at room temperature is only 0.0012 mm Hg, the length of the mercury column does not require much correction to be used as a measurement of the atmospheric pressure.



Measuring blood pressure. The pressure cuff is wrapped around the patient's arm and inflated. As the air pressure is slowly released, blood starts to flow through the partially pinched-off artery during an increasing fraction of the cardiac cycle. The sounds of that turbulent flow are heard through the stethoscope. The gauge in the photograph uses a calibrated spring instead of a mercury column.

to 760 mm Hg, and this has been defined to be the standard **atmosphere.** Another pressure unit!

If

1 atmosphere = 760 mm Hg = 760 torr

by definition, what is the *atmosphere* in our other units? First, how much head of water corresponds to 760 mm of mercury? Answer:

$13.546 \times 76 \text{ cm} = 1029.4 \text{ cm of water}$

Now you know that a water barometer would require a room with a very high ceiling!

You may hear it said that "a head of 10 meters of water is an atmosphere." Such a statement we now see to be 3% off, close enough for many purposes. If someone asks you how deep under water you have to swim before the pressure on your ear drums becomes twice the usual, the answer is 10 meters down.

Sick of units? Then skip this exercise, which is here mostly for handy reference.

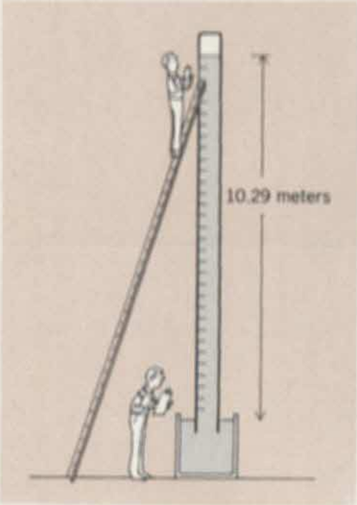


FIGURE 6.3 Water barometer. Now you know why they use mercury.

EXERCISE 5 A column of water 1029 cm high exerts a pressure of (1.029 kg/cm²) *g*; that's 1 atmosphere (see Fig. 6.3). Converting further, we get

$$1 \text{ atmosphere} = 1.008 \times 10^5 \frac{\text{newton}}{\text{m}^2}$$

$$= 1.008 \times 10^6 \frac{\text{dynes}}{\text{cm}^2}$$

Verify these conversions by recalling that

$$1 \text{ kg} \times \text{g} = 9.8 \text{ newtons}$$

and

$$1 \text{ gm} \times \text{g} = 980 \text{ dynes}$$

Again, you may hear people say that a million dynes per square centimeter is an atmosphere. Show that they are less than 1% off.

6.4 GAS LAWS

In Sec. 6.5 on deep-sea diving we shall be concerned with the way the volume of air in the lung changes with ambient pressure. You could probably guess it: Air density is directly proportional to air pressure, provided that the temperature remains constant. That simple proportionality is called **Boyle's Law.** For a constant mass of air, doubling the pressure means halving the volume: The volume is inversely proportional to the pressure.

Repeat: Boyle's Law says

pV = constant

p = pressure
V = volume

The implication is that the "constant" is a function of temperature and, of course, proportional to the amount of air.

It turns out that Boyle's Law is an excellent approximation for all gases as long as they are not near their condensation points, that is, near the temperatures and pressures at which they liquify or freeze. We know that liquids and solids are nowhere near as compressible as gases, so we have to expect the law to fail as a gas becomes closer to becoming a liquid or solid.

Even if you have never had a chemistry course, you are able to give a pretty good description of a gas on the microscopic level: little molecules banging around at random, spending most of their time traveling in straight lines and only a small fraction of their time colliding, interacting with each other. We are going to make that picture into a theoretical model for "explaining" Boyle's Law, called the **ideal gas.** That model neglects the molecular interactions completely. When a gas condenses into a liquid, it does so because of the attraction between the molecules—they like to stick together. When a liquid evaporates and becomes a gas, it is thermal motion of the molecules that breaks those bonds. You might think that the transitions from liquid to gas (evaporation) and from solid to gas (sublimation) as the temperature is raised should be a natural part of our picture. The ideal gas model pretends they don't exist!

Ideal Gas

We now derive the **"ideal gas law."** That is the relation between pressure and volume for a gas of noninteracting "point" molecules each of mass m. Please don't be put off by the notion of something having mass but no size. Remember, it is an idealization. It just says that the molecules are very small. For simplicity we shall assume that their collisions with the walls of the container are elastic. That assumption is not actually necessary, but it is convenient. If we measure pressure against a flat wall perpendicular to the x-direction, the elastic-collision assumption allows us to forget about the y and z components of a molecule's velocity. Those two components are unchanged by colliding with the wall, while the x-component of the velocity v_x is reversed in sign. An elastic collision with a massive wall just results in a "mirror reflection" of the velocity (see Fig. 6.4).

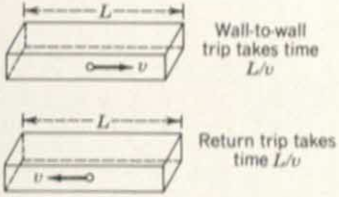

L
v
Wall-to-wall trip takes time L/v
L
v
Return trip takes time L/v

FIGURE 6.4 Kinetic-theory derivation of the gas laws. It takes the molecule a time $2L/v$ to make a round trip, bouncing off opposite walls.

Suppose that two such parallel walls are separated a distance L. Then the time the molecule spends traveling from one wall to the other is L/v_x. It returns to the first wall after a time $2L/v_x$. Each time the molecule collides with that wall it transmits a momentum $2mv_x$ to the wall. That is because it comes in with a momentum component $+mv_x$ and leaves with $-mv_x$, as a consequence of our convenient elastic-collision assumption. The average force the molecule exerts on the wall is the momentum imparted per unit time,

$$\text{force} = \frac{\text{momentum}}{\text{time}} = \frac{2\,mv_x}{2L/v_x} = \frac{mv_x^2}{L}$$

for each molecule. The pressure p on the wall is the force per unit area A of wall due to all the molecules (N of them) in the container having the same v_x:

$$p = N\frac{\text{force}}{A} = N\frac{mv_x^2}{LA}$$

If we make our container a straight cylinder, its volume is LA, its length times the area of the flat wall on the end. That gives us a relation between pressure p and volume V:

$$\text{pressure } p = N\frac{mv_x^2}{V}$$

This equation says that the pressure due to that "species" of molecule is inversely proportional to the volume of the container. Or we could say, the pressure is directly proportional to N/V, the number of molecules of that species per unit volume. That is the kind of statement we were after!

Of course, not all the molecules of a gas are going to have the same x-component of velocity v_x. Here recall that the kinetic energy of the molecule is $\frac{1}{2}mv^2$. At this point we do bring in the y- and z-components of the velocity because

$$v^2 = v_x^2 + v_y^2 + v_z^2$$

That's Pythagoras! Now we introduce a statistical notion. We assume that the velocities have random directions so that the average value of v_x^2 is the same as the average value of v_y^2 and of v_z^2. With this isotropy assumption, we can interpret our pressure formula in terms of the average kinetic energy of a molecule. The v_x^2 is one-third of v^2, on the average. So our "law" becomes

$$p = \frac{\frac{1}{3}NM(v^2)_{\text{av}}}{V} = \frac{N}{V} \times \tfrac{2}{3}\,(\text{average kinetic energy per molecule})$$

Our model has given us more than a "derivation" of Boyle's Law, that gas pressure is inversely proportional to volume. It also allows us to interpret the proportionality constant. For the ideal gas, the pV product is a measure of the kinetic energy of the gas. Since the pV product is so nearly constant for real gases at constant temperature, the ideal gas model leads us to associate temperature with average kinetic energy per molecule for real gases too.

A temperature scale based on the ideal gas law defines **absolute temperature** T to be proportional to the average kinetic energy per molecule. It is conventional to write

$$p = \frac{NkT}{V} \qquad \text{(Eq .1)}$$

so that

$$kT = \tfrac{2}{3}\,(\text{average kinetic energy per molecule})$$

The constant k is called **Boltzmann's constant,** and it really is a universal constant. It is the same for all gases. Its numerical value depends on the choice of units for energy (joules?) and the choice of units for temperature (Kelvin?). The convention is to choose the size of the degree such that there are 100 degrees between the freezing and boiling points of water. That convention gives the Kelvin (K) scale of absolute temperature. On the Kelvin scale, the freezing point of water is 273 K, the boiling point is 373 K. This choice makes Boltzmann's constant

$$k = 1.3807 \times 10^{-23} \text{ joules per Kelvin per molecule}$$

In older books this was written "joules per degree Kelvin." Nowadays it is fashionable to omit the word *degree* and treat Kelvins as just another energy unit, just like joules or ergs.

Since absolute temperature is a measure of average kinetic energy, **absolute zero** temperature would seem to imply the absence of all molecular motion. It should be pointed out that while there is no limit on how close to absolute zero things can be cooled, the fact that absolute zero can never actually be reached is called the **Third Law of Thermodynamics.**

Avogadro's Law

Our ideal-gas model tells us that the volume occupied by a certain number of molecules at a given pressure and temperature does not depend on the mass of the molecules. That is, it is the same for all gases. That prediction is known as **Avogadro's Law.** The number of molecules whose weight in grams is equal to the particular gas's molecular weight (carbon = 12, etc.) is, of course, the same for all substances. That number is called **Avogadro's Number.** It is

$$N_{\text{Avogadro}} = 6.02 \times 10^{23}$$

the number of molecules in a **mole.** The volume occupied by one mole of ideal gas at 0° C (= 273 K) and 1 atmosphere pressure is 22.4 liters.

Near the beginning of this chapter there is the statement that air is 20,000 times more compressible than water. You are now in a position to check that. To increase the density of water by 5 parts in 100,000 requires 1 atmosphere of added pressure. To increase the density of air—essentially an ideal gas at a pressure of 1 atmosphere—by 5 parts in 100,000 requires only 5/100,000 of an atmosphere additional, since density and pressure are proportional. The ratio of compressibilities is indeed 1/0.00005 = ____________.

EXERCISE 6 How much (i.e., by what percentage) does the density of air decrease when its temperature is raised from 27°C to 28°C (i.e., from 300 K to 301 K)?

The chemists like to write the ideal gas law in terms of the number of moles of gas, n, rather than the number of molecules of gas, N. The ratio is, of course, Avogadro's Number:

$$\frac{N}{n} = N_{\text{Avogadro}}$$

Then they give a special name to the product of Avogadro's Number times Boltzmann's constant:

$$N_{\text{Avogadro}}\, k = R$$

R is called the **ideal gas constant,** and the law can be rewritten

$$p = \frac{nRT}{V}. \qquad \text{(Eq. 2)}$$

EXERCISE 7 Calculate the ideal gas constant R using the fact, which you already know, that a gas at a pressure of 1 atmosphere and a temperature of 273 K (that is called S.T.P., or Standard Temperature and Pressure) has a volume of 22.4 liters. Your result will come out in liter atmospheres per mole per Kelvin. Then convert it to more familiar units: joules per mole per Kelvin. You will need the conversion factor 1 atm $= 10^5$ N/m^2.

Mixture of Gases

In the derivation of the ideal gas law using **"kinetic theory,"** we talked about a "species" of molecule having a certain mass m and a certain x-component of velocity, v_x. Later we averaged over all possible values of v_x and gave a physical interpretation to the statistical average of the kinetic energy per molecule $\frac{1}{2}mv^2$. If we have a mixture of gases, we have different values of the mass m. Suppose that there are two or more gases in the same volume. Again using kinetic theory, it can be shown that the average molecular kinetic energy is the same for all the gases in equilibrium. This just says that all the gases come to the same temperature. Their pressures are additive: Each species of molecule hits the wall independently of the other species. In air, for example, 78% of the pressure is due to nitrogen. That is because 78 out of every 100 molecules of air are N_2 molecules. When the total pressure is 1 atmosphere, the partial pressure of nitrogen is 0.78 atm. Another 21% of the molecules are oxygen. So oxygen contributes a partial pressure of 0.21 atm. All other gases make up the remaining 1%.

In Sec. 6.6, the **Law of Atmospheres,** we learn that the relative concentrations of nitrogen and oxygen change as you go up into the upper atmosphere. That is because of the different masses of N_2 (molecular weight 28) and O_2 (molecular weight 32). Oxygen, being heavier, tends to stay lower down. The higher you go, the smaller the fraction of oxygen. The partial pressure of oxygen falls off faster with increased altitude than the partial pressure of nitrogen.

In Sec. 6.5, Deep-Sea Diving, we find essentially the opposite effect. At a depth of 50 meters, the air we breathe is at 6 atmospheres. So it occupies only one-sixth the volume it did at the surface, according to Boyle's Law. If your scuba tank used ordinary compressed air, the partial pressure of oxygen 50 meters down will be 6 times as great as the pressure the lungs require to oxygenate the blood. As a consequence, you will have an enormous excess of oxygen in your breathing air. Most of the oxygen you inhale down there you end up exhaling again.

6.5 SPECIAL TOPIC: Deep-Sea Diving

You have probably tried swimming under water. If you had a cold or a stopped-up nose, you may have felt the pressure on your ear drums: As you go down, the weight of the water above you exerts pressure, which your skin and soft tissue transmit all through your insides. Indeed, your body is full of liquids and not very compressible, so you don't feel yourself getting smaller. Air, on the other hand, is very compressible. If no additional air gets into the middle ear as you go deeper, the added pressure makes the ear drum bulge inward. That can hurt. One of the first things a **skin-diving** teacher talks about is the importance of keeping the **Eustachian tubes** unclogged (see Fig. 6.5). They are the air passages between the pharynx and the ears. With these tubes open, as the air pressure in your respiratory system rises, more air can get into the middle ear. In this way the pressure on both sides of the ear drum is equalized without stressing it. (See Problem 60, Popping the Ears)

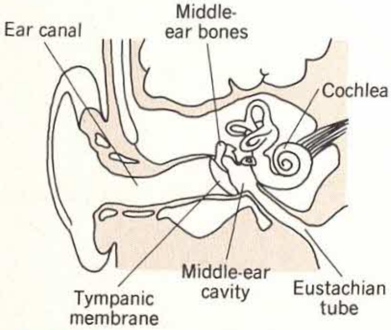


FIGURE 6.5 If the Eustachian tubes are clogged as you dive deeper, no air can get from your respiratory system into the middle ear, and your ear drum will bulge inward.

In skin diving, the total mass of air in the respiratory system remains constant as you go deep into the water. That air obeys Boyle's Law. If you double the pressure by going 10 meters down[7] holding your breath, you halve the volume of the lungs. Twenty meters down, the head of water gives 2 atmospheres of additional pressure, so the lungs are down to one-third their usual volume; 30 meters down, one-quarter, and so on. In some parts of the world, pearl divers work at 50-meter depths without breathing apparatus. With a head of water giving 5 atmospheres, the ambient pressure down there is 6 atmospheres, so the lungs have one-sixth the volume they have at the surface.

The additional pressure above atmospheric is called the **gauge pressure.** The *absolute* pressure is the gauge pressure plus the pressure of the open air. When a tire gauge says that your automobile tire is at 29.4 psi (pounds per square inch), you would translate that as 2 atmospheres:

1 atm = 14.7 psi

Scuba divers.

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[7] See Sec. 6.2: Every 10.3 meters of depth is one additional atmosphere.

That is gauge pressure. The absolute pressure in the tire is then 3 atmospheres. In diving, the head of water above you tells you the gauge pressure in your lungs: Each 10.3 meters (=34 feet) down is one more atmosphere. The pressure that Boyle's Law talks about is, of course, absolute pressure.

We need that terminology in connection with **scuba**[8] diving. As you descend with scuba, you keep inhaling air supplied from the tank and exhaling air into the mask, bubbling it out into the water through a valve (see Fig. 6.6). This air is at the ambient pressure at your depth, so your lungs remain roughly at their usual volume. When you are 10 meters down, your tank is supplying air at 1 atmosphere gauge pressure or 2 atmospheres absolute pressure—twice the pressure at the surface. In order to keep you breathing all the way down, the air pressure in the tank has to be greater than the water pressure at the lowest depth to which you intend to dive. Each time you expand your lungs to inhale, the air pressure in your mouth falls just a bit below the **ambient** pressure at your depth. A pressure-sensing diaphragm allows that small pressure difference to open the valve that lets air come from the tank through the air hose into your mouth. The air in the tank may have a gauge pressure of 2250 psi (= 153 atmospheres)! Many scuba tanks are designed to be filled to such high pressures. But the regulator mechanism keeps the air you inhale exactly at the pressure of the water outside the tank. (See Problem 61.)

Some Dangers of Breathing High-Pressure Air

If you are planning to go scuba diving, be sure to get expert instruction. This section uses scuba diving in order to teach physics, not vice versa. Many dangers associated with life under water will not even be touched on here. For example, we are not going to discuss the danger of drowning!

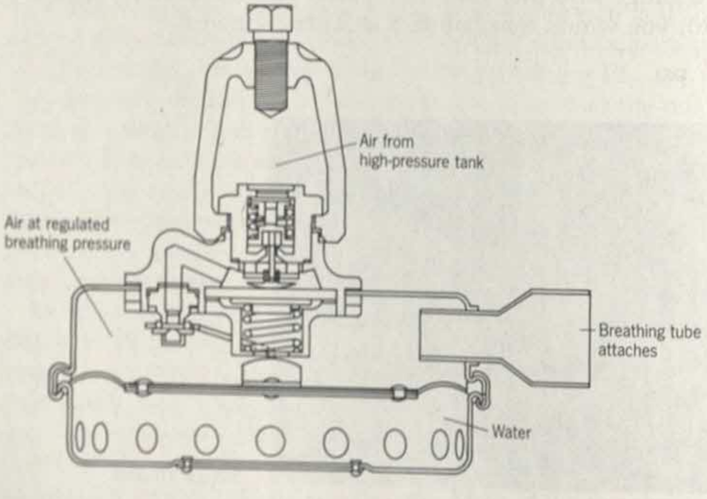

FIGURE 6.6 Scuba regulator—supplies the right pressure for breathing at every depth. The sensing diaphragm separating the middle (air) compartment from the lower (water) compartment moves up if the water pressure increases. The lever arrangement amplifies this motion to open the small valve on the left. This allows air from the small "buffer" chamber above it to bleed into the large air compartment until its pressure equals the water pressure. The "buffer" pressure acts through the coil spring in the center to close the big valve from the high-pressure tank at the top. The diver inhales through a tube attached to the air compartment at the right-hand side, removing air from this compartment. Breathing is capable of reducing the pressure there by 5 or 10 cm of water, enough to open the small valve again.

[8] Self-contained *u*nderwater *b*reathing *a*pparatus.

The painful ear drum is one example of a class of troubles called "squeeze." Although middle-ear squeeze can be a problem in either scuba or skin diving, the so-called *thoracic* **squeeze** is peculiar to skin diving. We talked about the way lung volume decreases in a breath-hold dive as you go deep down: It obeys Boyle's Law. But there is a volume below which the supporting structures start to resist further compression of the lungs. This volume is just a bit less than what you have left after you exhale all you can at atmospheric pressure. If you were to go down to a depth where the pressure tends to compress the lungs to a volume smaller than this minimum volume, the air in the lungs would be at a lower pressure than the surrounding tissue, and would essentially suck in the blood vessels that line the lung. At sufficient depth these could rupture. That phenomenon is called thoracic squeeze.

Although "squeeze" is a danger you could encounter on descending, there are other dangers encountered in coming up, especially in coming up too fast. If you have been breathing with a scuba at depth and come up without exhaling, then the pressure in your lungs will be greater than the ambient pressure. This is the direct opposite of squeeze. Here the lungs are being blown up like a balloon, and like a balloon, a lung could burst. Or you might develop an *air embolism*, an air bubble that forms in a blood vessel in the lung and travels from there to the heart and into the arterial system. When it gets into a small artery, it is likely to get stuck and plug up that artery. This is most serious if that happens to be an artery feeding the brain, the heart, or the lung. The formation of such an air bubble can be understood in terms of the design of the alveoli, the tiny structures that form the mass of the lung. The normal function of these membranes is to allow gases to diffuse easily into and out of the blood stream: oxygen in, carbon dioxide out. The equilibrium concentration of each gas dissolved in the blood is proportional to the partial pressure of that gas. That is called **Henry's Law,** and is somewhat analogous to the ideal gas law. With a pressure gradient across the membrane, the net rate of diffusion of air through the alveoli is enhanced, but it might not all go into solution inside. When the pressure is low on the bloodstream side, tiny bubbles form, as they do in a newly opened bottle of soda pop. They coalesce into larger bubbles. That is how an air embolism could occur if you don't exhale as you come up.

There is another danger associated with coming up too fast, called decompression sickness, or "the **bends.**" This is connected with the nitrogen, which, according to Henry's Law, dissolves in the blood while the nitrogen pressure is great, that is, while you are at depth. If you stay down long enough, much more nitrogen dissolves in the blood than at atmospheric pressure. If you come up slowly, that is, decompress gradually enough, the concentration of dissolved nitrogen will be "supersaturated," diffusive transport of nitrogen out through the alveoli will be enhanced, and the concentration of dissolved nitrogen will slowly return to its surface value without harm. But if you decompress too fast, then transport through the alveoli can't keep up and nitrogen will go out of solution in blood vessels in various parts of the body. There are many and quite frightening symptoms of this **decompression sickness,** one of them being pain, especially in the joints. The treatment, of course, is to recompress immediately. The characteristic time for the concentration of dissolved nitrogen to come to equilibrium with the air in the lungs is of the order of 36 hours. The nitrogen takes a long time to come back out into the lungs. That says: After a long dive, take a long time to decompress.

What physics have we learned in this essentially qualitative description of three of the many dangers incurred in deep-sea diving? We learned that the walls of a container in which pressure inside is less than the pressure outside may break. (When that happens suddenly, it is called **implosion.**) We learned that when the pressure inside is greater than the pressure outside, the container can *ex*plode. In retrospect, that all seems pretty obvious stuff. We learned Henry's Law for gases in solution, and that a decrease in pressure can result in supersaturation. We talked about nucleation of the gaseous phase out of the liquid phase as the pressure falls further. Also, we mentioned diffusive transport across membranes in the presence of a pressure gradient. Quite possibly, these phenomena are not new to you, but it may have been useful to meet them in the somewhat unfamiliar surroundings of Jacques Cousteau's "Silent World." ■

■ 6.6 SPECIAL TOPIC: The Law of Atmospheres

In this space age, we all know that our atmosphere extends less than a hundred kilometers up. Air pollution has become a fashionable worry, and we are becoming increasingly conscious of how thin that layer of air really is. Nothing keeps the air from escaping into outer space except its own weight, so its density gets progressively less the higher up you go. Mountain climbers have always known this; it gets harder to breathe at high altitudes. Our jet airplanes, cruising at altitudes around 10,000 meters, have pressurized cabins, so that the passengers are not aware how thin the air is up there. But at the beginning of every commercial flight the stewardess gives her demonstration lecture on the oxygen masks to use "in case the cabin pressure should drop." How low is the air pressure at such altitudes? The ideal gas law can give us an estimate.

It will be a rough estimate, because the temperature is not uniform all the way up. Variations in air temperature of 10 to 20 percent on the absolute temperature scale are normal. Because the pressure drops by a much greater fraction, it is a good "first-order" approximation to neglect the temperature variation, that is, to assume that the temperature is independent of the altitude. The value of the acceleration of gravity g may also be considered constant throughout the atmosphere. This means that air pressure is proportional to air density all the way up—according to Boyle.

Pressure Decreases Exponentially with Increasing Altitude

The pressure at the bottom of the atmosphere—at sea level—is a measure of the weight of all the air above it. Indeed, the pressure at any altitude measures the weight of all the air above that altitude. So the altitude at which the air pressure is just half of what it is at sea level is the altitude at which half of the total air mass is above and half below. Let us call that height $h_{1/2}$. According to Boyle's Law, the air density at $h_{1/2}$ is half of what it is at sea level. If we consider the next layer of air of the same thickness $h_{1/2}$, extending from altitude $h_{1/2}$ to $2h_{1/2}$, that layer weighs only half as much as the lower layer of the same thickness. So the pressure at altitude $2h_{1/2}$ will be half of what it is at altitude $h_{1/2}$, or one-quarter of the sea-level pressure. Each time we go up another $h_{1/2}$ in altitude, the pressure drops by half.

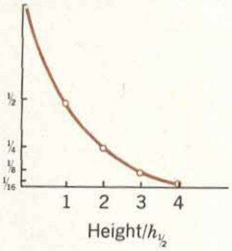

FIGURE 6.7 Exponential fall-off of pressure with altitude: Law of atmospheres.

A graph will help visualize that altitude dependence (Fig. 6.7). Here it is in a chart:

Altitude expressed in units of $h_{1/2}$	Pressure expressed in units of its sea-level value
1	$\frac{1}{2}$
2	$(\frac{1}{2})^2$
3	$(\frac{1}{2})^3$
.	
.	
.	
q	$(\frac{1}{2})^q$

The number q measures the altitude in units of $h_{1/2}$ so that

altitude $z = q\, h_{1/2}$

and the pressure $p(z)$ at altitude z can be written in units of $p(0)$:

$$p(z) = (\tfrac{1}{2})^{z/h_{1/2}}\, p(0) \qquad \text{(Eq. 1)}$$

This is sometimes called the **Law of Atmospheres.** Since the altitude is in the exponent, we say that the pressure falls **exponentially** with altitude.

Now we can fill in the curve between the points in the graph. Evidently, the pressure gradually approaches zero with increasing height. You might ask, does it ever really get to zero? Well,

at $4h_{1/2}$ it is $\frac{1}{16}$ of $p(0)$
at $5h_{1/2}$ it is $\frac{1}{32}$ of $p(0)$
at $6h_{1/2}$ it is $\frac{1}{64}$ of $p(0)$

Fill in the blanks:

at $7h_{1/2}$ it is ____________of $p(0)$
at $8h_{1/2}$ ____________
at $9h_{1/2}$ ____________
at $10h_{1/2}$ ____________
.
.
.
at $20h_{1/2}$ ____________of $p(0)$

Now we are down to less than one-millionth of sea-level pressure. Really zero? No, regardless of how high you go, there will be some stray air molecules up there. So it is hard to say where the atmosphere really "ends." One might say that it ends roughly where the density of air molecules is equal to the average interplanetary gas density.

But what determines that characteristic **half-value layer** thickness $h_{1/2}$? Intuitively, we know it has to be something about the weight of air. An air molecule of mass m has weight mg. How much does a cubic centimeter of air weigh?

$$\text{weight of } 1\ \text{cm}^3 = \left(\frac{\text{number of molecules}}{\text{cm}^3}\right) \times (mg)$$

The number of molecules per cubic centimeter depends on the temperature and pressure. According to the Ideal Gas Law (Eq. 6.4–1),

$$\frac{\text{number}}{\text{volume}} = \frac{p}{kT}$$

with weight mg,

$$\frac{\text{weight}}{\text{volume}} = \left(\frac{p}{kT}\right)(mg)$$

The weight of a thin layer,

$$\begin{aligned}\text{weight} &= \left(\frac{\text{weight}}{\text{volume}}\right) \times (\text{area}) \times (\text{thickness})\\ &= \frac{p}{kT}\, mg \times \text{area} \times \text{thickness}\end{aligned}$$

tells how much that layer contributes to the pressure underneath it:

$$\frac{\text{weight}}{\text{area}} = \frac{p}{kT}\, mg \times \text{thickness} \qquad \text{(Eq. 2)}$$

Repeat: The change in pressure through the layer is proportional to the pressure in the layer *and* to the layer thickness. Over a thickness h, the pressure changes by a fraction

$$\frac{\text{change in pressure}}{\text{pressure}} = \text{fraction} = \frac{mg}{kT}h \qquad \text{(Eq. 3)}$$

The quantity kT/mg has dimensions of length. Therefore, it will give us a rough idea of the scale of height over which appreciable changes in pressure occur. Since we are interested in breathing, let's concentrate on the oxygen. Put in the molecular weight of oxygen,

$$M = N_{\text{Avogadro}}\, m = 32 \text{ grams per mole,}$$

the value of the ideal-gas constant

$$R = N_{\text{Avogadro}}\, k = 8.3 \text{ joules per mole per Kelvin,}$$

a reasonable value for the average temperature of the atmosphere

$$T = 273 \text{ K,}$$

and

$$g = 9.8\,\frac{\text{newtons}}{\text{kg}}$$

The characteristic height comes out

$$\frac{kT}{mg} = \frac{RT}{Mg} = 7.2 \times 10^3 \text{ m} = 7.2 \text{ kilometers}$$

EXERCISE 8 Verify. This is good practice in keeping units straight. (HINT: You will need to convert the 32 grams to 0.032 kg in order to work in MKS units.)

In the next section this characteristic height for oxygen gas on earth is related to the quantity $h_{1/2}$, which is the altitude change over which the pressure drops by half. If you prefer to skip all that mathematics, the result is in Eq. 7.

Evaluation[9] of $h_{1/2}$

If it takes n layers of thickness d to decrease the pressure by one-half, then

$$h_{1/2} = nd \qquad \text{(Eq. 4)}$$

If the pressure decreases by the fraction f in each such d-layer, then each d-

---

[9] This section is a bit mathematical. In particular, it uses logarithms and a limiting process using successive approximations.

layer has a pressure that is a fraction $1-f$ of the pressure in the d-layer below it. So the relation between the numbers f and n is

$$(1 - f)^n = \tfrac{1}{2}. \qquad \text{(Eq. 5)}$$

What that equation says is, you have to multiply the pressure times $1-f$ every time you go up a height d. You have to do that n times to get from sea level up to the half-value altitude. Remember, both f and n depend on the layer thickness d that you chose. If you take d thin enough, the fraction f is proportional to d, while n is inversely proportional to d:

$$\begin{aligned} f &= \frac{mg}{kT}\,d && \text{from Eq. 3} \\ n &= \frac{h_{1/2}}{d} && \text{definition of } n\text{, from Eq. 4} \end{aligned} \qquad \text{(Eq. 6)}$$

To be sure, you have to take a layer thickness d small enough so that the pressure changes very little in each d-layer. Equation 2 does not have a well-defined meaning if the pressure changes appreciably within the d-layer. That means the fractional change f in each d-layer has to be small and the number ($= n$) of d-layers that it takes to make up a half-value layer is large. If you look at the pressure-altitude curve, small layer-thickness d means that the curve can be approximated by a straight line over the layer thickness d.

Our technique will be to choose a reasonably small value of d and find the corresponding fraction f using Eq. 6. For the value of f we then solve Eq. 5 for n. Equation 4 then gives us $h_{1/2}$. What may be unfamiliar to you is solving an equation in which the unknown is an exponent.

Here are the steps. Use Eq. 6 to substitute for f and n in Eq. 5.

$$\left(1 - \frac{mg}{kT}\,d\right)^{h_{1/2}/d} = \tfrac{1}{2}$$

Take the logarithm of both sides,

$$\frac{h_{1/2}}{d}\,\ell\text{n}\left(1 - \frac{mg}{kT}\,d\right) = \ell\text{n}\,\tfrac{1}{2},$$

and choose a sufficiently small value of d. For example, you might try taking $mgd/kT = 0.001$ and look up the logarithm to base 10 of 1-0.001 ($= 0.999$) using a log table or a fancy calculator. You get -0.000434, good to six decimal places, which is three significant figures. Log $\frac{1}{2}$ is -0.301, good to three figures. Solve for $h_{1/2}$, noting that $d = 0.001\ kT/mg$:

$$\begin{aligned} h_{1/2} &= \left(\frac{0.301}{0.000434}\right) d \\ &= \left(\frac{0.301}{0.000434}\right) \times \left(0.001\,\frac{kT}{mg}\right) \\ &= 0.693\,\frac{kT}{mg} \end{aligned} \qquad \text{(Eq. 7)}$$

EXERCISE 9 Repeat this arithmetic for a different value of d and see that you get almost the same result. The smaller you take d, the more accurate your answer becomes. If you can't decide, try $d = 0.0001\ kT/mg$ first, then a still smaller value.

In the Appendix, our result (Eq. 7) is derived by "solving" a differential equation. That is just a shortcut for students who are familiar with exponential functions and don't need the practice with the numerical approach.

For oxygen gas, we had the result $kT/mg = 7.2$ km for the characteristic length scale. That gives a half-value layer thickness

$$h_{1/2} = (0.693)(7.2\ \text{km}) = 5.0 \times 10^3\ \text{m}$$

for oxygen. We now have the answer to our question about the pressure at 10,000-meter altitude. For oxygen, 10,000 meters is twice $h_{1/2}$, so the oxygen pressure is down to 25% of its sea-level value. No wonder you need an oxygen mask to breathe up there. Incidentally, Mount Everest, the highest mountain on earth, is just short of 9000 meters. ■

6.7 ESSAY: Mountain Climbing*

If you have ever hiked in mountains above 3000 meters, you may have felt short of breath for the first day or two. Many people, when they spend their first night at high altitudes, experience nausea and have trouble falling asleep. But 3000 meters is only three-fifths of the half-pressure altitude $h_{1/2}$ for oxygen. So the oxygen pressure should be

$$(\tfrac{1}{2})^{0.6} = 0.66$$

of its sea-level value, down only 34%. Wouldn't your respiratory system simply adjust to this slight lack of oxygen by faster breathing? Indeed it does; but it takes a matter of hours, not just minutes. You might wonder why the body is built that way. It responds slowly to changes in oxygen level, rapidly to changes in carbon dioxide level.

When we exercise, we need more oxygen, and as we get "out of breath," we find ourselves panting—breathing rapidly. The receptors that govern the breathing rate sense the concentration of carbon dioxide in the blood. The more CO_2, the faster we breathe. Since the burning of carbohydrates makes CO_2 as a waste product, that is a fine regulatory mechanism for us to have. But at high altitudes the carbon dioxide pressure is down by a larger percentage than the oxygen pressure, resulting in lower blood levels of CO_2, especially during rest. Here is the calculation:

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*This section is optional and can be omitted without loss of continuity.

The molecular weight of CO_2 is 44 ($= 12 + 2 \times 16$). This makes the half-value altitude for carbon dioxide

$$h_{1/2} = 0.693 \frac{kT}{mg} = 3.65 \text{ kilometers}$$

EXERCISE 10 Verify!

At an altitude of 3000 meters, or 0.83 $h_{1/2}$, the partial pressure of carbon dioxide is

$$(\tfrac{1}{2})^{0.83} = 0.57$$

of its sea-level value—down 43%.

If you experience nausea during a high-altitude hike, a good trick is to breathe into a bag. It sounds quite backward, since there is even less oxygen pressure inside the bag than in the outside air, once you have exhaled into it. But your exhaling enhances the carbon dioxide pressure inside. In just a few seconds, incipient nausea will disappear and your breathing rate will pick up. WARNING: Above 4000 meters you will probably need an oxygen tank.

Earache

Driving on mountain roads, it can happen that you lose altitude rapidly. Within the space of a few minutes, the ambient pressure around you can increase markedly. Suppose that you don't keep open the air passages into your ears, the Eustachian tubes, during your descent. Then you may feel the excess outside pressure bending your ear drums inward. It can be very painful. Tricks for preventing it include yawning or frequent swallowing, to let air in through your Eustachian tubes. You can also learn to "pop" your ears. Now you know why some people—especially people with colds—complain of earache in an airplane coming in for landing.

EXERCISE 11. Calculate the pressure difference between the middle ear and the outside if the middle ear is still at the 1500-meter pressure and the outside is at sea-level pressure. Assume a molecular weight of 29 for air.

NOTE: The cabin is pressurized in commercial aircraft. The usual practice is to pump it up to the pressure at 1500 meters altitude while the craft is at higher altitudes.

APPENDIX TO CHAPTER 6: ALTERNATE DERIVATION OF THE LAW OF ATMOSPHERES.

Solution of a Differential Equation

Equation 2 says that a rise in altitude of Δz corresponds to a change in pressure of

$$\Delta p = \frac{-p}{kT} mg \, \Delta z$$

Letting Δz go to zero gives an equation for the derivative of pressure with respect to altitude z:

$$\frac{dp}{dz} = \frac{-mg}{kT} p \qquad \text{(Eq. A.1)}$$

By scaling the units of z to make them dimensionless,

$$\xi = \frac{mg}{kT} z$$

this differential equation can be put in the form

$$\frac{dp}{d\xi} = -p \qquad \text{(Eq. A.2)}$$

Its solution is conventionally written

$$p(\xi) = p(0)e^{-\xi} \qquad \text{(Eq. A.3)}$$

where e is the base of the natural logarithms 2.718. Saying it in words instead of algebraic symbols, this says that the pressure falls exponentially with increasing altitude, decreasing to $1/e$ (= 0.368) of its value in a height kT/mg.

Suppose that you wanted to plot the graph of Fig. 6.7 so that it became a straight line. You would plot the logarithm of the pressure versus the altitude; that is called a semilogarithmic plot. To see how that works, take the log of both sides of the equation:

$$\ell\text{n}\, p(\xi) = \ell\text{n}\, p(0) - \xi$$

or

$$\ell\text{n}\, p(z) = \ell\text{n}\, p(0) - \frac{mgz}{kT}$$

The $\log_e$ changes by unity when the altitude z changes by kT/mg. Perhaps logarithms to base 10 are more familiar:

$$2.3 \log p(z) = 2.3 \log p(0) - \frac{mgz}{kT}$$

where

$$\ell n\ 10 = \frac{1}{\log e} = 2.3$$

is the ratio of the natural to the common logarithm.

You will recall the feature of the logarithmic scale, that a given interval on the log scale corresponds to a given ratio of the arguments. For example, if the common logarithms of two numbers differ by 1, the numbers are in a ratio 10:1. How much do the logarithms of two numbers differ if their ratio is 2:1? [Answer: log 2 − log 1 = 0.301] So if the pressure is to fall by one-half, the dimensionless quantity mgz/kT has to increase by

$$(0.301)(2.3) = 0.693$$

Repeat: The altitude change in which the pressure falls to half is 0.693 kT/mg. That is the height we have called $h_{1/2}$.

SUMMARY

One liter of water weighs 1 kilogram.

Water is not very compressible; air is.

Water does not expand very much when heated; air does.

The density of the human body is roughly that of water.

Archimedes' Principle asserts that the buoyant force exerted on an object by a fluid is equal to the weight of fluid the object displaces.

The fraction of a floating object that is submerged is equal to its specific gravity.

The pressure at the bottom of a column of fluid is proportional to its "head." So it is conventional to measure pressure in units like "centimeters of water" or "millimeters of mercury."

One atmosphere is defined as 760 mm of mercury (also pronounced 760 torr). This comes to 1029 cm of water, or 1.009 × 10^5 newtons/m^2. Newtons per square meter are also called pascals.

Some of the dangers encountered in deep-sea diving are squeeze, over-pressure in the lung, air embolism, and bends.

Henry's Law states that the equilibrium concentration of a dissolved gas is proportional to the partial pressure of that gas above the liquid.

The Law of Atmospheres states that the pressure of the air falls off exponentially with altitude.

The half-pressure altitude for each constituent of the air is inversely proportional to its molecular weight.

Breathing rate is regulated by the carbon-dioxide level in the blood.

Yawning or swallowing help keep your Eustachian tubes unblocked while ambient pressure is rising, and can prevent earache.

KEY TERMS

Density Mass per unit volume. Sometimes the term *mass density* is used, to prevent confusing it with *number density*. The density of water is 1 gram per cubic centimeter.

Number density Number of molecules per unit volume. At 1 atmosphere and 273K, the number density is $2.7 \times 10^{19}/\text{cm}^3$.

Specific gravity Density relative to that of water. A dimensionless measure of density. The specific gravity of water is 1, by definition. Objects whose specific gravity is greater than 1 do not float in water.

Compressibility Fractional decrease in volume per unit increase in pressure. The compressibility of air is 0.00001 per pascal. The compressibility of water is 5×10^{-10} per pascal.

Thermal expansion coefficient Also called the volume coefficient of thermal expansion. The fractional increase in volume per unit increase in temperature. The thermal expansion coefficient of air is 0.004 per degree C.

Archimedes' Principle The buoyant force on an object placed in a fluid is equal to the weight of fluid it displaces.

Pascal's Principle The pressure exerted by a fluid is the same in all directions.

Pressure Compressive stress on the walls of the container. Force per unit area exerted normal to the wall. The SI unit of pressure is the pascal.

Pascal Newton per square meter. Unit of pressure.

Atmosphere Unit of pressure. 1 atmosphere = 1.009×10^5 pascals = 760 mm of mercury.

Gram weight Unit of force equal to the weight of 1 gram = 1 gram × g. The acceleration of gravity g is, of course, not the same in all places. Here on earth, set $g = 9.8$ newtons per kilogram.

Centimeters of water Unit of pressure. 1 cm of water = 980 dynes per cm^2 = 98 newtons per m^2.

Torr Millimeter of mercury. 760 torr = 1 atmosphere. 1 torr = 1328 pascals.

Sphygmomanometer Pressure cuff. Device for measuring blood pressure. The cuff is pumped up with air until blood flow is cut off in the artery being measured. As the air is let out, the sound of turbulent flow is first heard for part of the cardiac cycle at the systolic pressure, then for all of the cycle at the diastolic pressure. What is read is the head of mercury.

Barometer Device for measuring the ambient air pressure. The standard laboratory instrument is the mercury barometer.

Torricellian vacuum The empty space in the glass tube above the column of mercury in the barometer.

Boyle's Law The density of a gas is directly proportional to the pressure, other things being equal (temperature, chemical composition). For a constant amount of gas, this is simply written pV = constant.

Ideal gas A gas obeying Boyle's Law. A model of a gas in which the size of the molecules and all intermolecular forces are neglected.

Ideal gas law $pV = NkT$. Here p = pressure, V = volume, N = number of molecules, k = Boltzmann's constant, and T = absolute temperature.

Boltzmann's constant $k = 1.38 \times 10^{-23}$ joules per Kelvin. Used to convert temperature units to energy units.

Absolute temperature Also called ideal-gas temperature. A measure of the average kinetic energy per molecule of ideal gas. When temperature difference is measured in degrees Celsius, the unit of absolute temperature is the Kelvin. Conversion: 0° C = 273.15 K.

Absolute zero The lowest possible temperature. The temperature at which the kinetic energy of the molecules would extrapolate to zero.

Third Law of Thermodynamics Absolute zero temperature can not be attained in a finite number of cooling steps. For the First and Second Laws see Chapters 8 and 9.

Avogadro's Law Equal volumes of gas contain the same number of molecules at given temperature and pressure, regardless of the chemical composition.

Avogadro's Number 6.02×10^{23} = the number of molecules in a mole.

Mole Gram molecular weight. Amount of a substance whose mass in grams is numerically equal to its molecular weight. A mole of oxygen is 32 grams of oxygen.

Ideal gas constant Commonly called R, equal to 8.314 joules per Kelvin per mole. The ideal gas law for one mole of gas may be written $pV/T = \text{constant} = R$.

Kinetic theory A mathematical model in which the laws of mechanics are applied to idealized "molecules" in order to derive approximate laws for the behavior of gases and liquids.

Fluid Gas or liquid.

Isotropic The same in all directions. Gases and liquids are usually isotropic; crystalline solids often are not.

Skin diving Going several meters down in the water without any breathing apparatus.

Scuba Self-contained underwater breathing apparatus. Includes an air tank, a regulator, and a breathing tube.

Eustachian tube Air passage connecting the middle ear with the throat.

Gauge pressure Pressure above atmospheric pressure. This is the pressure read on a tire gauge, for example. When the gauge pressure is 2 atmospheres, the absolute pressure is 3 atmospheres.

Squeeze If a body cavity (middle ear, lung) is at a pressure below the ambient pressure, the walls of that cavity tend to collapse inward.

Ambient "Of the surroundings."

Henry's Law The equilibrium concentration of a gas dissolved in a liquid is proportional to the partial pressure of that gas above the liquid.

Bends Decompression sickness.

Decompression sickness A pathology brought on by coming up too fast after a deep or long dive. Nitrogen dissolved in the blood comes out of solution as the pressure is lowered and forms bubbles in the blood vessels.

Supersaturated A solution in which the concentration of solvent is greater than the equilibrium concentration at that pressure and temperature. Supersaturation is an unstable state. A supersaturated solution of a gas tends to nucleate bubbles.

Nucleation Birth of a droplet or bubble from a "nucleus" of just a few molecules sticking together.

Implosion Sudden caving in of the walls of a container in which the pressure is much less than outside. The opposite of an explosion.

Exponential decay Equal intervals on the abscissa correspond to the same percentage drop of the ordinate.

Law of Atmospheres Atmospheric pressure decays exponentially as you go up.

Half-value layer The thickness of atmosphere in which the pressure drops by half.

REVIEW QUESTIONS

- The text says that 1 liter of water has a mass of 1 kg. That is true at 1 atm ambient pressure and 4°C. How many percent off is it at 20°C, given that the thermal expansion coefficient of water is 0.001% per degree C.?
- If your specific gravity is 1.01 and your mass is 65 kg, what is your volume? If you increase your volume by 2 liters by taking a deep breath, what does your specific gravity become?
- Use Archimedes' Principle to prove that the fraction of a floating object that sticks up above the water is the same if the object is turned upside down.
- In the formula $p = \rho g h$, what value would you put in for ρ (Greek rho) to find the pressure at the bottom of a swimming pool?
- If the swimming pool is 4 meters deep, would it be correct to say that the gauge pressure at the bottom is 400 cm of water? Why is the absolute pressure at the bottom more like 1430 cm of water?
- Describe how to convert from centimeters of water to millimeters of mercury. Why are both of these pressure units in use in hospital surroundings?
- Boyle's Law may be stated, "pressure is proportional to density," or else "pressure is inversely proportional to volume." Explain why those two statements are equivalent.
- The kinetic-theory derivation of the ideal-gas law assumes that molecular size is negligible. Explain how such an obviously nonsensical assumption leads to such a quantitatively accurate law.
- In what pressure and temperature ranges does the ideal gas law break down? Which of the assumptions of the kinetic-theory model are badly violated there?
- Review the argument that the pressure due to one molecule is proportional to that molecule's kinetic energy. Did you need an isotropy assumption to complete the argument?
- Review the argument that the total pressure is proportional to the number of molecules in the container.
- Where in the derivation of the ideal gas law did we use the assumption that the molecules make collisions only with the walls, not with each other? As you know,

this assumption is grossly violated at all but the lowest gas densities. How can such a bad assumption turn out to result in useful predictions?

- Explain why the value of Boltzmann's constant depends on our convention that there are 100 degrees between the freezing and boiling points of water.
- Acetone is a large molecule. Large molecules occupy more space than small molecules. Yet Avogadro's Law says that a mole of acetone vapor occupies the same volume as a mole of helium. Explain this apparent paradox.
- If you pump a gram of nitrogen into a container containing a gram of oxygen, the pressure in the container more than doubles. If you pump a gram of nitrogen into a container containing a gram of helium, the pressure increases by less than 15%. Explain quantitatively.
- If you are breathing with a scuba tank at a depth of 20 meters, the gauge pressure in the tank has to be at least 2 atmospheres. Argue that in order to go to twice that depth, the gauge pressure in the tank has to be at least double that.
- In swimming deep under water, beginners may experience middle-ear squeeze. Explain why it hurts and how to avoid it.
- Explain why it is important to come up slowly, especially from a very deep dive.
- When you take the cap off a bottle of soda pop, you can tell that the concentration of dissolved carbon dioxide inside is supersaturated. What evidence do you see? Why was it not supersaturated before you took the cap off?
- The weight of the air in a vertical column 1 square inch in cross section is about 14.7 pounds. The lowest 3 miles of that column weigh close to half that. Yet you know that the atmosphere extends a lot more than 6 miles up. Explain.
- In deriving the Law of Atmospheres we said that the fractional change in pressure in climbing through a layer of air of given thickness is inversely proportional to the absolute temperature. Review that argument.
- The half-pressure altitude came out 0.693 kT/mg. Where does that 0.693 come from?

PROBLEMS

✶ **1. Density of Water** The text states that 1 liter of water has a mass of 1 kg.

(a) What is the mass of a volume of water contained inside a cube 1 meter on a side?

(b) How many cubic meters of water have a mass of 10^4 kg?

(c) One mole of water is 18 grams of water. How many liters of water is that?

(d) One mole represents 6.02×10^{23} H_2O molecules. What is the volume per molecule?

(e) What is the side of a cube having that molecular volume?

NOTE: That gives a rough idea of the distance between oxygen atoms.

✶ **2 Density of Concrete** Concrete is about three times as dense as water. Jargon: The specific gravity of concrete is 3.0.

(a) What is the mass of 1000 cm^3 of concrete?

(b) How much volume does 1 kg of concrete occupy?

(c) If a 1000-cm^3 container is half-filled with concrete, then up to the top with water, what is the total mass of the contents?

✶ **3. Specific Gravity** The specific gravity of seawater is about 1.025. That means its density is 1.025 grams/cm^3.

(a) How much does 1 liter of seawater weigh?

(b) How many liters of seawater weigh 1 kg.?

✶✶✶ **Accuracy of Specific-Gravity Measurement** Assume that the difference in density between seawater and pure water is proportional to the salinity (concentration of salt). How accurately would you have to measure the specific gravity to measure the salinity correct to three significant figures? Answer in the form "correct to ______ parts per million."

✶✶✶ **5. Hieron's Crown** A pure gold coin weighing 30 grams displaces 1554 cubic millimeters of water. A pure silver coin weighing 25 grams displaces 2381 mm^3 of water. The crown weighs 523 grams and displaces 29150 mm^3.

(a) Argue that the crown cannot possibly be pure gold.

(b) Assuming that it is a Au–Ag alloy, what fraction of its mass is silver?

(c) What fraction of its volume is silver?

✶ **6. Lung Volume** On inhaling, you might take in 3 liters of air. If your specific gravity is 1.01 before you inhale, what is it after you inhale? Assume a body mass of 75 kg. Repeat the calculation putting in your own weight.

✶ **7. Compressibility** The text states that the compressibility of water is 0.005% per atmosphere. This says that when the pressure increases by 1 atmosphere, the density of the water increases by 5 parts in 100,000.

(a) When the pressure is changed from 1 atm to 1000 atm, what does the density of water become? Express in kg/liter or grams/cm^3. (HINT: Notice that you can hardly tell the difference between a pressure change of 999 atm and 1000 atm.)

(b) If the initial volume of water had been 1 cm^3, what would be the final volume?

(c) Argue that we might just as well have said that a pressure increase of 1 atm. corresponds to a volume decrease by 5 parts in 100,000. (HINT: How accurate is the statement that $1/(1 - 0.00005) = 1 + 0.00005$?)

(d) In the argument of part (c) you assumed that squeezing on the water decreases its volume but leaves its mass unchanged. How do you know that assumption is correct?

✶ **8. Compressibility of Air and Water** Increasing the pressure of air by one-hundredth of an atmosphere increases its density by 1%.

(a) By what percentage is the density of water increased when its pressure is increased by 0.01 atm?

(b) How many atmospheres does it take to increase the density of water by 1%?

✶ **9. Archimedes' Principle** Balsa wood has a specific gravity of 0.15.

(a) What fraction of a block of balsa wood is submerged when it floats?

(b) A 10 cm × 10 cm × 10 cm block is floating upright. How deep down is the bottom surface?

(c) What is the buoyant force on the bottom surface? Express in kilogram-weight, that is, units of kilograms × g.

NOTE: Those old-fashioned units have the advantage here that you don't have to keep multiplying by 9.8 newtons/kilogram (= g).

(d) How heavy a piece of lead would have to be placed on top of this block before it will sink? Again answer in kilograms × g.

✶ **10. Archimedes' Principle** A hollow cylindrical float made of sheet steel has a total volume of 1 m^3 and weighs 25 kg × g. You might think of one of those oil drums they use to make rafts.

(a) What fraction of its volume is submerged?

(b) What volume of water is displaced by the submerged fraction?

(c) How much does that much water weigh?

✶✶ **11. Archimedes' Principle** An oak slab (specific gravity 0.7) has sides 20 cm × 20 cm × 2.5 cm. You can intuit that the stable equilibrium position for floating would be with the large face horizontal.

(a) How deep down in the water is the bottom surface?

(b) What is the total buoyant force on the 20 cm × 20 cm face at that depth?

(c) What is the force per square centimeter?

(d) If this block is placed in a container of water that was previously full to the brim, what minimum volume of water spilled over as the block was carefully floated? Does this answer depend on the shape of the piece of wood? Explain.

✶✶✶ **12. Live and Dead Fish—Archimedes' Principle** Some species of fish can change body volume with the aid of a swim bladder. The fish can compress the gas-filled bladder by muscle contraction in order to sink.
(a) Assume that a certain fish has a specific gravity of unity when its swim bladder occupies 10% of the body volume. By what fraction must the volume of gas in the swim bladder be decreased in order to give the fish a specific gravity of 1.02?

NOTE: When its specific gravity is unity, its density is the same as that of water, so the buoyant force is just equal to its weight. With specific gravity less than unity, the fish will tend to float upward; with specific gravity greater than unity it will tend to sink.

(b) Live fish appear to be in static equilibrium at any depth, that is, they have the same average density as the water they inhabit. Dead fish float to the surface as decay sets in, releasing gas bubbles inside the body of the fish. By what percentage has the body volume been inflated by gas if the specific gravity is 0.985?

✶ **13. Weighing Under Water** The specific gravity of silver is 10.3. This means that a given volume of silver has a mass 10.3 times that of the same mass of water. If you weigh a silver ring under water, what fraction of its "true weight" will the scale register? What happened to the rest of its weight?

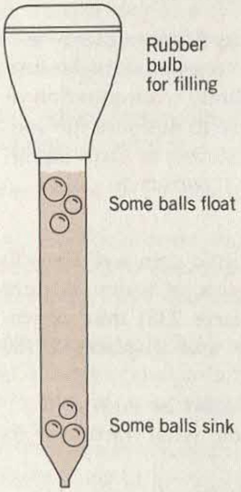
✶✶ **14. The Buoyancy of Air** If you weigh an object in a vacuum instead of in air, the scale reading should be slightly higher. The weight of 1 liter of water would be increased ________%.

✶✶ **15. Weighing a Bag of Water** If you weigh a plastic bag containing 1 liter of water, the scale registers 1 kg. Evidently, the plastic bag is very light. What will the scale register if you perform the weighing under water?

✶✶ **16. Force to Hold You Under Water** Assume a specific gravity of 0.97 and a body mass of 75 kg. How much downward force does it take to hold your body totally submerged? Answer either in kg × *g* or in newtons.

QUESTION FOR FURTHER THOUGHT: When you swim under water, part of the force of your stroke goes into keeping you from floating to the surface. The deeper down you are, the more / less force it takes to keep you from floating up.

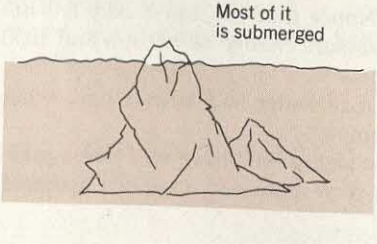
✶✶ **17. Hydrometer** This instrument is a glass tube containing colored balls of various densities (see the figure). The heaviest one floating gives a good reading on the density of the liquid being tested. A hydrometer is used to tell how much antifreeze (ethylene glycol) has been added to the water in your car's radiator. The specific gravity of ethylene glycol is 1.12. What would be the specific gravity of a mixture of 1 quart antifreeze and 1 quart water if the volume of the mixture were exactly 2 quarts? In fact, the specific gravity turns out to be 1.068. This is evidence that the volume of the mixture is about 0.7% less than the sum of the mixing volumes. Verify.



PROBLEM 17. Hydrometer—Color of the last ball that floats tells the specific gravity of the liquid.

✶✶ **18. Icebergs** The metaphor says that there is 9 times as much iceberg under the water as above water (see figure). If the specific gravity of seawater is 1.025, what is the specific gravity of the ice?

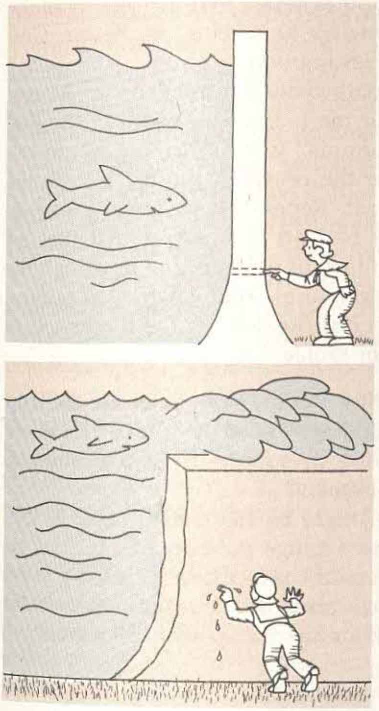
NOTE: Specific gravity of a substance is the ratio of its density to the density of pure water at 4°C. Your answer comes out a bit higher than the textbook specific gravity of ice (0.9168) because the metaphor exaggerates a bit.



PROBLEM 18. Iceberg—Less than one-ninth of its volume is visible above the surface of the sea.

✶ **19. Finger in the Dike** There is a hole 1 meter below the water level! So the water pressure at the level of the hole is 9.8×10^3 newtons per square meter (= 0.98 newtons/cm^2). If the area of the hole is 1 cm^2, the little boy can keep the sea out by putting his finger against the hole (see figure). How hard does he have to push? Answer in newtons.

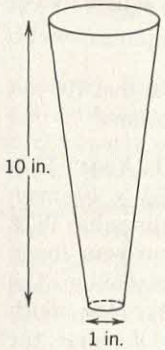
NOTE: The pressure given is gauge pressure.



PROBLEM 19. His finger in the dike saves the country.

✶ **20. Units of Pressure** A 0.5-liter graduated cylinder is 25 cm high. This means that its cross-sectional area is 20 cm^2. (Verify: 25 cm × 20 cm^2 = ________ cm^3.) When the graduate is full, the water inside weighs 0.5 kg × g (= 4.9 newtons). Verify that the pressure on the bottom is 2450 newtons/m^2. It may be expressed simply as "25 cm of water."

✶✶ **21. Sloping Sides** A 0.5-liter German beer glass 25 cm high has sloping sides; that is, it is conical, with a narrow (5 cm^2) base (see figure). When full, the pressure on the bottom is "25 cm of water" or 2450 newtons/m^2. That represents a force of 1.225 newtons, which is only one-quarter the weight of the beer.



PROBLEM 21. German beer glass.

(a) Check that.
(b) The rest of the weight of the $\frac{1}{2}$-liter of beer is evidently borne by the sloping walls. Argue that the walls can exert a net upward force even though the stress on the liquid is everywhere normal to the glass surface.

✶ **22. Toothpaste Tube**
(a) A force of 1 newton applied by 2 cm^2 of thumb to the side of the toothpaste tube raises the pressure of the toothpaste by ________ newtons/m^2.
(b) Express in centimeters of water.
(c) Express in atmospheres.

✶ **23. Bottle Corking** A corking machine is capable of pushing down with a force of 200 newtons to force the cork into the bottle.
(a) If the cork has a cross-sectional area of 2.5 cm^2, to what gauge pressure can its insertion raise the liquid inside the bottle? Express in atmospheres. (HINT: 1 atm = 1.013×10^5 newtons per square meter. Don't forget the cm^2 to m^2 conversion: $(1\text{ cm})^2 = (10^{-2}\text{m})^2 = 10^{-4}\text{m}^2$.)
(b) If the bottom of the bottle has an area of 100 cm^2, how much force must it withstand? Express in tons. (1 ton = 2000 pounds; 1 pound = 4.445 newtons.)
(c) Examine the bottom of a French wine bottle to see how its design allows the glass to withstand such large forces.

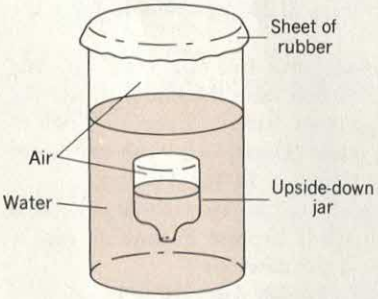
✶✶ **24. Stepping on the Garden Hose**
(a) How much can that raise the pressure inside? Assume an area of contact of 20 cm^2 of shoe sole on a 75-kg gardener. Express in atmospheres.
(b) The hose is attached to the water tap by a threaded fitting 2 cm in diameter. How much additional force does stepping on the hose apply to that fitting? (HINT: Cross-sectional area of the hose?)

✶✶ **25. Weight of Air on the Roof** A small house might cover an area of 700 square feet. If atmospheric

pressure is 15 pounds per square inch, each square foot of the roof has more than a ton of air above it.
(a) Verify.
(b) Why does the architect not have to take that 700-ton load into account in the design of the roof beams?

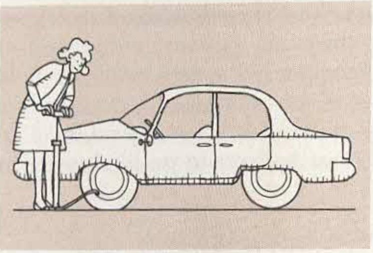
✶✶ **26. Can the Vacuum Hose Pull Your Skin Off?** Remember that 1 atmosphere is 1.013×10^5 newtons/m^2. The rubber hose used for vacuum lines has thick walls, to keep it from collapsing. If you put your finger over the end, you can feel your skin being sucked in. The inside diameter of the hose is 6 millimeters. How much force is on the skin of your finger? (HINT: Of course, the air pressure in the vacuum line is not zero. But even a very primitive vacuum pump can pump down to less than 0.01 atm. The *difference* between the outside and inside pressures is what the skin feels.)

✶✶ **27. Cartesian Diver** The partially filled upside-down jar (the diver) in the diagram is just light enough to float. But when the pressure inside is raised by pressing on the flexible top of the graduated cylinder, the jar becomes heavy enough to sink. Argue that the buoyant force on the jar is roughly proportional to the volume of the air inside it. Why does this decrease when the rubber cover is depressed?



PROBLEM 27. Cartesian diver—when you press on the rubber sheet, the diver descends.

✶✶ **28. Tire Pressure** A certain car weighs 2400 pounds. This means that each tire supports 600 pounds.
(a) If the gauge pressure in each tire is 30 psi (pounds per square inch), how many square inches of each tire have to be in contact with the road?
(b) If the gauge pressure drops to 15 psi, the contact area doubles. Can you tell by looking at the tire? About how much is the tire deformed? This is intended to be a simple challenge to your physical intuition and an exercise in expressing something quantitative in your own words. How might you measure "deformation"? Obviously, there is more than one answer!



PROBLEM 28. One tire is down to 15 psi. It has twice the contact area of the tire at 30 psi.

(c) In solving part (a), you assumed that the rigidity of the rubber contributed negligibly to holding up the car. Give some evidence for this assumption.
(d) Optional (for the mathematically minded). Make reasonable assumptions for the radius and width of the tire, and use the "sagitta formula" $y = \frac{1}{2}s^2/R$ (Sec. 3.2) to calculate how much lower the center of the tire is with the car's weight than without. Repeat for the lower pressure. The axle comes about half an inch lower. All that air makes less than half an inch of difference to how high the car is off the ground (see figure). That is why you can use a bicycle pump to lift a car! (Remember the discussion of mechanical advantage in Problem 4.2.)

✶✶ **29. How Do They Pump Tap Water to the Top Floor?** Tall office buildings need to have elaborate plumbing. Imagine trying to supply the whole building using a pump in the basement!
(a) A 50-story building might be 150 meters high. A 150-meter head of water has a gauge pressure of ________ atmospheres. Not only would conventional plumbing burst or leak water under such pressures, but the water would come out of a ground-floor tap at a speed of 55 meters per second!
(b) Verify. Remember, such a jet, if pointed straight up, would shoot 150 meters high.
(c) QUESTION FOR FURTHER THOUGHT: If you wanted to have the same water pressure in all the bathrooms, every floor would have to have its own water pump. If each pump is designed to lift water just about 3 meters (i.e., one floor), what happens to the bathroom on the top floor if any one of the lower pumps fails? That design of a pumping system would be called a "series connection."
(d) In a system with "parallel connection" the failure of one pump would not affect the floors above. Why would you expect such a system to be more expensive than the series type?

✶ **30. Blood Pressure**
(a) Argue that when you are standing up, the blood pressure in your toes is about one-fifth of an atmosphere

greater than in your brain. (HINT: What value do you use for the specific gravity of blood?)

(b) Argue that it is the other way around while you are standing on your head.

✶✶✶(c) QUESTION FOR FURTHER THOUGHT: In a blood-pressure measurement using a sphygmomanometer (pressure cuff), does it matter whether your arm is raised or lowered?

✶✶✶ **31. Blood Pressure in the Swimming Pool** Blood pressure is conventionally expressed as two numbers. "Normal" might be "120 over 80." The 120 is called *systolic;* the 80 is called *diastolic.* The two values correspond roughly to the maximum and minimum values during the cardiac cycle. The units are millimeters of mercury.

(a) Express 80 mm Hg in "centimeters of water."

(b) An argument is proposed that some of your smaller blood vessels might collapse while you are swimming. Your leg might be more than a meter below the surface, being squeezed by the surrounding water. But your face is above the surface so that the pressure in your circulatory system cannot be higher than usual. The water pressure outside your leg can easily be greater than the quoted "80 mm Hg." If the pressure outside the blood vessel is greater than the pressure inside, it will tend to collapse. What is the logical flaw in the arument? (HINT: The argument *is* fallacious. There is absolutely no danger of blood-vessel collapse due to immersion in water.)

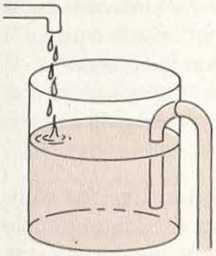
✶✶ **32. Inverted Bottle** Here is a parlor trick best done in the kitchen. Put a card or piece of wax paper over the mouth of a full bottle. Then carefully turn the bottle upside down. Because the pressure of the air in the room is greater than the pressure of the water in the bottle, the water stays in! Do the computation to prove the inequality for an ordinary-size bottle.

NOTE: Surface tension does contribute to make this trick work.

✶✶ **33. Warm-Water Heating** In small houses, heating systems used to be built that used gravity to circulate warm water from the furnace through the radiators. Suppose that the highest radiator is 10 meters above the furnace. The average temperature on the warmer side (pipes leading from furnace to radiator) is 10°C higher than on the colder side (pipes leading from radiator to furnace). How much pressure difference is sustaining the flow of water? (HINT: How much less dense is warm water than cold water?)

NOTE: To get fast enough flow with such a small pressure difference requires large pipes.

✶✶ **34. Siphon** To rinse laboratory glassware repeatedly, a syphon arrangement is often used. Water flows continuously into the rinsing bucket from the tap. When the level rises above the bend in the tube, the entire tube fills with water. The pressure difference between the two ends of the tube now sustains a flow in the tube, which finally empties the bucket. Then the cycle starts over. Explain how the period of the cycle (the repetition time) can be adjusted by regulating the flow from the tap.



PROBLEM 34. Syphon.

✶✶ **35. Capillary Rise** Surface tension will keep a liquid from running out the bottom of a narrow tube. The water "likes" to wet the glass. The force with which the glass pulls on the water is proportional to the length of the water-to-glass-to-air interface.

Call the surface tension force per unit length of contact T. That curve of contact has a length $2\pi r$. If the water surface makes an angle θ with the glass, then the force supporting the weight of the column of water is $2\pi T \cos\theta$. In equilibrium, a column of height h can be supported:

$$2\pi r T \cos\theta = \rho g \, \pi r^2 h$$

(a) In a glass capillary 0.6 mm in diameter the water rises to a height of 4 cm. How high will water rise in a 0.3 –mm capillary?

(b) Under what conditions can the equation

$$T = \tfrac{1}{2}\rho g r h$$

be used to determine the surface tension of a liquid by measuring its capillary rise?

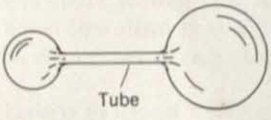
✶ **36. Soap Bubble** The excess pressure (gauge pressure) inside a soap bubble of radius r is given by the formula $p = 4T/r$, where the surface tension of soap solution is $T = 0.025$ newtons per meter.

(a) What is the gauge pressure inside a soap bubble 6 cm in diameter?

(b) What is the biggest soap bubble you ever blew?
(c) Was its size limited by your ability to blow hard or by the instability of the skin?

✱✱✱ 37. **Soap Bubble, Derivation of $p = 4T/r$** Think of the pressure as tending to make the bubble come apart, say at its equator. Each hemisphere experiences a net push of $p \times \pi r^2$. The skin pulls back with a force equal to T times twice the length of the equator, $T \times 4\pi r$.
(a) In equilibrium, those forces add to zero. Solve for p.
(b) Why did we take the area to be πr^2, the area of a circle, instead of $2\pi r^2$, the area of a hemisphere? (HINT: The pressure gives rise to force perpendicular to the surface.)
(c) The inside of the bubble contributes a surface-tension force equal to T times the length of the "inside equator"; the outside of the bubble contributes a force equal to T times the length of the "outside equator." Why are we justified in setting those equal and simply saying "twice the length of the equator"?

✱✱ 38. **Two Connected Soap Bubbles** Use the equilibrium formula $p = 4T/R$ to show that the larger bubble in the figure will grow at the expense of the smaller one (see figure).



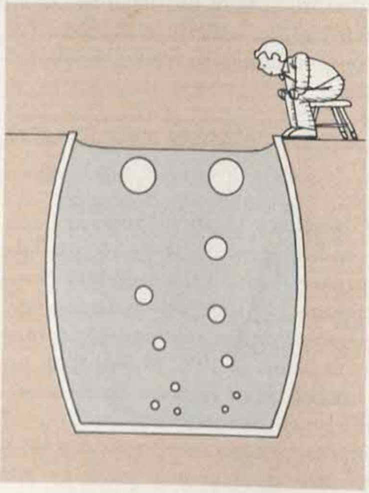
PROBLEM 38. Two connected soap bubbles.

✱ 39. **Boyle's Law** Doubling the pressure means halving the volume.
(a) Does this imply that halving the pressure doubles the volume?
(b) Suppose that you have air at 1 atmosphere and it occupies a volume of 1 liter. How much pressure do you have to apply to squeeze it into a volume of 0.25 liter?
(c) If it were allowed to expand into a 4-liter volume, what would be the pressure?

✱✱ 40. **Boyle's Law and Balloons** A helium balloon is filled at ground level, so the pressure inside is roughly 1 atm. It is released and gains altitude. A few kilometers up, the air pressure is only half an atmosphere. So the pressure of the helium is also roughly 0.5 atm.
(a) What happens to the volume of the balloon?
(b) How would you design a balloon to take account of such volume changes? What would happen to a toy balloon?
(c) We kept saying that the pressure inside is "roughly" the same as the pressure outside. That means we neglected the tension of the balloon material. Estimate the excess pressure that tension can provide in a toy balloon and express it as a fraction of an atmosphere. Be guided by the fact that 1 atm = 14 pounds per square inch. Back up your estimate with a sentence or two about how strong that thin rubber sheet might be.
(d) Our calculation has assumed that the temperature way up there is the same as down near the ground. Actually, it is likely to be colder. What effect would this have on the answer to part (a)?

✱ 41. **Boyle's Law** When you are skin diving at a depth of 2 meters, the volume of your lungs is ________% less than at the surface. Assume that you have not exhaled on the way down. (HINT: The answer is *not* 20%. What is 1 divided by 1.2?)

✱✱ 42. **Boyle's Beer** Boyle's father was a brewer. The story is that the young Boyle arrived at his discovery by watching the bubbles grow as they rose to the surface in the large beer vats. We now ask the question, how much bigger did the bubbles actually get in Boyle's father's beer vats? Suppose that the vat is 2 meters deep. A difference in head of 2 meters is a pressure difference of one-fifth of an atmosphere. (Approximate 10 meters = 1 atm, beer having roughly the density of water.) So the pressure down at the bottom is 1.2 atm.
(a) If a bubble forms at the bottom and starts to float up when it has a diameter of 1 millimeter, what would be its diameter when it gets to the top? Remember, the volume of a sphere is $(\frac{4}{3})\pi r^3$.



PROBLEM 42. Boyle's beer—of course, the cartoon exaggerates the growth of the bubbles. To double in volume, they would have to rise through 30 feet, according to Boyle's law. But maybe some small bubbles coalesce into big ones.

(b) Even in a tall glass, the bubbles do seem to grow visibly as they rise to the top (see figure). Someone suggests that what may be happening is that gas (CO_2) goes out of solution most easily at a surface, for example, the surface of a bubble. Explain how that phenomenon would explain the growth of the rising bubbles.
(c) Attempt to explain why an open bottle of beer goes flat, while a capped one remains carbonated.

NOTE: Don't just say the gas can get out of the open bottle. . . .

✶ **43. Boyle's Law**
(a) By increasing the pressure by 1%, you decrease the volume by ________%. (HINT: Increasing pressure by 1% means multiplying it times 1.01.)
(b) By decreasing the pressure by 1%, you increase the volume by ________%. (HINT: Decreasing pressure by 1% means multiplying it times 0.99.)
(c) By increasing the pressure by 0.1%, you decrease the volume by ________%.
(d) By increasing the pressure by 2%, you decrease the volume by ________%.
(e) By increasing the pressure by 50%, you decrease the volume by ________%.
(f) By decreasing the pressure by 50% (i.e., cutting it in half), you increase the volume by ________%.
(g) By increasing the pressure by 100% (i.e., doubling it), you decrease the volume by ________%.
(h) By increasing the pressure by 25%, you decrease the volume by ________%.
(i) By decreasing the pressure by 25%, you increase the volume by ________%.

✶ **44. Ideal Gas Law** A bicycle tire might have a total volume of 2 liters. So the mass of air inside can be calculated, assuming the molecular weight of air to be 29.
(a) Verify that the answer is 2.59 grams at 1-atm pressure. You needed to remember that 1 mole of gas occupies 22.4 liters at S.T.P.
(b) What is the mass of air inside when the tire is pumped up so the pressure is 4 atm? Note that this means 3 atm above "atmospheric," or 3-atm "gauge pressure." The tire gauge reads zero at atmospheric pressure, of course.
(c) Can you tell that the inflated tire is heavier by lifting the bicycle? Support your answer by one convincing sentence.

✶ **45. Why the Ideal Gas Law Is So Good** The density of liquid helium is 0.15 grams per cubic centimeter. We are going to use that number to estimate what fraction of the volume of helium *gas* is empty space, what fraction is "occupied." Assume that the atoms in the liquid are packed tightly together. That is surely an extreme assumption but will serve for an estimate. Suppose that the atoms are spheres and calculate what fraction of the volume of helium gas is actually occupied by the atoms at S.T.P. (HINT: That is a density of 4 grams/22.4 liters. If you don't like spheres, make the atoms cubes. The arithmetic is easier that way. The answer is different, but it still makes the point that the occupied fraction is very small. That is why the "ideal gas" assumption, that the atoms are "points," is so good.)

✶ **46. Put It Away Hot?** Fresh from the dishwasher, the jar is at 67°C (= 340 K). So is the air inside. You put on an air-tight cover. Then it cools to 300 K.
(a) At what pressure is the air inside after it has cooled? Express as a fraction of an atmosphere.
(b) Assume atmospheric pressure = 10^5 N/m^2. If the jar lid has a top area of 100 cm^2, how much force is needed to pry it off? (HINT: The air above is pushing down with a force of 10^5 N/m^2 $\times$ 10^{-2} m^2 = 10^3 N. The air inside is pushing up with a force = ________.)

✶ **47. Ideal Gas Law** Prove that the density of an ideal gas (grams per cm^3) at a given pressure and temperature is directly proportional to the molecular weight.

✶✶ **48. Ideal Gas Law**
(a) Prove that the number density of an ideal gas (number of molecules per cm^3) at a given pressure and temperature is independent of molecular weight.
(b) Is this also true for a mixture of gases? Yes or No? Defend your answer cogently.

✶✶ **49. Perfume Atomizer in the Airplane Luggage** Explain why the atomizer might leak if your suitcase is put in the baggage hold but not if you carry it aboard and keep it with you in the cabin.

Perfume atomizer.

★★ **50. Slow Burning of Hydrogen** The burning of hydrogen to form water, $2\ H_2 + O_2 \rightarrow 2\ H_2O$, need not occur explosively. By adsorbing the gases on certain metal surfaces it can occur isothermally, that is, at constant temperature. What happens to the pressure of a 2:1 mixture of hydrogen and oxygen as it is converted to water vapor? Answer in the form "It is reduced / increased by ________%."

★ **51. Steam Engine** By what percentage is the pressure of an ideal gas (water vapor) increased in being heated from 293 to 373 K, that is, from room temperature to the boiling point?

★ **52. Density of the Air** On a hot day, the air temperature might go from 293 up to 303 K. Assume that the air pressure remains constant. By what percentage does the density of the air increase? / decrease?

★ **53. Humid Air** Ordinary air is about four-fifths nitrogen and one-fifth oxygen. That means 1 molecule in 5 is oxygen. By doing a weighted average of the respective molecular weights 28 and 32, we obtain an effective molecular weight of 29 for air.

(a) Verify.

(b) On a wet day, the air might have as much as 1 part in 50 of water vapor (molecular weight 18). Comparing at the same temperature and pressure, how much more/less dense is wet air than dry air? Answer in percent.

★★ **54. Helium Balloons** Helium ($M = 4$) is used in balloons for taking scientific apparatus into the upper atmosphere.

(a) How many kilograms of helium are needed in a balloon to lift 1 tonne (1000 kg) of apparatus? Neglect the weight of the plastic skin. Assume S.T.P. ($T = 273$ K, $p = 1$ atm), $M_{\text{air}} = 29$.

(b) Argue that on a hot day you would need more / less / the same amount of helium, but that it would occupy more / less / the same volume.

★★ **55. Kinetic Theory** The speed of sound in air is 330 m/sec at S.T.P. (Standard Temperature and Pressure means 273 K and 1 atmosphere). Since the size of a molecule is so much smaller than the average distance between molecules, that number gives us an order-of-magnitude estimate of average molecular speed. Consider a 1-meter cube of air and concentrate on an N_2 molecule traveling in the x-direction at 330 m/sec. Its mass is 28 grams/6.02×10^{23}.

(a) Calculate the momentum it imparts to the wall when it collides elastically. Why is this twice the momentum it carries?

(b) How long does it take to make its 2-meter round trip between collisions on the same wall? Note that we are neglecting collisions between molecules (ideal gas).

(c) What is the average force that one molecule exerts on that one wall?

(d) Knowing that 6.02×10^{23} molecules of N_2 (that's 28 grams) occupy 22.4 liters at S.T.P., how many molecules are there in 1 m^3?

(e) Suppose that one-third of all the molecules contribute, on the average, to the force on the wall perpendicular to the x-direction. Calculate the pressure (= force/1 m^2) on that wall.

NOTE: You expected to get a pressure of 1 atm. (= 10^5 N/m^2). Your calculation gave you a smaller pressure than that. This is because the speed of sound underestimates the effective molecular speed by a factor $(1.4/3)^{1/2}$.

★★ **56. More Kinetic Theory** We now turn the preceding problem around. You will use your knowledge that, at S.T.P., 1 mole of ideal gas occupies 22.4 liters. You apply this to N_2 gas to calculate the mean kinetic energy of a molecule. Then you set that average kinetic energy equal to $\frac{1}{2}mv^2$ to find the so-called r.m.s. (root mean square) speed. Not surprisingly, this will turn out to be a bit faster than the speed of sound.

(a) Using kinetic theory, we proved that $pV = (\frac{1}{3})Nmv^2$. For N = Avogadro's Number, set $p = 10^5$ newtons/m^2, $V = 22.4 \times 10^{-3}\ m^3$, and $Nm = 28 \times 10^{-3}$ kg, and solve for v.

(b) Where did the $\frac{1}{3}$ in that formula come from?

(c) Why would you expect the speed of sound to be slower than the r.m.s. speed you calculated in part (a)? (HINT: Do not try to justify the factor $(1.4/3)^{1/2}$. That argument requires knowledge beyond the scope of this course.)

★★ **57. Guessing the Speed of Sound** Knowing the speed of sound in air (330 m/sec, average molecular weight = 29), estimate the speed of sound in helium gas (molecular weight = 4). What assumptions do you make?

★★ **58. Brownian Motion** Molecules are too small to see, even with a microscope. So we only know about their random thermal motion by inference. Tiny dust particles, however, consisting of billions of molecules, are big enough to see under a low-power microscope. Yet they are light enough so that the statistical fluctuations resulting from collisions with gas molecules are observable. The unpredictable shivering motion is called *Brownian motion.*

(a) Assume that a Brownian particle of mass 10^{-14} kg is in *thermal equilibrium* with the air in which it is suspended. This means that the average kinetic energy of its random motion is the same as that of the air molecules. Calculate its "r.m.s. speed" at 300 K. (HINT: Set $\frac{1}{2}MV^2$ of the dust particle equal to $\frac{1}{2}mv^2$ of a typical air molecule. Use the kinetic-theory result $\frac{3}{2}kT = \frac{1}{2}mv^2$ and put in the mass of a nitrogen molecule.)

(b) About how big is a 10^{-14}-kg dust particle? Get an order of magnitude only. "Dust" might have a density somewhere between 1 and 5 gm/cm^3. Assume a spherical shape for simplicity, so its volume is $\frac{4}{3}\pi r^3$, and calculate the diameter 2r. Does this seem about right for a particle of cigarette smoke?

(c) What does cigarette smoke look like under the microscope? (HINT: You won't find the answer in the text.)

✶ **59. Deep Down the Water Is Denser** Deep-sea work may be done as much as 1 kilometer below the surface.

(a) What is the pressure (in atmospheres) at that depth?

NOTE: The excess pressure over surface atmospheric pressure is called *gauge* pressure. This is the pressure the filling-station attendant talks about for automobile tires. He usually expresses it in pounds per square inch (14.7 psi = 1 atm). What you want to know is how much *more* pressure than atmospheric your tire has. The *true* pressure inside is 1 atmosphere greater than the gauge pressure.

(b) The compressibility of water is 5×10^{-5} per atmosphere. Repeat: When the pressure is increased by 1 atmosphere, the volume of the water decreases by 1 part in 20,000. (Verify that.) By what percentage has the density of water increased at a depth of 1 km?

(c) In the text it is stated that the pressure increases by atm when you go 1034 cm deeper down in the water. How should that figure be modified 1 km down?

✶✶✶ **60. Popping the Ears** If your Eustachian tubes are clogged with mucus, they can sometimes be unclogged by increasing the pressure in the respiratory system. You hold your nose and "push." The volume of your mouth + throat + bronchi + lungs can be changed by muscular control. The volume of your middle ear is changed only by the rather painful deformation of the ear drum. Argue that your ability to prevent middle-ear squeeze as you skin-dive depends on the respiratory system having such a large volume compared to the middle ear.

✶✶ **61. Scuba Regulator Valve** The design of such a regulator involves quite an impressive bit of engineering. On the one hand, the opening at the valve has to be large enough to allow an adequate flow of air for an active diver. To appreciate this requirement, just try jogging while inhaling through a soda straw! On the other hand, the force on the valve due to air pressure in the cylinder is proportional to the area of the opening. Suppose that the opening is 0.01 square inch in cross-sectional area (diameter just over one-tenth of an inch) and the air in the tank is at 2250 psi.

(a) Calculate the force of the air in the tank on the valve. See Fig. 6.6.

(b) When you start to inhale, an equal and opposite force is needed to open the valve. Yet the sensing diaphragm moves in response to the small pressure difference you create on expanding your lungs to start inhaling. You don't want that pressure difference to be more than 15 or 20 centimeters of water, around one-quarter psi. Otherwise breathing takes too much effort. For 0.25 pounds per square inch of pressure to push with a force of 22.5 pounds, the sensing diaphragm of the regulator would have to have a surface area of 90 square inches[10] (= 22.5 pounds/0.25 psi). Such a diaphragm would be impractically large. So the mechanical coupling between the diaphragm and the valve is designed to have considerable leverage, that is, mechanical advantage. A large displacement of the center of the diaphragm causes a much

PROBLEM 61. Try breathing through a straw. You don't need the swimming pool. . . .

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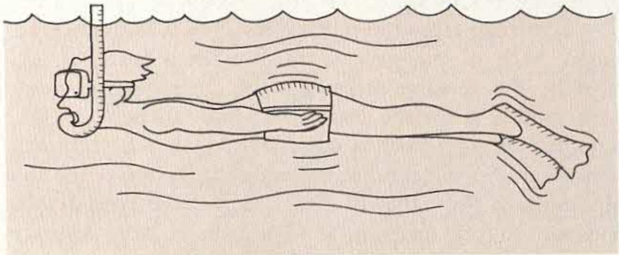
[10] Why this mixing of English and metric units? We chose pounds per square inch (psi) here, because that is what is stamped on the scuba tanks and read on pressure gauges sold in the United States. But we also wanted to make contact with the medical literature on the respiratory system, which talks about pressure in cm H$_2$O.

smaller motion at the valve. On the other hand, the motion at the valve must not be too small, otherwise there is not enough of an opening and the air cannot get through fast enough for comfortable breathing. Try breathing through a straw (see figure). By what fraction can you pinch it off before breathing becomes difficult? Repeat while exercising!

✶✶✶ 62. **Breathing Through a Snorkel** It is fun to look down into the water with goggles while breathing through a J-shaped tube whose end sticks up into the air (see figure). Explain why you could not breathe air at atmospheric pressure if you were several meters below the surface. Which of the following might happen?

1. The breathing tube would collapse.
2. Your diaphragm would not be strong enough to expand your chest cavity.
3. The blood vessels that line the lung would rupture.



PROBLEM 62. Breathing through a snorkel.

✶✶ 63. **Law of Atmospheres: Practice in Arithmetic** A layer of air has thickness d such that the pressure just above it is 0.05% less than the pressure just below it.

(a) How many such "d-layers" would you have to climb through to experience a 1% drop in pressure? (HINT: $(1 - 0.0005)^n = 0.99$.)

(b) A 10% drop in pressure? (HINT: $(1 - 0.0005)^n = 0.9$.)

(c) A 50% drop in pressure? (HINT: The answer is *not* 50 times the answer to part (a).)

✶✶ 64. **Law of Atmospheres: Composition** At sea level, the air is about 21% oxygen ($M = 32$) and 79% nitrogen ($M = 28$). What would you expect the relative percentages to be at an altitude of 10,000 meters?

✶✶ 65. **Half-Value Layer**

(a) By what fraction does the pressure of the atmosphere decrease as you go up 1 meter? (HINT: Use 6.6 Eq. 3.)

(b) Show that this equation would give nonsense if you put in a height difference $h = 7.2$ kilometers.

(c) Remember that 6.6 Eq. 2 makes sense only for a thin layer, since the pressure p has only a vague meaning if it changes appreciably within the layer. Show that a layer is "thin" if its thickness is much less than kT/mg.

(d) Try using 6.6 Eq. 3, 4, and 5 with a d-layer thickness of 5 kilometers to calculate $h_{1/2}$. Again, you get a pretty bad answer for the half-value layer. Now try 1 km; your answer gets closer to the right answer.

(e) Argue that the more accuracy you want in your answer, the thinner you need to take your d-layer.

✶ 66. **Evaluation of $h_{1/2}$**

(a) Use 6.6 Eq. 7 to calculate the half-value altitude for nitrogen gas (molecular weight = 28). Stay with $T = 273$ K for comparison with oxygen ($M = 32$).

(b) By what percentage is $h_{1/2}$ increased on a warm day, with an average atmospheric temperature of 293 K?

✶✶ 67. **Ozone** Ozone is O_3, triatomic oxygen. You can sometimes smell it near electrical machinery that has been making sparks. You may know that ultraviolet light absorbed in air can produce ozone from atmopheric oxygen.

(a) Calculate $h_{1/2}$ for ozone ($M = 48$).

(b) On the basis of your calculation, would you expect to find more ozone high up in the atmosphere or low down near the earth?

(c) The "ozone layer" is near the top of the atmosphere. Explain this fact, using your knowledge that (1) there is ultraviolet (UV) in the sun's radiation and (2) ultraviolet is strongly absorbed by air.

✶✶ 68. **About the Number e** In the Appendix the number e, the base of the natural logarithms, is used without being defined. Here is a numerical procedure for calculating that number, which may be considered a definition: If the fraction f by which the pressure decreases in a thin layer is $1/n$, then the fraction $(1 - f)^n$ by which the pressure decreases over n layers is $1/e$. As in the text, the number n of thin layers has to be large for the meaning of this prescription to become precise, that is, the fractional decrease $1/n$ per layer has to be very small. Repeat: $1/e = (1 - n^{-1})^n$ becomes a more and more exact definition of e the larger the number n becomes.

(a) Try to compute e using $n = 10$.

(b) Repeat using $n = 20$.

(c) If your calculator is sufficiently versatile, try really large values of n, like 100 or 1000. The larger n becomes, the smaller the deviations in the result. The mathematical jargon for this sort of successive-approximation procedure is to talk about the limit as n goes to infinity.

★★ **69. Powers of 2 or Powers of *e*?** Equation 6.6-1 may be written $p(z) = p(0)\,2^{-z/h_{1/2}}$. In the Appendix following Sec. 6.7, it was written $p(z) = p(0)\,e^{-z/h}$, where the characteristic height h ($= kT/mg$) turned out to be related to $h_{1/2}$ according to $h_{1/2} = 0.693\,h$. Show that this should properly be written $h_{1/2} = (\ell n 2)\,h$; that is the number 0.693 is written as a shortcut for the natural logarithm of 2.

★★ **70. Exponential Decay** A property tax charges your bank account one-twentieth of a percent per month.
(a) If you neither deposit nor withdraw and the bank pays no interest, in how many years will half of your money be gone? (HINT: $(1 - 0.0005)^n = \frac{1}{2}$, where n = number of months.
(b) Argue that your balance can be expressed in the form $B(t) = B(0)\,(\frac{1}{2})^{t/\text{half-life}}$.
(c) Show that this is equivalent to $B(t) = B(0)\,e^{-0.006\ t/\text{year}}$ The banker would say your tax rate is six-tenths of a percent per year compounded monthly.

☐ QUESTIONS FOR FURTHER THOUGHT

α. **Density of Muscle and Fat** After short-term weight loss the density of the human body is imperceptibly changed. After long-term weight loss the density is generally increased. Explain. (HINT: Short-term weight changes are usually due to changes in the water content of the body. When you drink a glass of water, what happens to your weight? To your volume? To your density?)

β. **Archimedes' Principle and Surface Tension** The proof of Archimedes' Principle used the implicit assumption that the submerged object experiences the same forces on its surface as did the volume of fluid it displaced. This means that the molecules of the fluid act the same way whether their neighbors are molecules of fluid or molecules of something else (the submerged object). Surface-tension forces are intermolecular forces that violate that assumption: They are species-dependent. For a floating solid, surface tension acts where the liquid–solid interface meets the air. Surface-tension forces are proportional to the *length* of the solid–liquid–gas interface. Pressure forces, on the other hand, are proportional to the *area* of the solid–liquid interface. For very small objects, then, surface tension will become comparable to buoyant force. Argue that for such small objects, Archimedes' Principle becomes only approximate. Which way do surface-tension forces pull? Answer by giving examples.

γ. **How to Weigh a Ship** Instead of saying "This ship weighs 10,000 tons," it is conventional to say "This ship displaces 10,000 tons." Invent a way to "weigh" a ship.

δ. **Liquids in a Spacecraft** Of course, you can't pour water, but you could squeeze it out of a plastic bag or bottle. If you squeeze hard, a jet of water will come out. If you squeeze very gently, the water will bead into drops, because of its surface tension. What shape will the drops assume? If a drop hits a wall, do you think it will bounce off, break up in a splash, or adhere?

ε. **Water Drops on the Car** After you have polished the car, water stands on the shiny surface in small drops. When the surface is dull, water can form a film on it. How does surface tension account for this observation?

ζ. **Surface Tension and "Wetting"** In a narrow capillary tube you can observe the upper surface of a column of water. It is concave upward. In other words, it meets the glass in an acute angle. A column of mercury has a convex upper surface, that is, it meets the glass in an obtuse angle (see figure). Does this tell you anything about the surface tension of mercury? Do your observations of drops of mercury rolling on a flat surface confirm that conclusion?

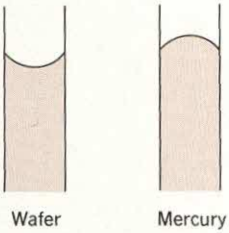
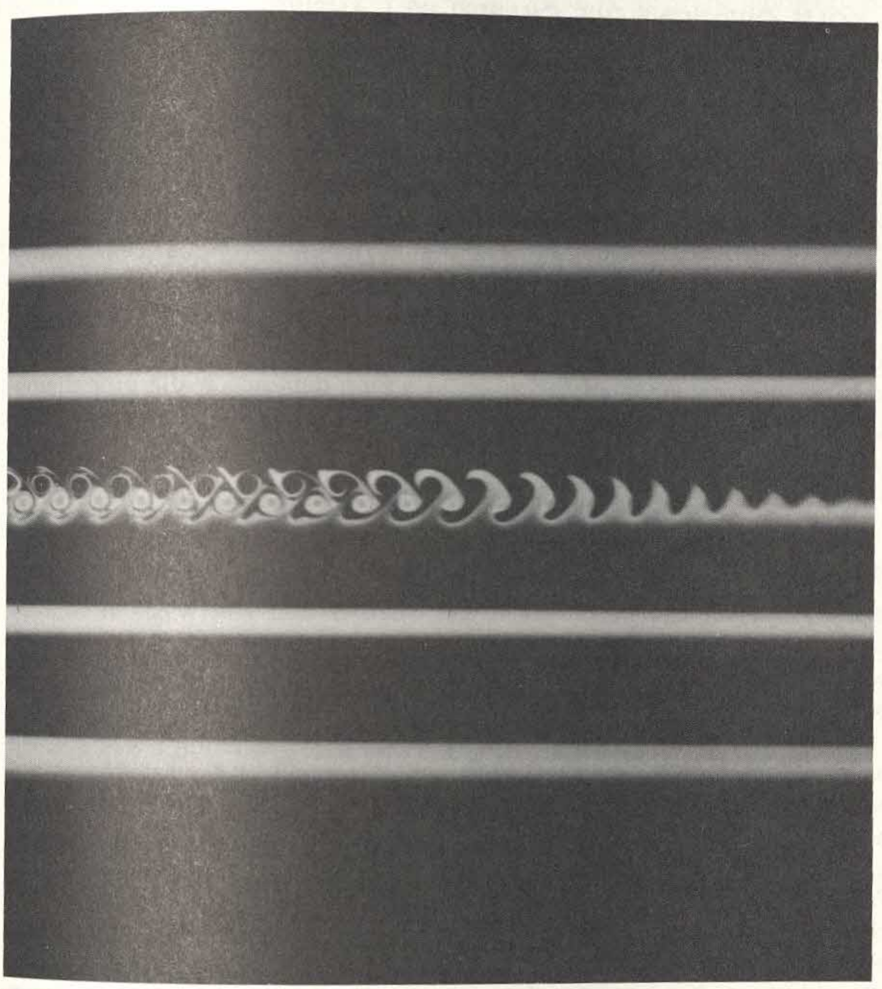


FIGURE 6.8 Surface tension and wetting—water versus mercury.



Vortex patterns in a stream of air flowing past a sharp edge.

7

FLUID FLOW

SOME QUESTIONS THIS CHAPTER WILL ANSWER

Argue that the force with which a gram of water from the shower hits your shoulder is proportional to the speed of the water.

Argue that the force you feel on your shoulder in the shower is proportional to the *square* of the speed of the water.

What is Bernoulli's Principle? Use it to explain that in order to double the speed at which water flows from your tap you have to quadruple the height of your water tower.

What is a *coefficient of viscosity?*

How does the speed of the water in a pipe depend on the pipe diameter?

Out in the open sea, why do long waves travel faster than short waves?

What is a semipermeable membrane? What is its role in osmosis?

7.1 SHOWER

Taking a shower is a good way to get clean. When you wash in running water, the motion of the water helps get the dirt off. Is it the momentum of the water that matters, or the kinetic energy? You probably want to say it's the force.

We are not actually going to answer the question. Be we are going to talk about the momentum and the kinetic energy carried by a fluid stream, and the force it can exert. Those three concepts are familiar to you from earlier chapters. In this chapter, they will be used in dealing with such diverse subjects as blood flow and ocean waves. Chapter 6 (Pressure of Air and Water) used only one of those concepts: force. That chapter could have been called **hydro***stat***ics,** because it is about *stat*ionary fluids. Then this one would be called **hydro***dynamics*, because it is about fluids in motion.

The shower offers us a familiar situation for thinking about the momentum of a fluid. Water comes out of the shower head at high speed. After it hits your shoulder it runs off rather slowly. In the first exercise, we look at the momentum transfer. Notice the technique: We focus on a cubic centimeter of water.

EXERCISE 1

(a) Each jet of a shower might have a cross section of 1 mm^2. If the water moves at 10 m/sec, how much water comes out through each jet per second? Argue that the answer is 10 cm^3/sec.

(b) Each cubic centimeter of that water has a mass of 1 gram ($= 10^{-3}$ kg), hence a momentum of ______ kg m/sec. The rate at which each jet transmits momentum is therefore ______ kg m/sec^2. One way to see that is to realize that it takes one-tenth of a second for each gram (each cm^3) of water to come out.

(c) When the jet hits my shoulder, almost all of its forward momentum is lost, that is, transmitted to me. The rate of momentum transfer per unit time is the force felt by my shoulder. That is ______ kg m/sec^2 or ____ newtons for each jet.[1]

(d) A shower head might have 50 such jets. Would my shoulder feel that as a heavy weight? Is it more than a pound?

EXERCISE 2

(a) A fire-hose nozzle has a cross section of 2 cm^2 and a water velocity of 30 m/sec. How much force would its jet exert if it hit you broadside? Is that more than your own weight? (HINT: Go through steps (a), (b), and (c) of Exercise 1. Note that 6 kg of water comes out each second!)

(b) If the cross section of the jet is doubled, keeping its speed constant, its force on impact is doubled. But if the speed is doubled keeping the cross section constant, not only does that double the number of cm^3 per second of water coming out, but it also doubles the momentum of each cm^3. So the momentum transfer per unit time is multiplied times ______. The force of the jet is quadrupled.

(c) Repeat the argument for any fluid, using the symbols

A = cross-sectional area
v = speed of fluid
ρ = density of fluid
Q = volume of fluid transported per unit time
F = momentum transported per unit time

First, the geometry:

$$Q = Av$$

To see that, think about the volume of a cylinder of fluid emitted in time t. It has a base of area A and a length vt. Next, the dynamics:

$$\text{momentum of unit volume of fluid} = \rho v$$

Unit volume has mass ρ. Finally, the momentum transport:

$$\begin{aligned}\text{momentum per unit time} &= \text{momentum per unit volume}\\ &\quad \times \text{ volume per unit time}\\ F = \rho v Q &= A\rho v^2\end{aligned}$$

Now check your answer to part (a) using that formula. Why can't you take a drink from a fire hose?

EXERCISE 3

Now ask how much kinetic energy the water transports

(a) Consider the shower jet of Exercise 1. It puts out 10 cm^3 of water per second, moving at 10 meters per second. Each cm^3 of water has a mass of 1 gram and a kinetic energy of

---

[1] Our old friend $F = ma$ can be read: Force equals time rate of change of momentum. To make that obvious, remember that acceleration is time rate of change of velocity, and momentum is mass times velocity.

$$\tfrac{1}{2}mv^2 = \tfrac{1}{2}(10^{-3}\text{ kg}) \times (10\frac{\text{m}}{\text{sec}})^2 = \underline{\qquad\quad}\text{ joules.}$$

But with 10 cc coming out each second, the transport of kinetic energy is 0.5 joules/sec. How was the result obtained?

(b) The 50 jets of the shower are pouring energy at my shoulder at the rate of 25 joules per second, or 25 watts. What would happen to that number if the speed of the water were doubled? Please don't say it would double! Remember, the number of cm^3 per second would double, and the kinetic energy of each cm^3 would be multiplied times ______. So the kinetic energy transported per second would be multiplied times ______.

(c) Let u = kinetic energy per unit volume
$= \tfrac{1}{2}\rho v^2$

Then

$$\begin{aligned}\mathcal{P} &= \text{kinetic energy per unit time}\\ &= \text{kinetic energy per unit volume} \times \text{volume per unit time}\\ &= uQ\\ &= \tfrac{1}{2}\rho v^2 Q = \tfrac{1}{2}\rho v^2 Av = \tfrac{1}{2}A\rho v^3\end{aligned}$$

Check to see that this formula agrees with your answer to part (b).

(d) The symbol $\mathcal{P}$ was chosen because kinetic energy transferred per unit time is equal to the power it takes to pump all that water. Calculate that power for the fire hose of Exercise 2.

The force goes as the square of the velocity. The power goes as the cube of the velocity. Both are powerful results and have intrinsic interest. For you, their prime importance may well lie in the confidence they give you in your ability to apply the methods of classical mechanics in new situations.

Let's review what these three exercises have done for us. They have shown us that the concepts of force, momentum, and kinetic energy are easily applied to a continuous medium. Concepts developed by talking about that abstraction, the single particle—or "mass point"—such concepts turn out to be easily adaptable to as shapeless and deformable a thing as a liquid or a gas. We solved two problems:

1. Finding the force exerted by a liquid jet on an object with which it collides inelastically.
2. Finding the power required to pump a liquid jet.

We did not need any principles we did not know before. To do Exercise 1, we used $F = ma$. For Exercise 3 we used the work-energy theorem.

7.2 BERNOULLI'S PRINCIPLE

When you wanted to shoot your water pistol a long way, you squeezed hard on the plunger (Fig. 7.1). More pressure inside means faster stream velocity outside. That can be made quantitive using the concept of potential energy. When the plunger is pushed in on one end, the water comes out of the hole on the other end. The work done by the plunger goes into the kinetic energy of the fluid forced out. Here is some arithmetic:

You are squeezing on the plunger, applying a pressure of 10^4 newtons per m^2. That means 10^4 N/m^2 above atmospheric pressure. If the plunger has a cross-sectional area of 1 cm^2 ($= 10^{-4}$ m^2) you are exerting a force of 1 newton on the water in the cylinder. Push the plunger in 1 cm, and you have done an amount of work given by

$$\text{force} \times \text{distance} = 1 \text{ newton} \times 10^{-2} \text{ m} = 10^{-2} \text{ joules}$$

You have forced 1 cm^3 of water out through the tiny hole at the end, and the work you did had to go into the kinetic energy of that cm^3 of water:

$$\text{kinetic energy} = \tfrac{1}{2}mv^2 = 10^{-2} \text{ joules}$$

Now solve for the velocity of the water jet! The mass of 1 cm^3 of water is 1 gram:

$$\tfrac{1}{2}(10^{-3} \text{ kg})\, v^2 = 10^{-2} \text{ joules}$$
$$v = ?$$

Did you get 4.47 m/sec?

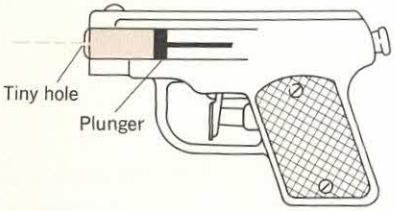


FIGURE 7.1 Water pistol. If you know the force pushing in on the plunger, you can calculate the speed of the emerging stream.

How much harder would you have to squeeze in order to double the velocity of the emerging jet? Twice as hard? If you squeeze twice as hard, then you do twice as much work on each cm^3 of water. According to the work-energy theorem, twice the work gives twice the kinetic energy. But twice the kinetic energy ($\tfrac{1}{2}mv^2$) does not give twice the velocity. So the conjecture about "twice as hard" was wrong. If you want to double the velocity, you quadruple the kinetic energy. That means you have to quadruple the work done on each cm^3. So four times the pressure is needed on the plunger. That is the essence of **Bernoulli's Principle:** Doubling the velocity of the stream outside requires quadrupling the pressure difference between inside and outside.

Here it is in symbols, for generality: Call the inside-outside pressure difference Δp. Call the cross-sectional area of the plunger A. In displacing the plunger by a distance x, the force ΔpA does work ΔpAx. But Ax is the volume V of fluid displaced. So ΔpV is the work done on that volume. That has to be set equal to the kinetic energy it gains by virtue of that work:

$$\Delta\tfrac{1}{2}mv^2 = \Delta pV$$

The mass m on the left and the volume V on the right are proportional:

$$\text{mass per unit volume} = \frac{m}{V} = \rho = \text{density}$$

If the fluid is water, which is rather incompressible, taking the density ρ as constant is a good approximation here. We get

$$\Delta \tfrac{1}{2}\rho v^2 = \Delta p \qquad \text{Bernoulli's Principle}$$

In words: The increase in kinetic energy per unit volume of fluid ($= \tfrac{1}{2}\rho v^2$) occurs at the expense of a decrease in potential energy per unit volume, p. The interpretation of the pressure, which is force per unit area, as an energy per unit volume, is a fringe benefit of the argument. You may feel more comfortable putting in a minus sign. After all, a pressure *drop* gives a velocity *increase*.

That argument was for a water pistol. Having the jet outside at atmospheric pressure helped make it simple. Also, the hole is very small compared with the cylinder, so the fluid in the cylinder is essentially at rest. Now make the same reasoning apply to a more general situation, where the fluid is enclosed in a tube of variable cross section. You may want to think about the aorta, the large and rather elastic artery that carries oxygenated blood from the heart. Or you may be interested in pathological situations like constrictions in blood vessels, or stenotic heart valves. Since all the blood that enters a blood vessel at one end has to leave it at the other end, the speed of flow has to be faster at a constriction. What Bernoulli's Principle tells us is that the pressure is less where the speed is greater.

EXERCISE 4 In the **aorta** of a certain dog, the top speed of the blood in the ascending aorta is 100 cm/sec. In the descending thoracic aorta, it is only 50 cm/sec. Find the difference in pressure between the two locations, assuming that they are at the same height. (HINT: Making the speed and pressure measurements at the same altitude avoids having to worry about gravitational potential energy. Take $\rho = 1$ gram/cm^3, roughly the same for blood as for water.) [Answer: The pressure in the descending aorta is higher by 375 newtons/m^2.]

EXERCISE 5 If the animal is turned so that this part of the descending aorta is 20 cm lower than the ascending aorta, show that the pressure difference is increased by 1960 N/m^2. (HINT: This is a Chapter 5 review exercise. It is included here for practical reasons: Changing the dog's position can dramatically change the pressure readings. A mercury manometer is easily read to ± 1 mm Hg, that is, ± 133 N/m^2. Verify this conversion of units, bearing in mind that the density of mercury is 13.59 gm/cm^3.)

This exercise reminds you that the gravitational potential energy per unit volume of fluid at height h is given by ρgh. (See Chapter 5.) We can now state Bernoulli's Principle more generally and take account of possible height differences. A concise statement is that the quantity

$$\tfrac{1}{2}\rho v^2 + p + \rho gh = \text{(kinetic energy + potential energy) per unit volume}$$

is a constant throughout the volume of the moving fluid. Repeat: The pressure is lower in the *faster* places and in the *higher* places.

Now let's go over the energy bookkeeping that led to this result. First, we should justify it. Then, we should criticise it. The statement that the work done on an object goes into increasing its kinetic energy is called the work-energy theorem. It is a consequence of the law of motion, $F = ma$. If the force F acts through a distance Δx, it does work $F\,\Delta x$:

$$F\,\Delta x = ma\,\Delta x$$

Let Δx be small. Then the average velocity is $\frac{1}{2}(v_1 + v_2)$ and we can write

$$\Delta x = \tfrac{1}{2}(v_2 + v_1)t$$

as well as

$$a = \frac{v_2 - v_1}{t} \qquad \text{by definition of acceleration}$$

Multiply those together to obtain

$$a\,\Delta x = \tfrac{1}{2}v_2^2 - \tfrac{1}{2}v_1^2$$

You may remember that result from Chapter 1, Kinematics. For the dynamics, all we need to do is to multiply through times the mass m:

$$\text{work} = F\,\Delta x = ma\,\Delta x = \tfrac{1}{2}mv_2^2 - \tfrac{1}{2}mv_1^2$$

This is the work-energy theorem.

EXERCISE 6 The water level in a water tower is 40 meters above the ground.

(a) How fast would water come out of a tap at its base? [Answer: 28 m/sec]

(b) Show that a jet of water that fast, aimed vertically upward, would shoot up to a height of 40 meters. Neglect friction.

(c) What is the water pressure at the base of the water tower? Give the gauge pressure, that is, the excess over atmospheric pressure. [Answer: 40 meters of water, or ______ N/m^2]

Now the critique. We used this theorem to say that all the work done by the pressure goes into kinetic energy of the fluid's motion. That is true, provided that there are no other forces around. But work done by pressure could, for example, go into stored elastic energy. Or it could go into breaking chemical bonds. Or it could go into kinetic energy of random motion, which we call heat. The work-energy theorem is still true, but the bookkeeping becomes more complicated. It has to take account of the work done against elastic forces or cohesive forces or viscous forces. The presence of viscosity, in particular, implies that it takes a force just to maintain the motion of a fluid, without any acceleration, without any change of potential energy. The pressure drops along a horizontal line of flow even though the speed of the flow is unchanged. In other words, it takes a pressure difference just to keep the fluid flowing. Viscous forces will be the subject of the next section.

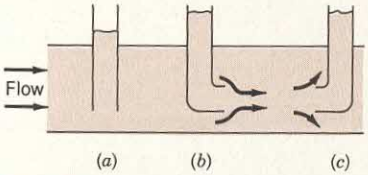


FIGURE 7.2 Manometer tubes sticking into a blood vessel. Pressure reading in *(a)* is lower than in *(b)* and *(c)*.

EXERCISE 7 One way to measure pressure inside a blood vessel is with a manometer (see Fig. 7.2). The manometer tube can stick right into the blood vessel. Bernoulli's Principle tells us that it makes a difference which way the opening of the tube faces. The height of the mercury column in *(a)* will be less than in *(b)* and *(c)*, where the opening is at a stagnation point of the flow pattern, if the opening is small enough. It doesn't matter whether the opening faces upstream or downstream. The pressure is the same in *(b)* and in *(c)*. Being normal to the flow, the open end sees a region with zero flow velocity. If the midstream speed of the blood is 120 cm/sec, how much higher are the mercury columns in *(b)* and *(c)* than in *(a)?* A tube having two openings, one at a stagnation point and one at a high-velocity point, can be used to measure the high velocity by reading the pressure difference between the two openings. This is called a **Pitot** tube (pronounced *pea-toe*).

7.3 VISCOSITY AND POISEUILLE'S LAW

To keep water flowing through a long, narrow tube requires pumping. To keep a thick sugar syrup flowing through such a tube requires harder pumping. Syrup is more *viscous* than water. All fluids—liquids and gases—have some viscosity. You can see why if you think about the atoms and molecules rubbing against each other. It takes negligible force to *start* two adjacent layers of fluid slipping relative to each other. That is what distinguishes a fluid from a solid. But once the layers start slipping, there is a force resisting the slipping that is proportional to the rate of slip. The force needed to keep deforming the fluid is proportional to the rate of deformation. To come back to the long tube, doubling the flow requires doubling the pressure drop between the ends.

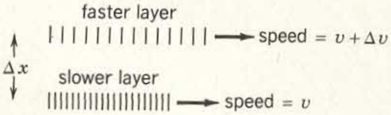


FIGURE 7.3 The faster layer pulls the slower layer to the right with a force $\eta\frac{\Delta v}{\Delta x}$ for every square meter of layer area. The Greek symbol eta (η) is the coefficient of viscosity. It is measured in newton seconds per square meter.

If you look up the **coefficient of viscosity** of water at room temperature, it is given as 1.0 centipoise, or 1.0×10^{-3} newton per square meter per second. This is what that means: If two layers of water whose centers are 1 millimeter apart flowed relative to each other at 1 millimeter per second, the faster layer would be pulling on the slower one with a shearing force of 10^{-3} newton for every square meter of layer (see Fig 7.3). The viscosity of glucose is 10^{16} times greater than that of water at room temperature, but only 20,000 times greater at the temperature of boiling water. The viscosities of some other liquids are listed in Table 7.1. All except the liquid hydrogen are room-temperature values.

In reading that explanation of coefficient of viscosity, you may have noticed that the "1 millimeter between layers" and the "1 millimeter" per sec-

TABLE 7.1

Coefficient of Viscosity	
Water	1.0×10^{-3} N sec/m^2
Methanol (methyl alcohol)	0.6×10^{-3}
Acetone	0.3×10^{-3}
Liquid hydrogen (−253°C)	0.01×10^{-3}
Ethylene glycol (permanent antifreeze)	20×10^{-3}
Castor oil	1000×10^{-3}

ond relative speed canceled out. We could have written "thousandth of an inch" or any other distance unit. In Fig. 7.3 it is called Δx. The point is that the coefficient has units of momentum per unit area. Repeat: N sec/m^2. We need that for the argument in the next paragraph, a so-called *dimensional* argument.

Question: How does the pressure **gradient** needed to maintain a certain flow in a tube depend on the diameter of the tube? The answer to the question is called Poiseuille's Law. Our proof is entirely dimensional, so it will not tell the whole story. But it is simple. First, convince yourself that doubling the *length* of the tube would require doubling the pressure difference to keep up a given flow. The pressure drop between the ends is proportional to the length of the tube. For a tube of constant cross section, the pressure gradient is constant too. That is, there is the same pressure drop per inch of tube anywhere along the tube. If the coefficient of viscosity is to be a constant, this implies that the pressure gradient is proportional to the flows.

$$\frac{\text{pressure difference between the ends}}{\text{length}} = (\text{constant}) \times \text{volume of fluid flowing per unit time}$$

The constant is called the **hydraulic resistance** per unit length of tube. We already know that it is proportional to the coefficient of viscosity. We want to know how it depends on the diameter of the tube, supposed circular in cross section. Assume that the dependence is some power law:

$$\text{resistance per unit length} = (\text{some number}) \times (\text{coefficient of viscosity}) \times (\text{diameter})^n$$

Our job is to find the exponent n. Write down all the units (dimensions):

$$\begin{aligned}\text{pressure difference per unit length} &= \text{newtons/m}^2 \text{ per meter}\\ &= \text{newtons meters}^{-3}\end{aligned}$$

$$\text{flow} = \text{volume per unit time} = \text{meters}^3 \text{ seconds}^{-1}$$

$$\begin{aligned}\text{resistance per unit length} &= \frac{\text{pressure difference per unit length}}{\text{flow}}\\ &= \text{newton seconds meters}^{-6}\end{aligned}$$

$$\text{coefficient of viscosity} = \text{newton seconds meters}^{-2}$$

$$\text{diameter of tube} = \text{meters}$$

Repeat the "power-law" assumption, keeping only the units:

$$\text{newton seconds meters}^{-6} = \text{newton seconds meters}^{-2} \times \text{meters}^{n}$$

The only possible value for the exponent n is

$$n = -4$$

Stated completely, **Poiseuille's Law** is

$$\text{pressure drop per unit length} = \frac{128}{\pi}\,(\text{coefficient of viscosity})\,\frac{(\text{flow})}{(\text{diameter})^4}$$

Our dimensional argument cannot give us that factor of $128/\pi$. A proper theory does, and the law agrees well with experiment as long as the speed of flow is slow enough. Slow enough means that there is no turbulence: All the fluid moves parallel to the walls of the tube, not in whirls or loops. The transition to turbulent flow is treated in Problem 27.

EXERCISE 8 Calculate the pressure drop between the ends of a 12-meter-long garden hose of inside diameter 1.5 cm, carrying a water flow of 0.2 liters per second. What fraction is this of the pressure head of a 10-meter-high water tower?

EXERCISE 9 Rewrite Poiseuille's Law in terms of the tube radius instead of the diameter. (HINT: The $128/\pi$ is replaced by $8/\pi$.)

EXERCISE 10 Remember the definition of hydraulic resistance. We could rewrite it,

volume of fluid per unit time
= (constant) × pressure drop per unit length

The "constant" this time is not the hydraulic resistance.

(a) How is it related to the hydraulic resistance?

(b) That constant, called the **hydraulic conductance,** is proportional to the fourth power of the tube radius r. (O.K.?) The cross-sectional area is πr^2. So the hydraulic conductance goes as the ______ (what power?) of the cross-sectional area.

Is Poiseuille's Law surprising? It says that for a given pressure gradient, the flow is proportional to the *square* of the cross-sectional area of the tube. In a thinner tube, not only does less fluid get through, but its average speed is slower too. You know that intuitively if you have tried drinking a milk shake through a cocktail straw.

7.4 OSMOSIS

If dried raisins are soaked in water, they puff up and look a lot like the grapes they used to be. The transport process by which the water flows in through the grape skin is called osmosis. This is also the process responsible for the functioning of the kidney, and for getting nutrients and waste products into and out of your capillaries. **Osmosis** is not the same as diffusion. The essential feature in osmosis is a membrane that lets the water molecules through easily and presents a barrier to bigger molecules.

The inside of the raisin has a high concentration of sugar. The large sugar molecules can't go through the pores of the membrane (the skin). You know that water molecules can get through, since the membrane stays intact during the drying process. When the raisins are soaked, the concentration (number of molecules per unit volume) of water outside the membrane is greater than the concentration of water inside. Because of their thermal motion, the water molecules carry momentum. Inside they are scarcer, so there is a momentum deficiency inside. Water will flow in through the pores of the membrane to relieve this momentum deficiency. In other words, water outside experiences an **osmotic pressure** tending to make it flow in through the pore. This osmotic flow will continue until the hydrostatic pressure inside exceeds the hydrostatic pressure outside by the amount of the osmotic pressure difference. As a matter of fact, the osmotic pressure can be defined by an equilibrium measurement of the hydrostatic pressure difference. In other words, the osmotic pressure "of the sugar" is the hydrostatic pressure that has to be applied to the inside to keep water from flowing in through the membrane.

Before this equilibrium is reached, the rate of flow is proportional to the *difference* between the hydrostatic and osmotic pressure differences. This difference of differences has been appropriately called the **driving pressure** for osmotic flow. As the raisin swells up with water, the sugar concentration inside decreases. The osmotic pressure is roughly proportional to this solute concentration. So as the raisin swells and the sugar is diluted, the osmotic pressure decreases. At the same time the hydrostatic pressure inside increases as the membrane is stretched. Both of these changes decrease the driving pressure, which goes to zero at equilibrium.

What characterizes osmosis is a membrane **permeable** to a **solvent** and relatively impermeable to a solute. The technical term is *semipermeable* membrane. Other solutes to which the membrane is permeable may also change the concentration of water molecules, but they don't result in any momentum deficiency. To the membrane, they look just like solvent molecules. During osmotic flow they will get swept through the pores along with the solvent.

Figure 7.4 may help make the mechanism more intuitive. All the molecules are in random thermal motion. Water molecules incident on the opening of the pore from the left push other water molecules in the pore to the

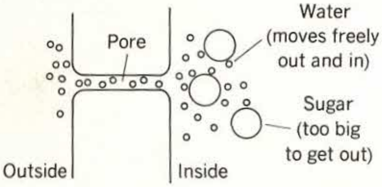


FIGURE 7.4 Mechanism of osmosis. The water will tend to flow in the pore from outside to inside until the hydrostatic pressure inside is higher than that outside by the difference in osmotic pressure.

right. But more will be incident from the left than from the right, because the large solute molecules on the right get in the way. The osmotic pressure is the result of this asymmetry.

That it is a "real" pressure is shown by experiments in which hydrostatic pressure is applied on the left without any solute on the right. The rate of water flow through the membrane is the same whether caused by hydrostatic pressure or by osmotic pressure. The visible difference is that with hydrostatic pressure, the membrane deforms. With osmotic pressure it doesn't. To understand that, remember that the thermal motion of the solute molecules does contribute to the hydrostatic pressure on the right!

Many textbooks confuse *osmosis* and *diffusion*. The confusion is tempting. Whenever a concentration gradient exists, there is diffusive transport down the gradient. The rate of diffuse transport is proportional to the gradient. It all sounds very much like osmosis. Dissolved sugar at the bottom of the lemonade glass will, in time, distribute throughout the lemonade even if you don't stir. But it takes months. Diffusion is terribly slow. It is a purely statistical process. The individual molecules move randomly, independent of each other. In osmosis there is a membrane with a pore that has a pressure gradient along it. The molecules all move together, pulling each other along as they flow through the pore.

EXERCISE 11. According to the ideal gas law, $p = NkT/V$, different molecules in a gas contribute their partial pressures in proportion to their concentration (Sec. 6.4).

(a) Approximately one in every five air molecules is oxygen. So the partial pressure of oxygen in air is one-fifth of an atmosphere. Calculate the concentration of O_2 in moles per liter. (HINT: Molar volume at 1 atm and O°C is 22.4 liters.)

(b) The same law turns out to hold for the osmotic pressure of solutes in a liquid. There it is called the van't Hoff Law. Calculate the osmotic pressure of sugar inside a raisin if for every milligram of water there is a milligram of sugar (see Fig. 7.4). Assume that water does not change its volume when sugar is dissolved in it. Take the molecular weight of sugar as 180. Note that $N_{\text{Avogadro}}kT = 22.4$ liter atm per mole. (HINT: First calculate the sugar concentration (= 5.55 moles per liter).) [Answer: 124.4 atm] Isn't that impressive? That is a head of water higher than the Empire State Building tending to inflate that raisin.

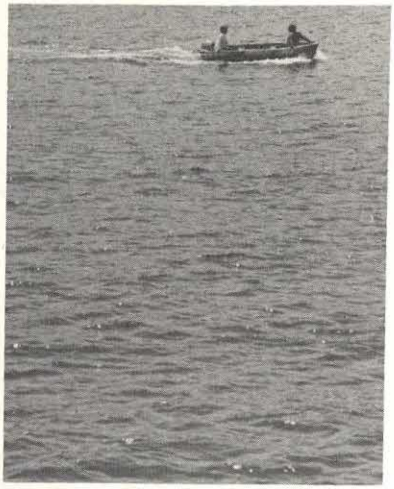
7.5 WAVES

In later chapters, this book talks a lot about waves: sound waves, light waves, even matter waves. But when we hear the word *waves*, most of us think first of water waves—waves in the ocean, in the bathtub, in the coffee cup. The ocean waves travel in a well-defined direction, so we call them

traveling waves or running waves. The waves in the coffee cup just go back and forth, so we call them **standing waves.** (More about that in Chapter 10.) Whether the waves are running or standing, it makes sense to talk about their speed.

Those long, rolling waves out in midocean look like they might be doing close to 20 miles an hour. The **ripples** in the coffee cup or in the roadside puddle are going less than one-tenth that fast. Even in the ocean some waves seem to overtake others. For the faster ones the distance between successive crests is longer. The crest-to-crest distance is called the **wavelength.** Our experience in boats tells us that waves with longer wavelength travel faster.

It is precisely those long, regular, deep-water waves that are most easily treated in a textbook. A snapshot of the water surface would look much like the textbook sine curve (Fig. 7.5*a*). If all you saw was one snapshot you couldn't tell which way the wave is traveling. Two successive snapshots or a movie would show you that the sine curve travels essentially without change of shape. That invariance of shape is actually the essence of what the mathematician means by the word *wave*.



Ripples excited by a light breeze. A strong wind would excite the (longer-wavelength) gravity waves.

If you observe a piece of paper floating on the surface, you find that it executes uniform circular motion. At the crest of the wave it moves forward; then it starts to move down. At the trough it moves backward; then it starts to move up (Fig. 7.5). Since each bit of water executes these vertical circles, the wave does not transport any water in a whole period. But it is easy to convince yourself that it transports momentum. Since the higher parts of the wave are moving forward, and the average pressure under higher parts is greater, more momentum is carried forward under the crest than is carried backward under the trough. A similar argument shows that the wave carries kinetic energy, even though the average displacement of the water over time is zero.

It is surely obvious to you that the important force responsible for the wave motion is the weight of the water—gravity. What we are going to show is that the speed of the wave is proportional to its **period.** That is not so obvious. The proof uses only our old friend $F = ma$ and the observed fact that the particles of water go round in circles at a constant rate. The proof is going to focus on a small volume of water near the surface, in a part of the wave where the surface slopes downhill. There the average pressure pushing that water forward exceeds the average pressure pushing it backward. The excess forward force is responsible for the transport of momentum.

Each little volume of water near the surface goes round in a circle of radius R; call its angular velocity ω. Just so you don't have to leaf back to Chapter 5, we remind you that the time to go around the circle once, the period T, is $2\pi/\omega$. The speed of the motion around the circle is ωR, the centripetal acceleration is $\omega^2 R$. Focus attention on the part of the wave where the downward slope of the surface is maximum. At that point the velocity vector of the water is directed vertically up (Fig 7.5*a*), and the acceleration vector is directed forward. Take a small volume $\Delta x\ \Delta y\ \Delta z$ near the surface. The pressure on the left-hand face exceeds the pressure on the right-hand face because the surface is higher on the left than on the right by (slope) · Δx. That maximum slope is $\omega R/v$, where v is the speed of the wave. (See Problem 41.) So the pressure force to the right on the left-hand face exceeds the pressure force to the left on the right-hand face by

$$\rho g\ \Delta y\ \Delta z \cdot (slope) \cdot \Delta x = \rho g\ \Delta x\ \Delta y\ \Delta z\ \frac{\omega R}{v}$$

This force we set equal to the product of the mass $\rho \Delta x \Delta y \Delta z$ and the forward acceleration $\omega^2 R$:

$$\rho g\ \Delta x\ \Delta y\ \Delta z\ \frac{\omega R}{v} = \rho\ \Delta x\ \Delta y\ \Delta z\ \omega^2 R$$

Just watch things cancel out as we solve for v:

$$v = \frac{g}{\omega}$$

That the volume $\Delta x\ \Delta y\ \Delta z$ cancels out is no surprise. That the density ρ of the water cancels out you also expect if you think of the wave as falling because of its weight. You remember that the light things and heavy things fall at the same rate. That the wave amplitude R cancels out might be a surprise. That cancellation says that high waves travel at the same speed as low waves. This property is called *linearity*. It depends crucially on the particle path being circular.

The formula for the wave speed that we have derived,

$$v = \frac{g}{\omega} = \frac{gT}{2\pi}$$

looks so simple that you might ask yourself, couldn't we have guessed it? Problem 39 shows by dimensional analysis that if you guess that the acceleration of gravity g and the period T are involved, the product gT is the only combination that has the dimensions of a speed. Of course, you can't get the 2π just by dimensional reasoning.

If you want to know how the wave speed v depends on the wavelength λ, one more step is needed. The wavelength is the distance the wave travels in one period:

$$\lambda = vT$$

Putting this into our formula, we have

$$v = \frac{g\lambda}{2\pi v}$$

or

$$v = \left(\frac{g\lambda}{2\pi}\right)^{1/2}$$

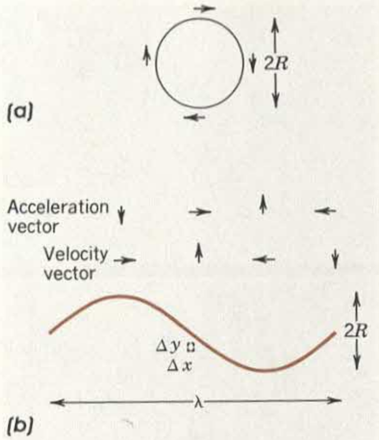


FIGURE 7.5 *(a)* Circular path of floating object for wave traveling to the right. Arrows show velocity vector at four places, that is, four times, one-quarter period apart. At wave crest, velocity is forward; one-quarter period later, velocity is downward, and so on. Acceleration vector is always directed toward the center of the circle. *(b)* Wave traveling to the right. At the crest of the wave, the water is moving forward, while its acceleration is downward. A quarter-wavelength to the right (earlier in phase), the water is moving upward, while its acceleration is forward, and so on. Velocity = ωR; acceleration = $\omega^2 R$.

In agreement with observation, this says that the long waves travel faster than the short ones. Where you see waves of different wavelengths, you can see the long ones overtaking the short ones.

Ripples in a coffee cup.

EXERCISE 12 20 miles an hour is about 9 m/sec. What is the wavelength of a wave having that speed? What is its frequency? Why are you unlikely to observe such waves in a pond?

NOTE: In shallow water the path of the surface particles is elliptical, not circular.

When we turn our attention to the **ripples** in the coffee cup, we might guess that surface tension is going to be the important restoring force, and gravity less important. Problem 47 shows that there the longer waves are slower than the shorter ones.

SUMMARY

Flow = volume per unit time = average velocity × cross-sectional area

Force of moving water = momentum transported per unit time
= density × speed × flow

Kinetic energy transported per unit time
= kinetic energy per unit volume × flow $\propto$ (speed)^3

Bernoulli's Principle says that the pressure is higher where the speed is slower. For horizontal flow, $p + \frac{1}{2}\rho v^2 =$ constant. For up-and-down flows, the pressure is higher lower down: $p + \frac{1}{2}\rho v^2 + \rho g h =$ constant.

The energy-conservation argument used in deriving Bernoulli's Principle has to be modified to take account of viscosity: Even with uniform flow, the pressure drops along the flow direction because of fluid friction. Double the flow – double the pressure gradient.

Viscous forces are shearing forces proportional to the rate of slip.

The coefficient of viscosity is shear stress divided by the velocity gradient. Its dimensions are momentum per unit area.

Castor oil is about a thousand times as viscous as water.

Poiseuille's Law says that the flow in a tube for a given pressure gradient is proportional to the fourth power of the tube radius. The hydraulic resistance of a tube is proportional to its length and inversely proportional to the square of its cross-sectional area.

Gravity waves have longer wavelengths than surface-tension ripples. They also travel faster. Their speed is proportional to the square root of their wavelength.

In osmosis, solvent flows through the membrane toward the side having the higher osmotic pressure of solute.

KEY TERMS

Hydrodynamics The science of fluids in motion.

Hydrostatics The science of fluids at rest.

Aerodynamics Hydrodynamics of gases. Describes the forces associated with the motion of objects of various shapes through air or other gases.

Bernoulli's Principle Along the path of flow of an incompressible fluid, any decrease in fluid pressure is equal to the increase in kinetic energy per unit volume plus any increase in gravitational potential energy.

Aorta The large artery leading out from the left ventricle of the heart.

Pitot tube Instruments used to measure flow velocity in terms of the pressure difference between an opening tangent to the stream and an opening perpendicular to the stream. The former (higher) pressure is called the static pressure, the latter (lower) pressure the dynamic pressure.

Venturi tube Measures the speed of flow of a fluid by monitoring the pressure drop at a constriction.

Viscosity Fluid friction. Resistance to a velocity gradient. Appears as a dissipative resistance to flow.

Coefficient of viscosity Ratio of shearing force per unit area to the velocity gradient giving rise to that shearing stress.

Kinematic viscosity Absolute viscosity (coefficient of) divided by density. The MKS units are m^2/sec.

Newtonian fluid A fluid for which the resistance to shear deformation is proportional to the rate of deformation. A fluid for which a coefficient of viscosity can be usefully defined. Blood, being a suspension of very elliptical corpuscles, is somewhat non-Newtonian.

Poiseuille's Law Relates the flow through a tube to the pressure gradient. The quotient of fluid flow through a round tube divided by the pressure difference between its ends is a constant equal to $\pi r^4/8\eta L$. Here r = tube radius, L = length, and η = coefficient of viscosity.

Gradient Measure of how rapidly a certain quantity changes in space. The pressure gradient in resting water is one atmosphere for every 10 meters you go down, or 0.1 atm/m. In vector analysis, the gradient vector points in the direction in which the changing quantity increases most rapidly.

Flow Volume of fluid transported per unit time.

Flux Volume of fluid transported per unit time per unit area.

Poise The cgs unit of viscosity = 1 gram/cm sec. The viscosity of water at 20°C is 1.0019 centipoise = 1.0019×10^{-2} dyne sec/cm^2.

Work-energy theorem The increase in kinetic energy of an object is equal to the work done on it.

Hydraulic conductance The ratio of the flow through a tube to the pressure difference between its ends. The hydraulic conductance of a garden hose might be 10^{-7} m^5/newton second. The reciprocal of hydraulic conductance is *hydraulic resistance*.

Systole The half of the cardiac cycle in which the heart contracts and blood flows out of it.

Diastole The half of the cardiac cycle in which the heart expands and blood flows into it.

Magnus effect The sideward Bernoulli force on an object spinning in air. This is what makes a curve ball.

Laminar flow Also streamline flow, or low-Reynolds-number flow, or nonturbulent flow. The flowing fluid may be described as composed of layers of constant velocity.

Turbulent flow Also high-Reynolds-number flow. Flow with an irregular velocity field having vortices distributed randomly throughout.

Reynolds number For flow in a tube of diameter d at velocity v of a fluid of density ρ and viscosity η, it is the dimensionless ratio $\rho dv/\eta$. At a critical Reynolds number (around 3000), laminar flow goes over into turbulent flow.

Ideal fluid Nonviscous fluid; a fluid without viscosity. Helium at temperatures below 2.2 K behaves like an ideal fluid.

Stokes' Law The resistance to motion at speed v of a sphere of radius R through a fluid of viscosity η is $6\pi\eta Rv$.

Vortex A region of fluid in which any volume element moves in circles and rotates as it travels. Plural: vortices.

Vorticity The measure of the strength of a vortex. Also, the existence of vortices.

Traveling wave A pattern that moves without essential change of shape

Standing wave Superposition of two or more periodic travelling waves going in different directions, so that no energy or information is transmitted. Characterized by *nodes*, places where the wave amplitude vanishes at all times.

Period The shortest time in which a pattern repeats.

Wavelength The shortest distance in which a wave pattern repeats. The distance a wave travels in one period.

Ripple A short-wavelength wave on the surface of a liquid. Surface tension is the force responsible for ripples, as distinguished from gravity waves.

Attenuate To diminish in amplitude. Waves are attenutated by viscous damping.

Osmosis The flow of a fluid (call it the solvent) through a membrane from the side of lower solute concentration toward the side of greater solute concentration. This transport mechanism is responsible for the sap rising in trees, also for the functioning of the kidney.

Osmotic pressure (of a solute) The hydrostatic pressure that must be applied to that side of the membrane containing the higher solute concentration in order to just prevent inflow of solvent.

Driving pressure The difference between hydrostatic pressure difference and osmotic pressure difference. When the flow of solvent through the membrane stops, the driving pressure is zero.

Permeability (1) The ability of a membrane (or other barrier) to let a certain fluid (solvent) flow through. (2) The ratio of the transmembrane flux of a certain fluid (solvent) to the driving pressure.

Permeable/Impermeable Able / not able to let molecules pass through. A membrane, by definition, is semipermeable; that is, it is more permeable to some molecules than to others. In the case of the raisin swelling up in water, the skin is permeable to water, impermeable to sugar.

Solvent When one atomic, ionic or molecular species is dissolved in a fluid, that fluid is called the *solvent*. The dissolved species is called the *solute*. In the raisin, the water is the solvent; the sugar is the solute.

REVIEW QUESTIONS

- Two shower jets transmit twice as much momentum as one. But if the speed of the water in each jet is doubled, keeping the cross section constant, the force exerted by each jet is multiplied times ________.
- If one jet puts out 100 grams of water every 10 seconds and has a 1-mm^2 cross section, how much force can that jet exert on my shoulder? (HINT: How fast is the water moving before it hits?)
- The rate at which momentum is transferred from the water to my shoulder is called the ________ on my shoulder.
- If the number of grams of water per second is doubled without changing the cross section of the jet, the kinetic energy transported per unit time is multiplied times ________.
- If the pressure on the plunger of a water gun is doubled, the speed of the jet is multiplied times ________.
- To double the speed of the jet, the pressure on the plunger must be multiplied times ________.
- If the kinetic energy per unit volume of water in a pipe is greater by 0.01 joules per cm^3 at a constriction than in the uniform part of the pipe, how much greater / less is the water pressure there? ________ newtons per m^2.
- A pressure "head" of 10.29 meters of water is 1 atmosphere. How fast would that make the water come out of the faucet?
- Fluids have no rsistance against compressive / shearing forces.
- The ____________________ force between layers of a fluid is proportional to their rate of relative slip.
- The coefficient of viscosity of most liquids is an increasing / decreasing function of temperature.
- When fluid flows in a long tube, doubling the pressure difference between the ends implies multiplying the rate of flow times ________.
- Doubling the diameter of the tube while keeping constant the pressure drop between its ends implies multiplying the rate of flow times ________.
- State Poiseuille's Law.
- In what units can a coefficient of viscosity be expressed?

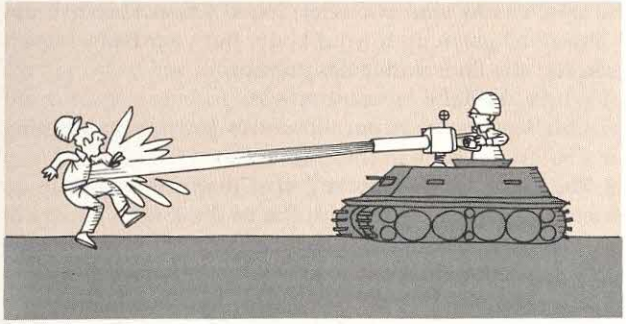
- Explain how osmosis works.
- It is known that a concentration gradient can persist in an unstirred solution for a long time. In other words, diffusion of solute through a solution is a very slow process. But if a membrane impermeable to the solute but permeable to water separates the regions having different solute concentrations, the difference disappears quickly: Water flows through the membrane into the region of lower solute concentration. You might think a membrane would impede the flow of water. Instead, it acts to speed it up. Explain.
- The speed of long ocean waves is an increasing / decreasing function of wavelength. In order to double the speed, the wavelength must be multiplied times ______.

PROBLEMS

✶ **1. Bathtub** Water comes out of the spigot at 1 meter per second. The stream has a cross-sectional area of 4 cm^2.
(a) How much water comes out per second? Answer in cm^3.
(b) How long does it take for a liter of water to come out? (HINT: 1 liter = 10^3 cm^3 for our purposes.)
(c) The water surface in the tub has an area of 0.6 m^2. How long does it take to make the surface rise by 10 cm? How does this compare with bathtubs you have known?
(d) How much water do you like to use for a tub bath? Answer in cm^3 or m^3 or liters. Compare with the volume of water used in 5 minutes by the 50-jet shower in Exercise 1.

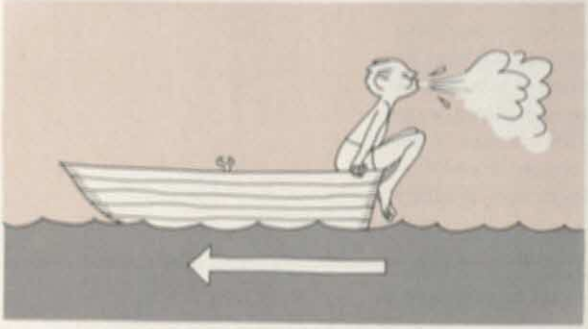
✶ **2. Garden Hose** The water jet has a cross-sectional area of 2.5 cm^2 and emerges at a speed of 3.2 m/sec.
(a) What volume of water comes out every second?
(b) What is the mass of that much water?
(c) What is the momentum of that water?
(d) If it hits a vertical surface and runs down it (inelastic collision), what force (momentum transfer per unit time) does the water transmit to the object it hits?

✶ **3. Water Cannon** The water jet is 10 cm^2 in cross section and moves at 20 m/sec (see figure).
(a) What volume of water is pumped in 1 second?
(b) What mass of water is pumped per second?
(c) What is the momentum of that water?
(d) Suppose that the water hits a man horizontally in the chest and runs down his clothes. What is the horizontal force the water exerts on the man?
(e) Express that force in pounds. (HINT: 4.5 newtons = 1 pound. Do you think that would knock a man down?)



PROBLEM 3. Water cannon.

✶ **4. Jet Boat** You fill your mouth with water (say 40 cm^3) and blow it out forcefully (say at 4 m/sec). (See figure.)
(a) How much momentum does this impart to the water?
(b) Suppose that you and your little boat have a total mass of 160 kg. How many mouthfuls does it take before you and your boat are moving backward at 1 meter per second? Neglect all frictional and viscous effects.
(c) Did you ever think your boat would move forward faster if you helped by blowing across the stern? What do you think of that idea now?



PROBLEM 4. "Jet boat"—the boat is propelled forward by your blowing water out of your mouth over the stern. How many mouthfuls before you get the boat going at 2 miles an hour?

★★ **5. Momentum and Energy** When the water jet of the shower hits your shoulder, its downward momentum is almost all given up to your body. But your body doesn't sink into the floor under this pressure.

(a) Where does the momentum go?

(b) Could you prove your answer to part (a) by standing on a bathroom scale in the shower?

(c) We know from Chapter 2 that momentum has to be exactly conserved. What happens to the kinetic energy of the water jet when it hits your shoulder?

(d) Assume that the "collision" with your shoulder is completely inelastic, that is, the water does not bounce off. Calculate the rate at which kinetic energy is lost in each jet of the shower in Exercise 1.

★★ **6. Warm-Air Ducts** A forced-air heating system is designed to move the warm air through pipes 20 cm in diameter at an average speed of 3 m/sec.

(a) What is the volume of air in 3 meters of pipe?

(b) What volume of air flows through the duct per second?

(c) There is a constriction in the duct where the pipe diameter is only 10 cm. How fast does the air move there?

(d) You solved part (c) by assuming that the same volume of air has to move past any point of the pipe per unit time, that is, the number of air molecules per unit volume is not appreciably less at the constriction than in the rest of the duct. You neglected the compressibility of the air. Now use Bernoulli's Principle to show that the pressure drop at the constriction connected with the increased air velocity there causes an error in the result of part (c) that is less than one part in a thousand. (HINT: You will need the density of the air. Approximate it as 1 kg/m^3. Use the pressure drop from part (c) to estimate the density of air at the constriction.)

★★ **7. Flow in a Pipe with a Constriction** Water flows in a pipe 2 cm^2 in cross-sectional area at an average speed of 1.5 m/sec (see figure).

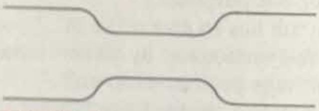
(a) What volume of water is transported by the pipe per second? (HINT: Be careful about the mixed units: some MKS, some cgs.)

(b) The pipe has a constriction where the cross section narrows to 1 cm^2. What is the average speed in the constriction? ________ m/sec.

(c) The kinetic energy per cm^3 of water is ________ times greater in the constriction than in the bulk of the pipe.

(d) At 1.5 m/sec, the kinetic energy of 1 cm^3 of water is ________ joules. We can write the kinetic energy per unit volume of water as ________ joules/m^3 = ________ ergs/cm^3.

(e) Pressure is a measure of potential energy per unit volume. According to the work-energy theorem, the potential energy per unit volume plus the kinetic energy per unit volume should be constant in the absence of frictional losses. The pressure in the constriction should be ________ newtons/m^2 greater / less than in the bulk of the pipe. (HINT: 1 joule/m^3 = 1 newton/m^2.)



PROBLEM 7.

★★ **8. The Heart as a Pump** The heart pumps about 5 liters of blood per minute.

(a) Express in cubic centimeters per second. For a pulse rate of 60 per minute, this gives you the volume pumped per heart beat.

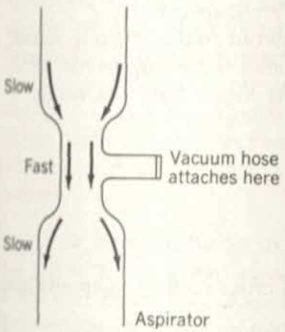
(b) The aorta has a cross-sectional area of 5 cm^2. What is the average speed of the blood in the aorta?

(c) During part of the cardiac cycle, the blood in the aorta actually flows at twice the average speed. Calculate the kinetic energy of 1 cm^3 of this fast-moving blood.

(d) During another part of the cardiac cycle, the arterial blood flows hardly at all. A cm^3 of slow-moving blood must be under greater pressure than the cm^3 of fast-moving blood. That is, it must have as much potential energy as the fast-moving cm^3 has kinetic energy. Recall that the product of pressure times volume represents potential energy, and calculate how much more pressure the slow-moving cm^3 is under than the fast-moving cm^3. Approximate the speed of the slow-moving blood as zero.

NOTE: This problem provided an opportunity to reiterate the argument presented in the derivation of Bernoulli's Principle.

★★ **9. Aspirator** In a water pipe the speed of flow is 2 meters per second and the water pressure is 10^4 newtons per square meter (approximately 0.1 atm) above atmospheric. At a constriction fhe speed is quadrupled to 8 m/sec (see figure). What is the water pressure there? Don't be troubled by getting a negative answer! Indeed, the pressure is below atmospheric, and a hose attached at the constriction will develop suction.



PROBLEM 9. Aspirator. The pressure is lower where the water moves faster, according to Bernoulli's Principle.

★★★ **10. Distensibility of the Arteries** The walls of the aorta, although thick and tough, are quite elastic. During systole, the part of the cardiac cycle during which the blood is discharged from the heart into the aorta, the blood pressure in the aorta rises. Because of its elasticity, the aorta increases in volume during this ejection phase. Later in the cycle (diastole), the pressure falls and the volume decreases. Evidently, this "capacitance" of the aorta and other arteries smooths the flow of blood further along in the circulatory system: The "beat" is strong in the arteries, quite weak in the veins. We say that the arterial system serves as a *hydraulic filter* to steady the flow of blood in the capillaries. If the arterial walls were rigid, flow in the capillaries would stop during diastole.

At the beginning of the ejection phase, the pressure in the left ventricle of the heart slightly exceeds that in the aorta, and blood flowing in the aorta is accelerated. As the arteries expand, this pressure gradient is reversed, and blood flowing in the aorta slows down.

(a) What happens to the kinetic energy of the blood that slows down? HINT: It goes into potential energy. Where is it stored?

(b) What happens to this energy during diastole?

★★ **Garden Hose** If you point the garden hose straight up, the jet of water reaches a height of 8 meters.

(a) Calculate the velocity of the water leaving the nozzle. (HINT: Neglect air friction.) If you have forgotten "How high can I jump?," you might turn back to Chapter 4.

(b)What is the water pressure behind the nozzle? Express in meters of water.

(c) Why does the size of the opening affect the velocity of the jet? Don't just answer, "The pressure is greater behind a smaller nozzle." Explain in terms of viscosity and speed of flow in the hose.

★★ **12. Garden Hose** The water escapes from the nozzle at 12.5 m/sec.

(a) If 200 cm^3 of water comes out per second, what is the size of the opening? Assume that all the water has the same speed of 12.5 m/sec. Give the area in cm^2 and the diameter in cm.

(b) The hose has an inside diameter of 1.43 cm. What is the average speed of the water in the hose? (HINT: How long is a cylinder of water 200 cm^3 in volume and 1.6 cm^2 in cross-sectional area?)

(c) How much lower is the pressure in the hose than in the water mains that feed it? Use Bernoulli's Principle and neglect viscosity. (HINT: You don't have to know the cross section of the water main. It is enough to know that not all the neighbors are watering their lawns at the same time; that is, the mains are so large that the speed of the water in them is negligible.)

★★ **13. Tennis, Ping-Pong, and Baseball** Because of the viscosity of the air, the layer of air immediately in contact with the ball must share its velocity. We talk about a *boundary layer* of air, which essentially adheres to the ball. The air adjacent to this boundary layer is also somewhat pulled along by it. The faster it moves, the lower the pressure.

(a) Show why with top-spin, this velocity is faster below the ball than above it. According to Bernoulli's Principle, the pressure will therefore be lower below and the ball will drop faster.

(b) Why does a ball with back-spin tend to "loft," that is, to drop slower than a nonrotating ball?

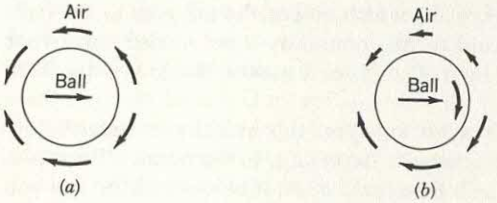
(c) If you want a pitch to curve to the left, which way should the ball rotate? Answer in the form "clockwise (or counterclockwise) as seen from the top." Explain.

✶✶ **14. Soccer** A good corner kick curves toward the goal. Which way should the ball be spinning for a kick coming from the left corner as seen by the attacking team? Clockwise?/counterclockwise?

Problem 7.14 Corner kick.

✶✶✶ **15. More on the Curve Ball** [Difficult] The transverse force on a spinning ball (called the **Magnus effect**) is due to the velocity of the boundary layer. It may be instructive to describe the force in the reference frame of the still air. This is an inertial frame, and usually the frame of the observer.

A nonspinning ball moving to the right sets up flow lines in the surrounding air as shown in *a* in the figure. The arrows are velocity vectors for the air moving to the left to get out of the way of the ball traveling to the right. But if the ball spins clockwise, as shown in *b*, the velocity of the boundary layer *adds* to the streaming velocity below the ball and *subtracts* from it above the ball (see figure).

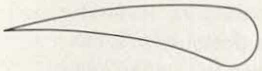


PROBLEM 15. Magnus effect. Why top spin makes the ball curve down.

NOTE: It is tempting to get this effect backward. Remember, the nonspinning ball has equal forces due to the moving air, pulling up on the top, down on the bottom. Argue that the Magnus force should be proportional to the angular velocity of the ball and to its radius. (HINT: The air velocity below is $V + R\omega$; the air velocity above is $V - R\omega$. The "Bernoulli pressure" is proportional to the square of the velocity.)

Aerodynamicists would talk about a clockwise circulation of the air around the ball. They would ascribe the transverse force to this circulation and say that the ball sheds vortices downstream.

✶✶ **16. What Keeps an Airplane Up?** Airplane wings have a cross section something like the figure shown here. Moving through still air, the relative speed of the air will be greater above the wing than below. According to Bernoulli's Principle, the pressure should therefore be greater below the wing than above, so the wing experiences lift.
(a) Argue that for a given cross section, the lifting force should be proportional to the wing spread.
(b) Why should the lift be greater in cold air than in warm?
(c) Why should the lift be smaller higher up in the atmosphere? Can you see how this might limit the height to which airplanes can fly?



PROBLEM 16. Air foil. Typical cross section of an airplane wing.

✶✶ **17. Blowing** What pressure difference is needed between the mouth and the outside to blow air out at 1 m/sec? Explain how you use Bernoulli's Principle to answer this. (HINT: The density of air can be obtained using the Ideal Gas Law, assuming an average molecular weight of 29.)

✶ **18. Viscosity Units** The **poise** is the CGS unit of viscosity, the dyne sec/cm^2, so that 1 centipoise = 10^{-2} gram sec^{-1} cm^{-1}. Show that this is equal to 10^{-3} MKS units.

NOTE: The SI unit is called the pascal second. (1 **pascal** = 1 newton per square meter.)

✶✶ **19. Poiseuille's Law** If water flows at an average speed of 10 cm/sec in a tube whose bore has a 1-cm^2 cross section, then 10 cm^3 of water flows past every second:

flow = (cross-sectional area) × (average speed)

This really defines what we mean by average speed. The water close to the walls moves hardly at all. The water near the middle of the tube moves faster than the average—twice as fast, to be exact. Use Poiseuille's Law to show that the average speed of flow is proportional to the cross-sectional area, for given pressure **gradient.**

✶ **20. Viscosity of Air and Breathing** The trachea is about 15 cm long and 2 cm in diameter.
(a) Calculate its hydraulic resistance (= difference in air pressure between the mouth and the lung divided by the air flow) from Poiseuille's Law. Use 1.8×10^{-5} N sec/m^2 for the viscosity of air at body temperature.
(b) A physiology textbook asserts that a pressure difference of less than 1 mm of mercury can move air out of the lung at 250 cm^3/sec. Can the book be right?

✶✶ **21. Work to Breathe** Use the pressure difference calculated in Problem 20 and an airflow of 0.25 liters per second to calculate the rate at which work is done in breathing. Express in watts. (HINT: Convince yourself that the product of a pressure times a volume is an energy.) The answer you get is surprisingly small. Of course, it has neglected the mechanical losses in moving the diaphragm and all the heat generated in the muscles involved.

✶✶ **22. Flow of Blood in the Capillaries** The capillaries are very thin blood vessels, but there are so many of them that their total cross-sectional area is around 5000 cm^2—much greater than that of the aorta, which is about 5 cm^2. All the capillaries are in parallel, which means that blood flowing through one capillary returns to the heart before going through another.
(a) If the average blood flow in the aorta is 80 cm^3/sec, what is the average speed of flow (cm/sec) in the aorta?
(b) What is the average speed of flow in a capillary?
(c) A typical capillary has a diameter of 8×10^{-3} mm (= 8 microns). How many capillaries are there?
(d) Use Poiseuille's Law to calculate the pressure drop in a capillary 1 millimeter long. The viscosity of blood is around 3×10^{-3} N sec/m^2.

NOTE: Most of the pressure drop (= 100 mm Hg) between the large arteries and the large veins occurs in the arterioles, the small blood vessels leading to the even smaller capillaries.

✶✶ **23. Work to Pump the Blood** The heart pumps about 5 liters of blood per minute. The average pressure difference between the intake (venous) and outflow (arterial) sides of the heart is around 100 mm Hg.
(a) At what rate does the heart do work on the blood? (HINT: Convince yourself that the work done in moving 1 cm^3 with a pressure of 1 dyne/cm^2 is 1 dyne cm (= 1 erg $= 10^{-7}$ joule).)
(b) Where does the energy go? Don't just say "into overcoming viscous forces," or "into heat." What gets warmed?

✶✶ **24. Motion of a Sphere** There is an elegant theorem in hydrodynamics for the motion of a sphere in an ideal (nonviscous) fluid. Because the fluid is ideal, uniform motion requires no force. But acceleration a requires a force $F = (m + m^*)a$, where m is the mass of the sphere and m^* is *half* the mass of the fluid displaced by the sphere. A lot of water has to move out of the way in order for the sphere to move. Calculate the downward acceleration under gravity of a diving bell 1 meter in diameter weighing 6000 newtons. Don't forget the buoyancy of the water!

✶✶ **25. Stokes' Law** The force needed to move a sphere through a viscous fluid at speed v is $6\pi\eta Rv$, where R is the radius of the sphere and η is the viscosity of the fluid.
(a) Show that a small sphere of mass m dropped in air reaches a terminal velocity given by $mg/6\pi\eta R$.
(b) Calculate the terminal velocity for a rain drop 1 mm in diameter. The viscosity of air is 1.8×10^{-5} kg m^{-1} sec^{-1}.

✶✶ **26. Settling of Suspensions** A bit of shaking is all it takes to suspend a finely divided powder in a liquid. Calculate the terminal velocity of a spherical particle one micron (= 1 micrometer $= 10^{-3}$ mm) in radius sinking in water. Take the particle density twice that of water.

✶✶ **27. Reynolds Number and Anemia** The units of the coefficient of viscosity η are newton sec/m^2. The literature sometimes quotes a related quantity, the kinematic viscosity $\nu = \eta/\rho$, which is the viscosity η divided by the density ρ of the fluid.
(a) In what units is it measured? Note that force and mass units (newtons and kilograms) have disappeared. That is why ν is called **kinematic;** it involves only space and time dimensions.
(b) Combine the average speed v of flow and the tube diameter d into an algebraic expression having the same units as the kinematic viscosity.

The dimensionless ratio $N_R = \rho dv/\eta$ is called the **Reynolds number.** When speeds are high enough so that this number goes much over 3000, smooth flow is unstable and tends to break into **turbulent** flow.
(c) An aorta has a diameter of 25 mm and carries a peak flow of 200 cm^3 of blood per second. What is the speed of

flow, averaged over the cross section? What is the Reynolds number? Assume the viscosity of blood to be three times that of water. You know its density. In severe anemia the viscosity of blood is drastically low. This condition can be detected with a stethoscope as a functional cardiac murmur: The turbulence makes a noise.

✶✶ **28. Turbulent Breathing** The diameter of a nostril is roughly 5 millimeters. In rapid breathing, 500 cm^3 of air may be inspired (breathed in) or expired (breathed out) in about 1 second. The viscosity of air is 1.8×10^{-5} N sec/m^2. Calculate the Reynolds number for a flow of 500 cm^3/sec. Assume that the other nostril is clogged. Why do you think heavy breathing makes a noise? Why is breathing through the mouth less noisy?

✶✶ **29. Fire Hose** In Exercise 2, that fire hose with the 2-cm^2 nozzle cross section and the 30-m/sec water jet put out 6 kg of water per second, having a momentum of 180 N sec.

(a) Verify. Argue that the fireman holding the nozzle would feel it recoiling with a force of 180 newtons.

(b) On the other hand, the viscous drag of the water against the walls of the hose exerts a forward force on the hose. Suppose that the viscous friction of the water against the hose results in a pressure drop of 4×10^2 N/m^2 over a 10-meter length of hose having a 20-cm^2 cross section. Which force is more important to the fireman holding the nozzle, the recoil from the nozzle or the forward drag of the water against the sides of the hose?

(c) Explain why, even though the pressure is greater at the hydrant than at the nozzle, the water does not accelerate in the hose.

(d) The water does accelerate as it passes through the nozzle. What is the difference in average speed behind and in front of the nozzle?

Problem 7.29 Firemen leaning forward against the recoil of the fire hose. The acceleration of water occurs at the nozzle and at the bend in the hose.

✶✶ **30. Fire Hose**

(a) What is the kinetic energy of each kilogram of water in the jet above?

(b) Since 6 kg come out per second, at what rate does the pump have to do work? Answer in watts. Neglect the viscous loss for this calculation.

(c) How much more power does the pump have to put out to compensate for the viscous drag in 10 meters of hose?

✶✶✶ **31. Washing** This chapter began by asking whether it is the momentum of the water or its kinetic energy that matters for getting clean in the shower. Since you can also get clean in a tub bath, the question was obviously not well stated. Washing involves dissolving some of the dirt, lowering the surface tension that causes some things to stick and physically rubbing off others. That rubbing can be done by a scrub brush or by the motion of the water. With moving water it is the viscosity of the water that gives rise to the "cleaning" force. Accordingly, one would answer that it is just the *speed* of the water jet that matters. Even a small jet will remove a piece of stubborn dirt if the water moves fast enough.

How much "cleaning force" (force parallel to the walls) does the water exert on the walls of a uniform water pipe 2 cm in diameter and 17 meters long if the water pressure drops 10^3 N/m^2 (about one-hundredth of an atmosphere) between the ends? Why don't you have to know the viscosity of water to solve this problem? (HINT: The speed of the water is the same at both ends of the pipe. See also Problem 10.45, Ultrasonic Cleaner.)

✶✶✶ **32. Osmotic Regulation of the Interstitial Fluid** The hydrostatic pressure in a human capillary might typically drop from 25 torr (above atmospheric) on the arterial end to 10 torr on the venous end. This is just due to viscous loss. After all, the capillaries are very narrow. (Recall that 1 torr = 1 mm Hg = 133.3 newtons/m^2.) The hydrostatic pressure in the interstitial fluid surrounding the capillaries is actually less than atmospheric, perhaps −6 torr gauge pressure. If there were no solutes in the blood to which the capillary wall is impermeable, osmotic flow of fluid would be inward throughout the length of the capillary. But protein molecules in the blood give an osmotic pressure of 28 torr. Protein molecules in the interstitial fluid give an osmotic pressure of 5 torr.

(a) Argue that on the arterial end of the capillary, there is a driving pressure of 8 torr tending to pump fluid into the interstitial space.

(b) Argue that on the venous end of the capillary, there is a driving pressure of 7 torr pumping fluid into the capillary.

(c) If the pressure drop along the capillary is linear, argue that you would expect more fluid to leave the capillary than enters it. A graph of driving pressure versus position along the capillary would present a convincing argument. Such an excess of outward flow is actually observed. The lost fluid gets back into the blood via the lymphatic system.

✶✶ **33. Edema** In right heart failure the blood from the veins is not pumped fast enough, so the pressure on the venous sides of the capillaries is increased. According to the previous problem, such an increase slows down the flow from the interstitial fluid into the capillaries. Interstitial fluid builds up and causes swelling (edema) in the lower part of the body. In left heart failure the venous pressure rises in the blood vessels of the lung. Explain how "pulmonary edema" (collection of fluid in the lungs) can result in the patient essentially drowning in his own body fluids.

✶✶✶ **34. Permeability of Membrane** In membranes with large pores the hydrodynamics of viscous flow should give a good value for the permeability. Poiseuille's Law is

$$\text{flow} = \frac{\pi}{128}\frac{(\text{diameter})^4}{\text{viscosity}} \times \text{pressure drop per unit length}$$

where "flow" means the volume of fluid per unit time through one pore. So the hydraulic conductance L_p of one pore is

$$\begin{aligned} L_p &= \text{flow/pressure difference} \\ &= (\pi/128)\,(\text{diameter})^4/(\text{viscosity})\,(\text{length}) \end{aligned}$$

That is a measure of how readily the pore allows flow. Permeability ω is defined as the flow per unit area per unit pressure drop. But it is usually measured in moles per unit time per unit area of membrane divided by the force per unit area, or moles per newton per second. Accordingly, the relation between the permeability and the hydraulic conductance per pore is

$$\omega = (\text{number of pores per unit area})\,(\text{moles of water per unit volume})\,L_p$$

Take the following values for a cellophane membrane:

permeability $\omega = 2 \times 10^{-9}$ moles/N sec
pore diameter = 10 nm
membrane thickness = pore length = 0.5 mm
viscosity of water = 0.7×10^{-3} N sec/m^2 at body temperature (37°C)
density of water = 1 gram/cm^3, molecular weight = 18, giving molar volume = 55.6 moles per liter. (Verify!)

(a) Calculate the number of pores per unit area of membrane.

(b) Knowing that each pore has a cross-sectional area $\pi r^2 = 8 \times 10^{-17}$ m^2, what fraction of the membrane is open-pore area? The answer is less than half a percent.

✶✶✶ **35. Hemolysis** Red blood cells have a sodium chloride concentration of about 0.15 M. This means that the Na$^+$ concentration is 0.15 moles per liter and the Cl$^-$ concentration is 0.15 moles per liter, giving a total ion concentration of 0.3 M. If the cells are placed in pure water, the osmotic pressure difference tending to force water inside corresponds to that of 0.3 moles of solute per liter. The red cells swell up with water and their cell membranes become leaky, allowing hemoglobin to escape. Such a loss is called *osmotic hemolysis*.

When drugs are injected into the body, they are usually dissolved in "isotonic saline," sodium chloride solution having an ion concentration equal to that of the blood.

(a) What would happen to the red cells if you injected a *hypotonic* solution, a solution whose ion concentration is less than that of the blood?

(b) Now use your imagination: How about a *hypertonic* solution, one whose ion concentration is greater than that of the blood?

✶ **36. Speed of Ocean Waves** To double the wave speed, the wavelength has to be doubled / quadrupled

✶ **37. Speed of Ocean Waves** When the wavelength is doubled, the wave speed is multiplied by 1.41 / by 2 / by 4

When the wavelength is increased by 1%, the wave speed is increased by 1% / by 0.5% / by 2%

✶✶ **38. Wavelength and Period** Those long ocean waves with 156 meters between crests travel at 15.6 m/sec.

(a) How long is the time between crests?

(b) What is the relation between the period T of a wave, its wavelength λ, and the wave speed v? Rewrite the relation in terms of the frequency $\nu = 1/T$. (HINT: The frequency of these long waves is 0.1 Hz.)
(c) How well does the 15.6 m/sec satisfy the formula $V = (g\lambda/2\pi)^{1/2}$?

★★ **39. Wave Speed by Dimensional Analysis** Suppose you intuit that the parameters on which the wave speed can depend are the acceleration of gravity g and the period T.
(a) Postulate an expression of the form

$$v = (\text{constant})\ g^x T^y$$

Using the dimensions of v, g, and T, show that exponents x and y can only take on the values $x = 1$ and $y = 1$.
(b) Suppose you suspected that the density ρ of the water also enters the formula. Postulate an expression

$$v = (\text{constant})\ g^x T^y \rho^z$$

and show that $z = 0$ is the only possibility; that is, your suspicion was wrong.
(c) Repeat the dimensional argument with the variables g and λ (= wavelength), and show that the exponents both have to be $\frac{1}{2}$.

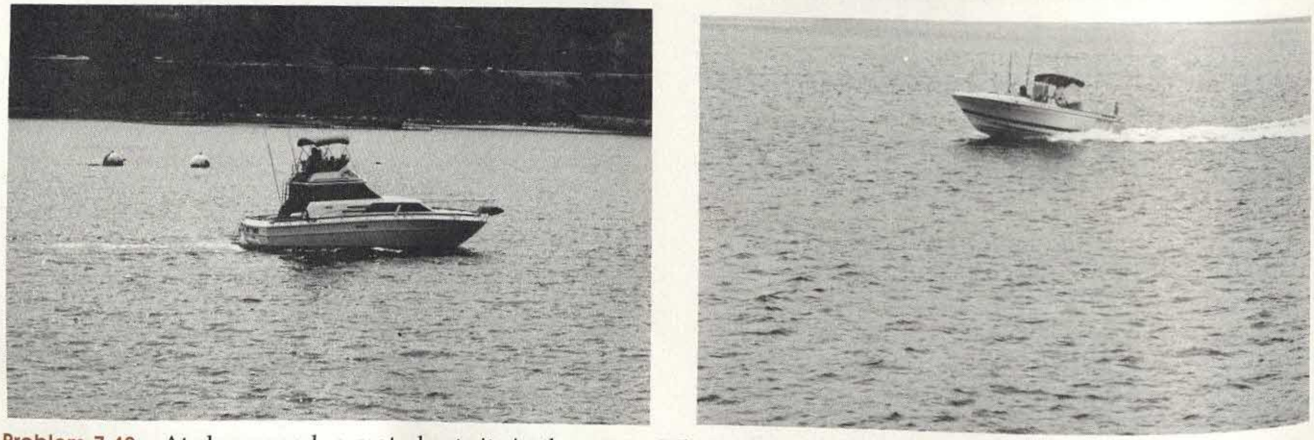
★★ **40. Maximum Speed Without Planing** Sailors know that big sailboats can go faster than little sailboats. This has nothing to do with the sails or the wind. The fastest speed a boat can attain without planing depends almost entirely on its length. The force of the water resisting the boat's motion increases dramatically above a critical speed v related to the boat's length L by the formula

$$\frac{v^2}{L} = 1.8\ \frac{\text{knots}^2}{\text{foot}}\ \left(= 5\ \frac{\text{ft}}{\text{sec}^2}\right).$$

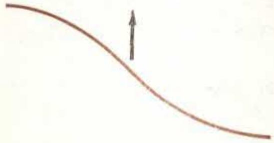
(a) Show that a 20-foot boat is thus limited to speeds below 6 knots (= 10 ft/sec).
(b) How about an 80-foot boat?
(c) Show that the critical speed v is the speed of a deep-water wave having a wavelength equal to the length of the boat.

NOTE: We surmise that the bow wave created by the boat's motion has the speed of waves whose wavelength is equal to the boat's length. To exceed that speed, that is, to overtake its own bow wave, requires great power. When a motor boat "breaks the wave barrier," its bow rises out of the water, and it is said to "plane."
(d) When an airplane "breaks the sound barrier," its speed exceeds the speed of the sound waves its motion generates. Argue that the "sonic boom" is analogous to the bow wave of a motor boat. (See Problem 36 in Chapter 10 on Sound and Hearing.)

★★ **41. Maximum Slope of the Wave** A particle in uniform circular motion has speed ωR. In a short time Δt it moves a distance $\omega R\ \Delta t$. At the place where the water wave slopes downhill steepest, the motion of the water is vertically up.



Problem 7.40 At slow speed, a motorboat sits in the water. When its speed exceeds the speed of its own bow wave, the bow rises out of the water: it planes.



PROBLEM 41. Wave moving to the right. For a sinusoidal wave, the point of maximum slope (point of inflection) is halfway between crest and trough.

NOTE: The wave moves forward; the water moves up (see figure).

How far back along the wave (Δx) is the level of the water surface higher by $\Delta y = \omega R\,\Delta t$? [Answer: $\Delta x = v\,\Delta t$, where v is the speed of the wave.] Convince yourself that the maximum slope is $\left(\dfrac{\Delta y}{\Delta x}\right) = \omega R/v$.

★★ **42. Dispersion** A medium in which waves of different frequencies travel at different speeds is said to be *dispersive*. For light waves in glass, for example, the speed of violet light is about 1% less than that of red, corresponding to not quite doubling the frequency. How much does doubling the frequency change the speed of gravity waves? Is it also a decrease?

★★★ **43. Waves in Shallow Water** [Difficult] For gravity waves in deep water the wave speed is given by the formula $(g\lambda/2\pi)^{1/2}$. But in shallow water of depth d much less than the wavelength, the formula becomes $(gd)^{1/2}$. For depths d comparable to the wavelength λ, the speed is given by that $(g\lambda/2\pi)^{1/2}$ multiplied by a function of the ratio d/λ.

(a) Is this a monotone increasing or decreasing function? Support your answer with an intuitive argument.

(b) That function is continuous. Show that for small values of d/λ, a 1% increase in d/λ corresponds to a 0.5% increase in the value of the function.

(c) For large values of d/λ, the function approaches the asymptotic value ________. A more detailed analysis shows the function is the square root of the hyperbolic tangent of $2\pi d/\lambda$.

★★ **44. Dimensional Analysis**

(a) For gravity waves in shallow water, it is reasonable to suppose that the acceleration of gravity g and depth d are the only parameters that determine the wave speed. Show that the speed has to be proportional to $g^{1/2}\,d^{1/2}$.

(b) For surface-tension ripples on the surface of a liquid of density ρ having surface tension T, it is reasonable to suppose that T, ρ and the wavelength λ are the only parameters determining the speed. Show how they can be combined into a formula having dimensions of m/sec. (HINT: T is in newtons per meter.)

45. (Experimental) Make some crude measurements of the speed of traveling waves in a bathtub or swimming pool. You can generate periodic waves by moving a hand or a foot.

(a) If you double the frequency, by how many percent is the wave speed increased? Decreased?

(b) Show that your measurements are consistent or inconsistent with the formula $(g\lambda/2\pi)^{1/2}$. Problem 43 shows why they might be consistent in the swimming pool and inconsistent in the bathtub.

(c) In the bathtub, it is easier to measure standing waves than running waves. The "fundamental" sloshing mode (lowest-frequency resonance) is easily generated by rhythmically sliding back and forth. Convince yourself that its wavelength is twice the length of the bathtub. Sketch!

★★ **46. A Conversation with the Swimming Coach** Do surface-tension forces matter in winning races? The question is whether it is important for the feet to break the water surface.

First let's calculate (estimate) how much force a fast swimmer is exerting in the backward direction to propel herself forward. A good athlete might put out as much as 800 watts of power for several minutes.

(a) If she is moving through the water at 2 meters per second, the water is pushing her forward with a net force of ________ newtons. That assumes 100% efficiency, of course. It neglects any energy she puts into up-and-down motion or splashing. That force is a ball-park figure, probably about twice too big. But it permits comparison with surface-tension forces.

(b) The surface tension of water is 0.073 newtons per meter. The perimeter of a human foot touching the surface is at most 0.5 meter, so the maximum surface-tension force available is ________ newtons. If surface tension is of any help in propelling the swimmer forward, it can at best contribute ________% of the force.

★★★ **47. Surface-Tension Ripples** [Hard] The formula for the speed of propagation of surface-tension ripples involves the surface tension of water, $T = 0.073$ newtons per meter, as well as its density $\rho = 1.0 \times 10^3$ kg/m^3 and the wavelength λ. It is $V = \sqrt{2\pi T/\rho\lambda}$. This formula is good only for very short wavelengths, so short that gravity is not important.

(a) If λ is doubled, what happens to the speed?
(b) For longer wavelengths, the formula has to be replaced by

$$V = \sqrt{\frac{g\lambda}{2\pi} + \frac{2\pi T}{\rho\lambda}}$$

Show that this goes over into the formula for the speed of gravity waves quoted in the text. How many percent off is the approximate formula for ripples of 2-mm wavelength?
(c) Assume that the speed V of the wind on the pond is the same as the speed of the surface waves it generates. Show that waves of two different wavelengths can travel together. Note that finding the two wavelengths for a given value of V involves solving a quadratic equation in λ. The smaller solution corresponds to the waves we call ripples.
(d) Solving the quadratic equation gives

$$\lambda = \frac{\pi V^2}{g}\left[1 \pm \sqrt{1 - \frac{4Tg}{\rho V^4}}\right]$$

Evidently, there is a minimum value of V for getting any waves at all, since you can't take the square root of a negative number. Find V_{minimum} from the data given. On days with little wind, you will notice glassy areas on a pond, separated from the more normal ripply ones. As the wind picks up, the boundary moves and eventually disappears.

☐ QUESTIONS FOR FURTHER THOUGHT

α. **Watching the Waves from the Shore** The waves seem to slow down as they get into shallower water. Indeed, this is what Problem 43 predicts. If the wave speed is slowed down by 30%, what has happened to the wavelength? What about the frequency? Can you be sure the frequency is unchanged?

β. **Real versus Ideal Fluids** A fluid with no viscosity at all is called an **ideal fluid.** There is a theorem that it is impossible to set an ideal fluid in rotation starting from rest. Just try setting those floating ice cubes in rotation by turning the glass! What is the role of the viscosity of air in making possible the flight of birds and airplanes? HINT: Remember the boundary layers.

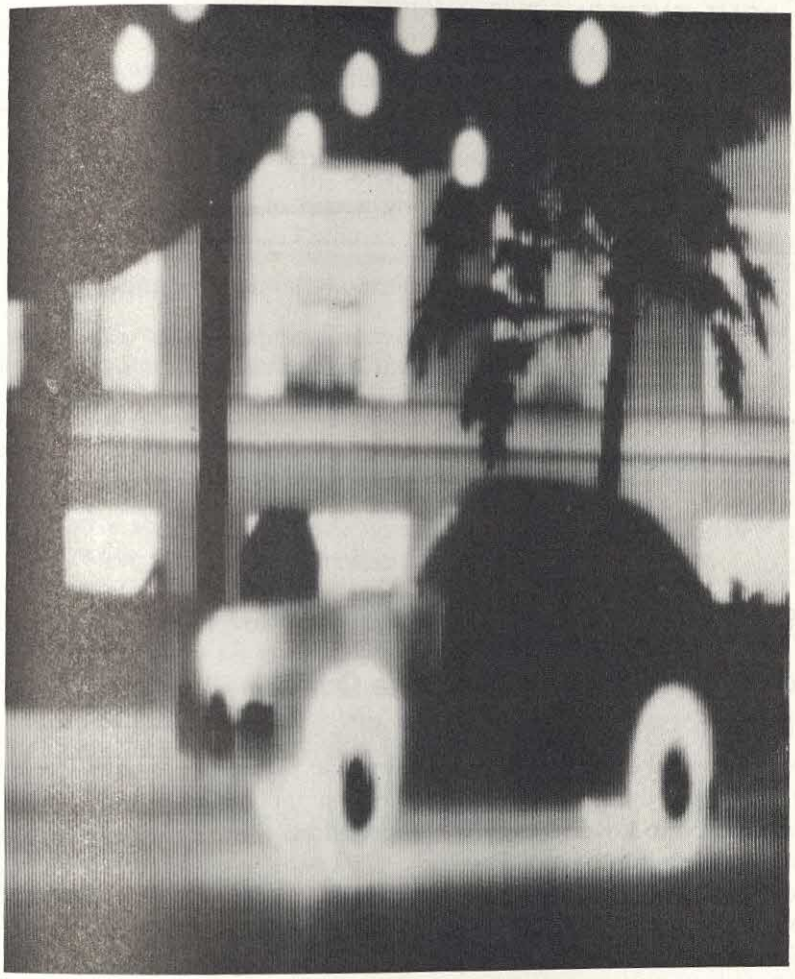
γ. When you turn the faucet on more, the water comes out faster. Yet the water pressure is the same in the city water supply, the length of pipe is unchanged, and so is the cross section of the opening. Explain how a constriction (the valve) controls the rate of flow. Would you say that Bernoulli's Principle is violated?

δ. **Momentum and Energy of a Wave** How does a speck floating on the water surface move as a wave goes by? It looks as though its path is nearly circular. As the crest of the wave approaches, the speck rises and moves forward. As the trough of the wave follows, the speck moves backward and falls.
(a) Argue that the momentum of any volume element of water is zero if averaged over a whole period of the wave.
(b) Argue that the wave nevertheless transports momentum in the direction of travel.
(c) If the wave hits a rigid wall, the average force on the wall will be zero when the time average is taken over a whole period. Argue that a wave hitting a soft or yielding wall exerts an average force on it in the direction of travel. The point is that the rigid wall reflects the wave completely; the yielding wall does not. Think about the water running down off a rock covered with algae.
(d) Argue that the wave transports kinetic energy. What happens to that kinetic energy in the seaweeds?

ε. **Sphygmomanometer** You recall how blood pressure is usually measured by the physician or nurse: The pressure cuff is inflated until the brachial artery is pinched off. Then the pressure is slowly released and the level **(systolic)** recorded at which the sounds of turbulent blood flow in the artery can first be heard in the stethoscope applied to the skin next to the artery. As the pressure is further reduced, blood flows in the artery for an increasing fraction of the cardiac cycle, but the periodic surging still results in the "Korotkoff sounds." The sounds stop when the pressure reaches the level **(diastolic)** where the artery is open for the entire cardiac cycle.

When the pressure in the cuff is just below diastolic, the artery, although open for the entire cardiac cycle, is slightly constricted. Of course, the speed of blood flow at the constriction is faster than in the rest of the artery. According to Bernoulli's Principle, the pressure at the constriction is therefore lower than the true arterial pressure. Why does this not result in too low a reading for the diastolic pressure?

5. **Getting Clean in the Shower** How should the shower head be constructed to get you clean fast? Is it better to have a lot of water at lower speed or less water at higher speed? There is some evidence that the tangential velocity of the water across the skin governs the rate at which sticky dirt is washed off. Argue that we probably prefer a shower that delivers the water at a speed slower than the best speed for getting clean. Argue that cost rather than optimal performance determines the design of residential showers.



Thermogram of a car that was made by using infrared-sensitive film. Dark areas are cold; light areas, such as lights, engine, and tires are warm.

8

HEAT AND TEMPERATURE

SOME QUESTIONS THIS CHAPTER WILL ANSWER

What is thermal conductivity?

Why are there no warm-blooded animals much smaller than mice?

Explain how the rate of melting of an ice cube depends on its size.

About how much does drinking a cup of hot coffee raise your body temperature?

What does the First Law of Thermodynamics say about perpetual-motion machines?

Argue that the volume coefficient of thermal expansion of a substance should be 3 times its linear coefficient of thermal expansion.

Name two advantages of the ideal-gas temperature scale over traditional scales like Celsius and Fahrenheit.

What is a triple point?

8.1 RATE OF COOLING

Warming the Body, Cooling the Coffee

If we want to be warmer, we put on more clothes. That works because the body is a source of heat. Individual muscle fibers are constantly contracting and relaxing, doing work while various sugars are being metabolized, broken down into lower-energy waste products. Most of that work is immediately converted into heat. The body has a sensitive thermostat, which keeps the blood's **temperature** very close to 37°C (= 98.6°F).[1] Except on the very hottest days, that is well above the temperature of the air around us. If the skin loses heat too fast, we feel cold. We respond with more muscular activity (e.g., shivering) and with contraction of capillaries close to the skin.

The rate at which we lose heat seems to depend on the amount of skin area exposed to air currents. Also, it certainly increases in a high wind. And the colder the air, the faster the heat loss. What clothes do for us is to slow down the air currents close to the skin. Air that has a chance to stay close to the skin for a while is warmed, and we lose heat more slowly. That is why we are advised to wear a number of thin layers of wool or cotton clothing for warmth. These fibers are highly kinked, providing tiny pockets of still air.

How fast does hot coffee cool off? A big pot cools slower than a small cup. A metal cup cools faster than a styrofoam cup. Here we are talking about something without a source of heat inside. The rate of heat loss is measured by the rate of temperature drop. But two cups side-by-side cool just as fast as one. Again, they cool faster if you blow on the surface, still faster if you stir. Here are two generalizations:

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[1] The conversion formula between Celsius and Fahrenheit degrees is given in Sec. 8.4.

1. The rate of heat loss is proportional to the surface area exposed to the colder surroundings.
2. The rate of heat loss is proportional to the temperature difference between the hot object and the colder surroundings.

EXERCISE 1

(a) How does blowing on the coffee increase the temperature difference between the hot surface and the air in contact with it?
(b) What about stirring?

You have probably learned somewhere that heat transfer occurs by **conduction,** by **convection,** and by **radiation.** Our two generalizations do not seem to distinguish between mechanisms. In any particular situation, all three mechanisms are involved at the same time. In the case of the hot coffee, convection is by far the most important mechanism, both in the air and in the liquid. The distinction between conduction and radiation can be fuzzy. Only when heat transfer is across empty space can we be sure that only radiation is involved. The reason for wanting to distinguish is that we characterize some materials as good thermal **insulators,** some as good conductors. That is the difference between the styrofoam and the aluminum cup.

Conduction

To make that difference quantitative, we shall define **thermal conductivity.** Every substance has one. Aluminum has a large one; styrofoam has a small one (see Table 8.1). But before becoming quantitative, some words of warning are in order. The two generalizations above, that rate of heat transfer is proportional to (1) area and (2) temperature difference, are rigorously true for conduction. They are reasonably good approximations when convection and radiation are involved. In an experiment designed to measure heat conduction, it is almost impossible to eliminate convection and radiation. Still, to define a conductivity, we assume that they have somehow been eliminated.

We assume that there is a hot reservoir and a cold reservoir, separated by a flat layer of the "conducting" material. Make the layer 1 millimeter thick and 1 square meter in area. If the temperature difference across the layer is 1°C, we say the **temperature gradient** is 1 degree per millimeter. If the conducting material is aluminum, 50×10^3 **calories** of heat would be conducted

TABLE 8.1 Thermal Conductivities, in Calories per Second per Meter per °C

	$K = \frac{-J}{\nabla T} \sim \frac{\text{cal/sec m}^2}{{}^\circ\text{C/m}}$
Air	0.0055
Fiberglass	0.01
Sawdust	0.01
Wool	0.01
Asbestos	0.02
Soil	0.03
Wood	0.03 to 0.06
Snow	0.05
Brick	0.15
Glass	0.2
Granite	0.5 to 1.0
Aluminum	50
Copper	92

across that 1-millimeter layer every second. But if the layer of aluminum between the same reservoirs were only half a millimeter thick, the temperature gradient would be 2 degrees per millimeter, and twice as much heat would flow per second: The rate of heat transfer is proportional to the temperature gradient (see Fig. 8.1). Our original temperature gradient was 1 degree/mm, or 10^3 degrees/m. That gave a heat flow of 50×10^3 calories per second per square meter of layer. Accordingly we say that the thermal conductivity is 50 thermal conductivity units ($= 50 \times 10^3/10^3$). In those same units, **styrofoam** has thermal conductivity around 0.01. We now work out the right units for the thermal conductivity.

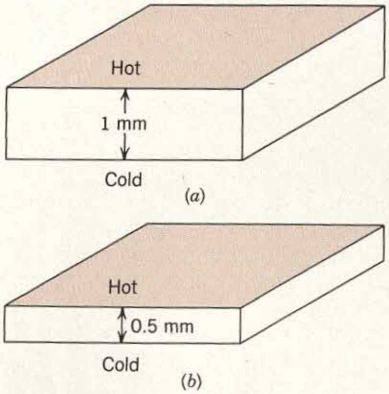


FIGURE 8.1 *(a)* Heat conduction—Heat flows through the aluminum sheet from the "hot" surface to the "cold" surface. The heat flow per unit area of sheet is proportional to the temperature gradient. *(b)* Through a sheet half as thick, heat flows twice as fast, because for given temperature difference the temperature gradient is double.

The heat flow J was expressed in calories/sec m². The temperature gradient ∇T was given in degrees/m. The law of heat conduction is that the two are proportional:

$$J \propto \nabla T \qquad \text{(Fourier's law)}$$

The thermal conductivity κ is the proportionality constant:

$$J = -\kappa\, \nabla T$$

Accordingly, the units of κ are

$$\frac{\text{calories/sec m}^2}{\text{degrees/m}} = \frac{\text{calories}}{\text{sec m degree}}$$

Don't be surprised that very few reference works use those particular units. A lot of them measure J to BTU per square foot per hour, and ∇T in °F per inch!

NOTE: 1 **BTU** = 0.2525 **kcal.** The − sign in the definition of κ just says that the heat flows *down* the temperature gradient.

EXERCISE 2 For aluminum, $\kappa = 50$ cal/sec m degree $= 0.50$ cal/sec cm degree $= 209$ watts/m degree. For copper, $\kappa = 92$ cal/sec m degree.

(a) Convert to the two other units.

(b) How much heat would a layer of copper 1 m^2 in area and 1 mm thick conduct per second if the temperature difference were 1°C?

(c) 2°C?

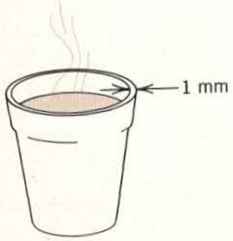


FIGURE 8.2 Styrofoam coffee cup. Thickness = 1 millimeter.

EXERCISE 3 If the layer thickness is doubled and the temperature difference is doubled, the temperature gradient ∇T stays the same.

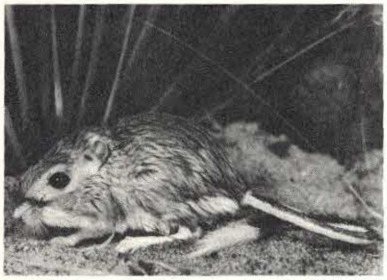
(a) What happens to the heat flow J?

(b) If the layer thickness is doubled and the temperature difference is halved, what happens to the heat flow J?

EXERCISE 4 The hot coffee is at 80°C. The styrofoam cup is 1 mm thick, and the outside is pleasant to touch, perhaps 50°C (see Fig. 8.2). If κ for styrofoam is 0.01 cal/sec m °C, what is the heat flow J across the surface, in calories per second per m^2?

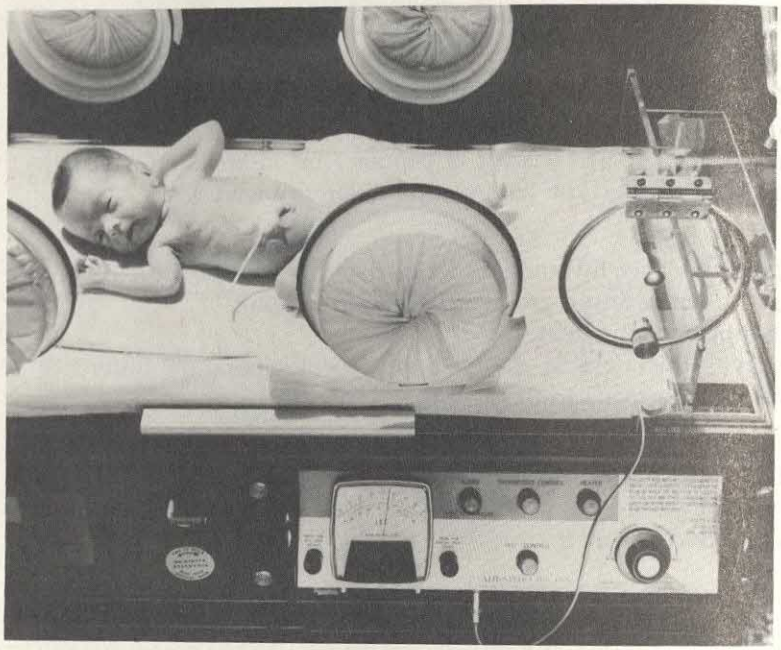
Size of Warm-Blooded Animals

The rate at which an animal can generate heat to keep warm depends on how much muscle it has. It is reasonable to guess that for different animals, the amount of muscle is roughly proportional to the body weight or volume. The rate at which the animal loses heat is proportional to its surface area. Right away you know why there are no **warm-blooded** snakes! The snake has a large surface-to-volume ratio. Warm-blooded animals tend to be built more compactly, just to save energy. But you also know why there are no warm-blooded animals much smaller than a mouse. A small animal just loses heat very fast. The smaller the animal, the greater its surface-to-volume ratio.



The kangaroo mouse is one of the smallest species of mammals. A warm-blooded animal much smaller than this just could not eat fast enough to keep warm.

To see how that goes, suppose that the mouse were a cube 2 centimeters on a side. Of course, there are no cubical mice. This silly assumption is a physicist's way of making the arithmetic easy. A 2-cm cube has a volume of 8 cm^3 (= 2 cm × 2 cm × 2 cm), and a surface area of 24 cm^2 (= six faces, each 2 cm × 2 cm). What happens if all the dimensions are halved? A 1-cm cube has a volume of 1 cm^3 and a surface area of 6 cm^2. You did this kind of arithmetic in Chapter 3, in "Digression on Scaling." There we worried about being strong enough to support our own weight. Here we are concerned with keeping warm. If the metabolic rate is proportional to body volume, then the 1-cm^3 mouse produces heat at one-eighth the rate of the 8-cm^3 mouse. But it loses heat at one-fourth the rate of the larger mouse. Evidently, the smaller mouse could not sustain much of a temperature difference between itself and its environment.



A premature baby in an incubator kept at body temperature. The surface-to-volume ratio of a baby weighing less than about 2 kilogram-weight is too large for the baby to sustain its body temperature in an environment 15° C colder.

The ratio of surface to volume has units of cm^2 divided by cm^3. For the 1-cm cube, it is

$$\frac{\text{surface}}{\text{volume}} = \frac{6\ \mathrm{cm}^2}{1\ \mathrm{cm}^3} = 6\ \mathrm{cm}^{-1}$$

For the 2-cm cube, it is

$$\frac{\text{surface}}{\text{volume}} = \frac{24\ \mathrm{cm}^2}{8\ \mathrm{cm}^3} = 3\ \mathrm{cm}^{-1}$$

Evidently, the surface-to-volume ratio is *inversely proportional* to the linear dimensions. Repeat: The smaller the animal, the greater its surface-to-volume ratio.

Now you know why babies are kept well wrapped in the winter and why premature babies have to be kept in an incubator. Small birds have to spend most of the day feeding. To keep warm, they have to consume several times their body weight in food per day. Contrast that with man, who eats less than 3% of his body weight per day. An animal much smaller than a mouse could not eat fast enough to keep its body temperature more than a few degrees above the temperature of its surroundings. (See Questions for Further Thought at the end of this chapter.)

EXERCISE 5 A cube of side a has volume a^3 and area $6a^2$. Its surface-to-volume ratio is $6/a$. A sphere of radius r has volume $\frac{4}{3}\pi r^3$ and area $4\pi r^2$.

(a) Calculate its surface-to-volume ratio.

(b) Find the surface-to-volume ratio of a sphere of volume 0.075 m^3. That is roughly the volume of an adult human.

EXERCISE 6

(a) Calculate the surface-to-volume ratio of a cube of volume V.

(b) Calculate the surface-to-volume ratio of a sphere of volume V.

(c) Compare the rate of heat loss of a warm sphere and a warm cube of the same volume.

8.2 HEAT CAPACITY AND SPECIFIC HEAT

Are you annoyed because terms like *calorie* and *degree* have been used in the foregoing pages without being defined? If you resorted to the Key Terms at the end of the chapter, you got a definition but little explanation. Indeed, for more about temperature scales you have to wait until Section 3. This chapter has been written with the conviction that you are no stranger to the thermometer. So much for °C and °F. Either **Celsius** degrees or **Fahrenheit** degrees are familiar to you. Physicists prefer °C because they are more international. You are likely to find them in use at the hospital, and certainly in the biology lab. But the calorie? That is familiar to you from diet charts. It has to do with food. What is its connection with heat? Actually, when they say that a glass of milk contains 100 Calories, they mean that in being metabolized, it gives up 100 *thousand* calories of heat. In other words, the nutritionist's **Calorie** is the physicist's **kilocalorie** (kcal).

But this chapter has also assumed that you accept the idea that **heat** is a form of energy. It has used the conversion formula

$$1 \text{ calorie} = 4.18 \text{ joules}$$

without making a fuss over it. That was big news in the eighteenth century, and it was called the **Mechanical Equivalent of Heat.** For you, the idea that work can be converted into heat is old hat. We showed back in Chapter 4 how doing work on an object gives it kinetic energy:

$$\text{work} = \Delta(\tfrac{1}{2}mv^2)$$

If we do work on a bunch of atoms, the fact that it all goes into increasing their total energy,

$$\text{work} = \Delta\sum_i \tfrac{1}{2}m_i v_i^2 \qquad \sum_i = \text{sum over all the atoms}$$

contains no new mystery. So you may wonder why an independent defini-

Waterfall. As the water loses altitude, its potential energy is converted into kinetic energy. When the water hits an obstacle, its motion becomes less organized, more random. We say that the kinetic energy appears as heat.

tion of the calorie as a unit of heat is necessary or even desirable. Nevertheless, here it is:

> The calorie is the amount of heat needed to heat 1 gram of water by 1°C. More precisely, it is the heat needed to raise 1 gram of water from 14.5 to 15.5°C.[2]

To get from that definition to the conversion formula, 1 calorie = 4.18 joules, required measurement. The work that needed to be done on the water to raise its temperature by one degree had to be measured. (See Exercise 7.) Not surprisingly, it turns out that it doesn't matter how the work is done. Stirring, or pumping, or throwing the water, or hitting it, all have the same effect. A given amount of work gives the same temperature rise. Again, not surprisingly, it takes twice as much work to heat 2 grams by one degree as to heat 1 gram.

EXERCISE 7 A stirrer agitates 1 liter (= 1 kg) of water. Moving at 360 rpm (= 6 revolutions per second) in a circle of radius 5 cm, it meets a resistance of 0.1 newton

(a) How much work does it do in an hour?

(b) How much does the water temperature rise? Assume that no heat is lost to the surroundings. [Answer: 0.162°C]

EXERCISE 8 A 10-gram bullet is brought to rest in 1 liter (= 1 kg) of water. It entered moving at 300 m/sec.

(a) Calculate the initial kinetic energy $\frac{1}{2}mv^2$ of the bullet.

(b) How many °C is the water warmed? Neglect the heat capacity of the bullet and any heat loss to the surroundings. [Answer: 0.1°C]

EXERCISE 9 How much warmer is the water at the bottom of a 100-meter waterfall than at the top?

Logically, we did not need the calorie as a separate energy unit for heat energy. The joule is a perfectly satisfactory unit. Is the calorie then just a historical vestige? Perhaps. But what it does for us is to give us a standard for **heat capacity.** The heat capacity of 1 gram of water is 1 calorie per degree C. That just means it takes 1 calorie to raise its temperature by one degree. We get that straight from the definition of the calorie. The heat capacity of 2 grams of water is 2 cal/°C. Since the heat capacity is proportional to the amount of water, it makes sense to talk about heat capacity per gram. That is called **specific heat.** Thus we say that the specific heat of water is 1 calorie per gram per °C.

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[2] Another unit, the average calorie, is 1/100 the amount of heat needed to raise the temperature of 1 gram of water from 0 to 100°C. That one is 4.186 joules. It is also called the International calorie. Then there is the thermochemical calorie, defined as 4.184 joules. If you need the fourth significant figure to interpret a measurement, you have to know which international conference the experimenter honored.

EXERCISE 10 Coffee is mostly water. Its specific heat is very close to 1 cal/gm °C. How many calories does one cup (= 225 cm^3 or 225 gm) of coffee have to lose to cool from coffee-pot temperature, 95°C, to drinking temperature, 45°C?

EXERCISE 11 The human body is said to be 80% water. Its average specific heat turns out to be about 0.83 cal/gm °C. How many calories are required to raise the temperature of a 75-kg adult by 1°C, as in a moderate fever?

EXERCISE 12 In being drunk, the cup of 45-degree coffee is cooled to 37°C. How much does it raise the body temperature of a 75-kg man? (HINT: The increase in mass of the man is negligible.) Can a clinical thermometer measure that small a temperature rise?

Water has a higher specific heat than most substances. That is one reason why water is used so widely as a coolant. Just for comparison, Table 8.2 lists the room-temperature specific heat of various other substances.

TABLE 8.2 Specific Heat at Room Temperature

	(Calories/gm °C)
Ethyl alcohol	0.56
Rubber	0.45
Wood	0.42
Air	0.25
Glass	0.2
Sand	0.19
Steel	0.11

EXERCISE 13 The specific heat of iron is 0.11 cal/gm °C. How much will 1 calorie raise the temperature of 1 gram of iron?

EXERCISE 14 The heat capacity of **monatomic** gases is very close to 3 cal/mole °C. This refers to the noble gases, such as helium, neon, and argon, and to metallic vapors. The number comes from theory: According to the Equipartition Theorem, each degree of freedom has average energy $\frac{1}{2}kT$, where k is Boltzmann's constant and T is the absolute temperature. Each atom of the gas has 3 degrees of freedom, corresponding to translational motion in the three dimensions of space. Accordingly, N atoms at temperature T have kinetic energy $\frac{3}{2}NkT$. The heat capacity is the energy per degree, or $\frac{3}{2}Nk$. If N = Avogadro's number, then $Nk = R = 1.98$ cal/mole K, the ideal gas constant.

(a) Verify that 1.98, using the data of Chapter 6.
(b) Show how the 3 cal/mole K is obtained.
(c) Calculate the specific heat of neon gas (atomic weight = 20).

EXERCISE 15 The heat capacity of **diatomic** gases, like O_2 and N_2, is given as $\frac{5}{2}k$ per molecule or $\frac{5}{2}R$ per mole. The theory is that in addition to the three translational degrees of freedom, there are two rotational degrees of freedom for each molecule.

(a) Show that this heat capacity comes to 5 cal/mole K.
(b) Calculate the specific heat of air (average molecular weight = 29). Compare with the value given in Table 8.1.
(c) How many calories does it take to raise the temperature of the air in a classroom 10 m × 10 m × 3 m by 1°C?

Are you ready for a definition? Try this one:

The *heat capacity* of an object is the heat needed to raise its temperature by a small amount ΔT divided by that temperature increment ΔT.

That kind of definition applies also to an object whose heat capacity changes with temperature. The reason for writing "ΔT" instead of simply "1°C" is that the heat capacity can change measurably even over a fraction of a degree. Also, heat capacity still has meaning near a boiling or freezing point. For example, the specific heat of water at 99.5°C—just half a degree below the boiling point—is 1.0067 cal/gm °C. But adding 1.0067 calories to a gram of water at that temperature will not raise its temperature by one degree. It boils at 100°C, and some of it changes to vapor. Right at the phase transition, heat capacity loses its meaning, since adding a small amount of heat does not raise the temperature. Another way to put that is to say the heat capacity is infinite there.

EXERCISE 16 How much heat is needed to raise the temperature of 1 gram of water from 99.5 to 99.6°C?

Latent Heat

At the boiling point, adding heat changes liquid to vapor. To vaporize 1 gram of water at 100°C takes 540 calories. If 540.5 calories are added to that gram of water at 99.5°C, only the first half of a calorie goes into raising its temperature—to 100°C. All the rest goes into vaporizing the liquid. As long as both liquid and gaseous water are present at a pressure of 1 atmosphere, the temperature will remain 100°C.

The situation is similar at the freezing point. You might prefer to call it the melting point. Just below the melting point of ice there is a meaningful heat capacity. As soon as ice starts to melt at 0°C, additional heat just makes more ice melt. Not until all the ice is melted can additional heat cause a temperature rise. When ice and water are in equilibrium at a pressure of 1 atm, the temperature is 0°C. It takes 80 calories to melt 1 gram of ice. You can see why the sea is such a good **thermostat:** In the winter, it can give up large quantities of heat to the colder air above it without getting any colder. A little bit of it freezes. In the spring, as that ice melts, the water cools the air in contact with it without itself warming up.

The heat energy needed for the **phase transition** is called a **latent heat.** The 80 cal/gm at the ice point is the latent heat of fusion. The 540 cal/gm at the steam point is called the latent heat of vaporization. They can be understood in terms of the breaking of chemical bonds. The H_2O molecules are held in tighter in ice than in liquid water. They form a crystal lattice. Heat added to ice goes into lattice vibrations of increasing amplitude. At the melting point, the vibrations are large enough to break the lattice structure, which gives way to the less-ordered structure of the liquid. At the boiling point, all bonds between H_2O molecules break. In evaporating, the average distance between H_2O molecules increases more than tenfold!

EXERCISE 17 How much heat is needed to transform 1 gram of ice at 0°C into 1 gram of steam at 100°C?

8.3 FIRST LAW OF THERMODYNAMICS

A phase transition with such a large volume change invites practical exploitation. One kilogram of water occupies a volume of 1 liter. One kilogram of steam at 100°C occupies 1700 liters. In a steam engine that huge expansion is used to convert heat into work. The steam pushes a piston or turns a turbine, cooling as it does work. You are more familiar with internal-combustion engines, like the gasoline engines used in cars or the Diesel engines used in trucks. These are also heat engines. Fuel is burned to heat up a gas, which expands. The hot gas pushes against a piston or a rotating vane and does work as it cools. We can define a heat engine as any device for converting heat into work.

In Sec. 8.2 we talked about converting work into heat. We paraded the old "work-energy theorem," that the work done on an object goes into increasing its kinetic energy $\frac{1}{2}mv^2$. The reverse process decreases the kinetic energy of the molecules of a gas in order to make them do work. That may be a little harder to analyze in detail. But you probably have enough faith in our ability to do "energy bookkeeping" so that you don't especially care about the mechanism of the energy conversion from one form to another.

That article of faith is called the **First Law of Thermodynamics.** The law just says that the heat that disappears reappears as work, and vice versa.

Perpetual Motion Machines

The popular statement of the First Law is that "You can't make a machine do work without feeding it." We did not have to wait for a chapter on heat to make that announcement. It might be regarded as another form of the work-energy theorem. But in a large sense, it is an article of faith. It says, if energy seems to appear out of nowhere, we just haven't looked hard enough for its source. The history of invention is full of perpetual-motion machines that looked fine on the drawing board. In fact, either they didn't work, or else they were fraudulent. The electric plug was hidden, or energy came in by radiation, or fuel was supplied in secret. Energy bookkeeping is always possible. In the history of physics, that faith has been so strong that physicists keep inventing new "forms of energy" to make the books balance.

☐ QUESTION FOR FURTHER THOUGHT The Rake's Progress

In a fairy tale, the devil offers a machine that turns stones into bread (see Fig. 8.3).

(a) Present an argument that such a device does or does not violate the First Law. (HINT: Both points of view can be defended.)

(b) A nuclear breeder reactor extracts energy from uranium and "breeds" plutonium, another nuclear fuel. Why is there no flaw in the energy bookkeeping? (See Chapter 19)

FIGURE 8.3 Turning stones into bread. Is the First Law of Thermodynamics violated?

8.4 TEMPERATURE SCALES

Fahrenheit to Celsius

If you grew up reading a Fahrenheit thermometer, changing to Celsius is a little bit like learning a new language. Figure 8.4 is the dictionary that allows quick translation (conversion). Here is how to memorize it:

The Celsius (centrigrade) scale has 100 degrees between the freezing point and the boiling point of water, 0°C and 100°C. The Fahrenheit scale has 180 degrees between those points, 32°F to 212°F. That is all you have to know. Everything else follows by proportions.

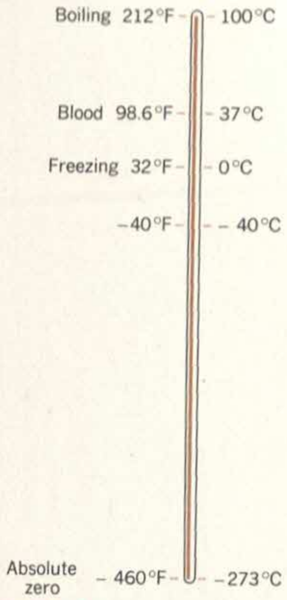

FIGURE 8.4

EXERCISE 18 Show that the formula

$y = \frac{9}{5}x + 32$

for converting Celsius to Fahrenheit degrees gives the right freezing and boiling points. Which is which?

EXERCISE 19 Prove the formula

$x = \frac{5}{9}(y - 32).$

EXERCISE 20 Prove the formula

$x + 40 = \frac{5}{9}(y + 40)$

EXERCISE 21 Tin melts at 231.9°C. Convert to °F.

EXERCISE 22 Hydrogen boils at −253°C. Convert to °F.

EXERCISE 23 Room temperature is often taken as 68°F. Convert to °C.

EXERCISE 24 Is there any temperature at which the Celsius and Fahrenheit readings are the same?

Mercury Thermometer

The thermometers you have read are probably mercury thermometers. There is a glass bulb at the bottom, containing the liquid metal mercury. Extending up from the bulb is a uniform capillary tube. Along the side of the tube are printed the numbers. Many household thermometers contain colored alcohol instead of mercury. In either case, you notice that the scale of degrees is uniform. The separation between the 30° and the 40° marks is the same as between the 40° and the 50° marks, and so forth, all along the tube. That may not seem remarkable to you, but here is what it means: It means that the volume of liquid is presumed to increase as much for a one-degree temperature rise at the bottom of the scale as at the top. Nature does not work quite that way. It is more accurate to say that the **coefficient of thermal expansion** of mercury is approximately constant over the temperature range of the instrument. That coefficient is 0.0001816/°C at room temperature. If the thermometer contains 0.1 cm^3 of mercury at room temperature, a rise of 1 degree C increases that volume by 0.00001816 cm^3. You see that the capillary tube has to be very thin to register such a small volume change.

EXERCISE 25 If the capillary has a diameter of 0.2 mm, how much does the top of the column move for 1 degree C? (HINT: Volume of a cylinder = height × πr^2.) [Answer: 0.58 mm]

Ten degrees above that, the volume of liquid is 0.10018 cm^3. OK? Another 1-degree rise gives a volume increase of

$$0.10018 \text{ cm}^3 \times 0.0001816 = 0.00001819 \text{ cm}^3$$

That is not exactly the same volume change as before, but the difference is in the fourth significant figure. If we really need great accuracy, that may matter. It may also matter that the volume coefficient of thermal expansion of mercury increases by almost one-tenth of 1% over 10 degrees. But we are again talking about that fourth significant figure in the volume change.

The next paragraph makes a big fuss over the **ideal-gas** scale of temperature. The volume coefficient of thermal expansion of an ideal gas is known from theory. The reason for parading the arithmetic for a mercury thermometer here is to show you that such an instrument is quite adequate for most purposes, even without fancy corrections. When better than four-digit accuracy is required, we need to be more careful in defining our temperature scale.

EXERCISE 26 For small temperature changes Δt, the volume expansion is given by the formula $V = V_{\text{initial}}(1 + \beta\,\Delta t)$. For alcohol at 20°C, the volume coefficient of thermal expansion β is 0.00112/°C. How much does the volume of 0.1 cm^3 of alcohol increase between 20°C and 21°C?

Ideal-Gas Temperature Scale

The zero on the Celsius scale is a completely arbitrary point, chosen by convention. The same goes for the zero on the Fahrenheit scale. Neither of these scales announces to the user that there really is such a thing as a lowest temperature, an absolute zero. But you know all about −273°C.

Absolute temperature is defined in such a way that it is a measure of the average kinetic energy of random motion of the molecules. Chapter 6 presented the kinetic theory of the ideal gas. In that connection, it was pointed out that a reasonable unit of temperature would be the joule per molecule. But historically, thermometers and degrees were invented before kinetic theory. That is why degrees are still with us as a temperature unit. Here is how the ideal-gas scale evolved:

It was considered desirable to have 100 degrees between the freezing point and the boiling point of water. Once that is accepted, the size of the degree is defined. The temperature values to be assigned to these points is then a matter of measurement. Measurement on an ideal gas? Well, it can be approached by using real gases at low pressures. But you see the problem. At the temperature of the freezing point of water, the volume coefficient of thermal expansion of an ideal gas turns out to be 1 part in 273.15 per degree. In order to have an ideal-gas law of the form

$$pV \propto T$$

that temperature has to be called 273.15. The unit is called the **Kelvin,** and we write 273.15 K. Some people say 273.15 degrees absolute. Nowadays the ° sign is no longer fashionable. After all, the Kelvin is an energy unit. The **absolute zero** of temperature corresponds to zero energy of random motion. Note that the size of the Kelvin is the same as the size of the degree C.

*Triple Point

To make very accurate temperature comparisons possible between different laboratories, it is necessary to be able to reproduce some standard temperature. The freezing point of water varies sensitively with atmospheric pressure. To get that 273.15 number, the pressure was specified to be exactly 1 atmosphere. So an accurate measurement of pressure is needed in order to reproduce that temperature. The situation is worse at the boiling point. There is a way around this: Look for a triple point.

At any given temperature, liquid is in equilibrium with vapor only at one pressure. We talk about the vapor pressure of the liquid. For example, the vapor pressure of water at 20° C is 0.023 atm. If there is less vapor than that above the liquid, it will evaporate. If there is more, then vapor will condense into the liquid. The vapor pressure of water at 100° C is 1 atm. That is why we call that the boiling point. Similarly, there is a vapor pressure above the solid. For example, the vapor pressure of ice at -10°C is 0.0026 atm. There is one temperature at which the vapor pressure of water and of ice are the same. At that point, all three phases—ice, water, and water vapor—are in equilibrium. Note that this **triple point** defines a unique temperature and a unique pressure. It is 273.16 K and 0.006026 atm. (= 4.58 mm Hg). Please don't try to memorize those numbers. They appear here just to convince you that there is such a thing as a triple point and that it can be achieved without an expensive vacuum pump. Modern thermometry uses a triple-point temperature of 273.16 K as a definition of the Kelvin as a unit of temperature.

☐ QUESTION FOR FURTHER THOUGHT The Triple Point of Carbon Dioxide (CO_2) is 216.55 K and 5.112 Atm.

At atmospheric pressure we see dry ice (solid CO_2) subliming, that is, going directly from solid to vapor without melting. What would you do in order to see dry ice melt?

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*This section can be omitted on first reading.

SUMMARY

Heat loss from the body prevented by clothes

Heat loss rate proportional to (1) area exposed and (2) temperature difference

Conduction, convection, and radiation

Definition of thermal conductivity

Law of heat conduction: heat flow $\propto$ temperature gradient

Small warm-blooded animals have to eat often

Surface-to-volume ratio is inversely proportional to linear dimensions

Specific heat of water is 1 calorie per gram per degree

Nutritionist's Calorie = kilocalorie

Mechanical equivalent of heat

Infinite heat capacity at freezing and boiling points

Latent heat

First Law: Heat that disappears reappears as work

Fahrenheit-Celsius conversion: 32°F = 0°C; 212°F = 100°C

Thermal expansion coefficients are around 10^{-3} per degree, or less, for liquids

Thermal expansion coefficients for gases = $1/T$. That is the ideal-gas scale

Triple point defines a unique temperature and pressure

KEY TERMS

Celsius, or °C Temperature scale used in most countries; also called Centigrade, because there are 100 degrees between the freezing and boiling points of water, called 0°C and 100°C, respectively.

Fahrenheit, or °F Temperature scale still in popular use in some English-speaking countries. The freezing point of water is 32°F; the boiling point is 212°F. That difference is 180 degrees, so the size of the degree Fahrenheit is $\frac{100}{180} = \frac{5}{9}$ of the size of the degree Celsius.

Kelvin, or K Also called the absolute temperature scale. The size of this unit of temperature is defined to be the same as the degree Celsius, but a temperature of zero K is the absolute zero of temperature, −273.15°C.

Temperature The measure of how hot something is, usually measured in *degrees*. The measuring device is called a *thermometer*. More precisely, temperature is a measure of the average energy of random motion.

Absolute zero The lowest possible temperature, the temperature at which one would extrapolate that all random motion would cease. A gas thermometer can be used to extrapolate to this unattainable state of total order.

Thermostat A device for regulating temperature. This is often a servomechanism that turns a heater on or off as the temperature gets cooler or warmer than desired.

In animals, on the other hand, thermostatic regulation is usually accomplished by varying the rate of heat loss. A thermostat can, of course, be simply a surrounding, a "bath" with a large heat capacity. This is the sense in which a frozen lake is a good thermostat for 0°C.

Thermal gradient Or Temperature gradient. A region with one cold side and one hot side. In technical use: The vector pointing in the direction in which the temperature increases most rapidly. Its magnitude is equal to the rate of increase, measured, say, in degrees per millimeter. Written ∇T, pronounced grad T. Don't confuse the symbol ∇ with the symbol Δ. The expression ΔT, pronounced delta T, just means temperature difference.

Metabolism The process by which living things break down nutrients for growth and energy supply.

Conduction Heat transfer via sound waves. The thermal motion of one atom is communicated to its neighbors via interatomic forces.

Radiation Heat transfer via electromagnetic waves.

Convection Heat transfer via mass transfer. In practice this often means air currents. Of the three heat transfer mechanisms, convection is the dominant one for heat loss from the human body and from the furnace. Radiation is the mechanism by which we get our energy from the sun. Conduction is what tells us the stove is hot *after* we touch it.

Insulator A material that does not let heat through easily. A material with low thermal conductivity. In technical use one often says *thermal* insulator, to avoid confusion with *electrical* insulator. But good thermal insulators are usually good electrical insulators too.

Styrofoam A very lightweight construction material made of polystyrene with tiny air cells throughout. You know this excellent insulator through the disposable hot-drink cups.

Thermal conductivity The characteristic property of a material that measures its ability to conduct heat. The quotient of heat flow divided by temperature gradient. Its reciprocal is called the thermal *resistivity*.

***R*-Value** Measure of the effectiveness of a sheet of thermal insulation. Quotient of temperature difference across the sheet divided by heat flow through it, in units of degrees Fahrenheit divided by BTU per hour per square foot. With attention to units, this is equal to thickness divided by thermal conductivity.

Heat In popular usage, heat is what is transported from a warm place to a cool place. In the eighteenth century, when it was discovered that heat is not a "substance" that is conserved, a new definition had to be found. Now heat is defined as the kinetic energy of random molecular motion. This definition makes it apparent that mechanical energy can be transformed into heat energy.

B.T.U. British thermal unit. The quantity of heat needed to raise the temperature of one pound of water by one degree Fahrenheit. 1 B.T.U. = 0.252 kilocalorie.

Calorie The quantity of heat needed to raise the temperature of one gram of water by one degree Celsius. If four-place accuracy is desired, distinguish between the 15°-calorie (= 4.184 joules) and the international calorie, defined to be 4.186 joules.

Kilocalorie, or Nutritionist's Calorie 1000 calories.

Warm-blooded animals Mammals and birds. The technical term is homoiothermic, because the temperature is the same regardless of the surroundings. Distinguish from the poikolothermic or cold-blooded animals, which assume the temperature of their surroundings.

Mechanical equivalent of heat 4.18 joules = 1 calorie.

Heat capacity The amount of heat needed to raise the temperature of an object by one degree.

Specific heat Heat capacity per gram or heat capacity per cm^3. For compressible materials, especially gases, distinguish between specific heat at constant pressure and specific heat at constant volume: $c_p > c_v$.

Monatomic Having one atom per molecule. The noble gases are monatomic: He, Ne, Ar, Xe, and Kr. So are the vapors of most metals.

Diatomic Having two atoms per molecule. For example, nitrogen (N_2), oxygen (O_2), hydrogen (H_2), and chlorine (Cl_2) are diatomic gases.

Translational degrees of freedom A point particle or a monatomic gas moves in three dimensions, so it has three translational degrees of freedom. Its energy can be written $\frac{1}{2}mv_x^2 + \frac{1}{2}mv_y^2 + \frac{1}{2}mv_z^2$, as the sum of three quadratic terms. A molecule with more than one atom may have rotational degrees of freedom as well, quadratic energy terms involving the angular velocity, also vibrational degrees of freedom.

Avogadro's number The number of molecules in 1 mole = 6.02×10^{23}.

Mole One gram molecular weight. For example, 1 mole of water (H_2O) is 18 grams of water, since hydrogen has atomic weight 1 and oxygen has atomic weight 16.

Latent heat The heat necessary to change (one gram of) a substance from one phase to another. The latent heat of melting (fusion) of water is 80 calories per gram.

Internal combustion engine Gasoline or diesel engine. A heat engine in which the fuel burns in the expansion chamber. Distinguished from a steam engine (external combustion engine), in which there is an external boiler.

First Law of Thermodynamics When mechanical work is transformed into heat or heat into work, the amount of work is always equivalent to the amount of heat.

Law of Dulong and Petit The specific heat of a solid is 6 calories per degree C per gram atomic weight. This has its theoretical basis in the Law of Equipartition of Energy. A good approximation for some substances, it predicts too large a value for many others.

Phase transition Usually a change from solid to liquid (melting), from liquid to gas (vaporization), or vice versa (freezing, condensation). There can also be different solid phases, with a density change at the transition. In liquid helium there is a "second-order" phase transition at 2.2 K with no density discontinuity and no latent heat but a discontinuity in the specific heat.

Capillary Means "thin as a hair." A thin glass capillary tube is used to make a mercury thermometer in order to get sizable change in length of the mercury column for small temperature changes.

Coefficient of thermal expansion The relative change in dimension per degree of temperature change. For example, dental amalgam expands in length by 25 parts per million per degree C. Don't confuse linear expansion coefficient α with volume expansion coefficient β. Note that $\beta = 3\alpha$. (See Chapter 3, Size, Strength, and Scaling.)

Ideal gas A gas obeying the ideal-gas law, that the product of pressure times volume is constant at constant temperature. Most real gases approach ideal-gas behavior at low pressures and high temperatures.

Triple point The temperature and pressure at which the solid, liquid, and gas phases of a single substance coexist. The point at which the vapor-pressure curve and the melting curve meet. The triple point of water, 273.16 K and 0.006026 atm, is used as a standard fixed point for the definition of the Kelvin temperature scale.

Third Law of Thermodynamics Absolute zero can never be attained. This "you-can't-get-there-from-here"-principle is more precisely stated, "Absolute zero cannot be attained in a finite number of steps."

NOTE: There is no limit to how close we can get.

REVIEW QUESTIONS

- Why does shivering help keep us warm?
- Warm clothes tend to be porous. Why?
- A big pot of coffee loses heat faster than a little pot, yet its temperature drops more slowly. Explain.
- Why does the aluminum cup cool faster than the styrofoam cup?
- Why does the stirred cup cool faster than the one left alone?
- Define the heat flow J. What are suitable units?
- What are suitable units for the temperature gradient ∇T?
- What are suitable units for the thermal conductivity?
- If the layer thickness is doubled and the temperature difference is halved, the temperature gradient is multiplied times ______?
- If the diameter of a sphere is doubled, its volume is multiplied times ______.
- If the volume of a sphere is doubled, its diameter is multiplied times ______.
- If the diameter of a sphere is doubled, its area is multiplied times ______.
- If the volume of a sphere is doubled, its area is multiplied times ______.
- If the diameter of a sphere is doubled, its surface-to-volume ratio is multiplied times ______.
- What is the heat capacity of 1 liter of water?
- Which is more, a calorie or a joule?
- Which has the larger specific heat, iron or water?
- Why can we not heat water above 100°C in an open pot?
- Which requires more heat, heating water from 99° to 100°C or boiling it off?
- Estimate the ratio of the volumes of the capillary and bulb of a fever thermometer.

- Where does the ideal gas expand more, in going from 273 to 274 K, or in going from 373 to 374K?
- How much does helium gas expand in warming from 5 to 6K at 1 atm? Answer in percent.
- Historically, two fixed points (freezing and boiling) were needed to define a temperature scale. Now only one point is needed? Explain.

PROBLEMS

✶✶ **Working in Cold Water** Air at 18°C is comfortable, 18-degree water seems cold.
(a) Why do we lose heat faster in water than in air? (HINT: Compare heat capacities per unit volume.)
(b) A "wet suit" is a rubber garment worn for working under water. It clings closely to the skin. As its name implies, it does not keep you dry. Why does it keep you warm, even though the water is touching your skin?

Problem 8.1 A diver in a wet suit.

✶ **2. Automobile Radiator** The car manufacturer chooses a metal with a high thermal conductivity: copper, $\kappa = 92$ cal/sec m °C. Suppose the thickness of the metal to be 0.5 mm, the area 0.5 m^2, and the rate at which the radiator has to get rid of heat to be 10^4 calories per second. The water temperature inside is 80°C. Calculate the temperature of the outside surface.

✶ **3. Igloo** How thick a brick wall would the Eskimo have to build to have the same amount of heat flow as through a 30-centimeter-thick wall of snow? Use the table of thermal conductivities (Table 8.1).

✶✶ **4. Igloo**
The wall is 30 cm thick and 20 m^2 in area. (See figure.)
(a) Calculate the heat flow through the wall of snow if the temperature difference between inside and outside is 30°C.
(b) Why should the inside temperature not be above 0°C?
(c) Could one person's heat output supply the heat conducted through the wall? Show your calculation. Do you base it on daily caloric intake?

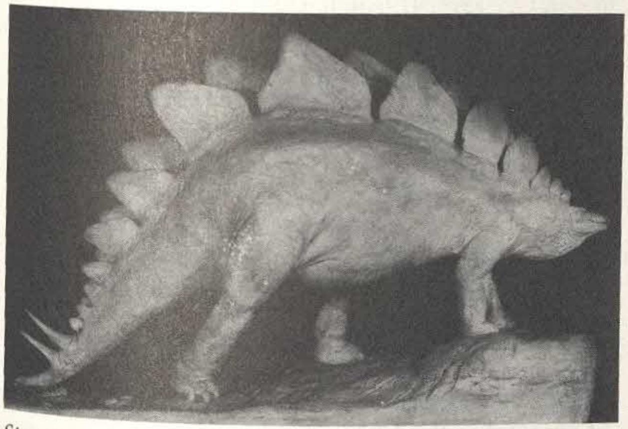
PROBLEM 4. Igloo.

✶✶ **5. Home Insulation** Wooden houses are usually built with about 10 cm (= 4 inches) of air space between the outside wall and the inside wall. Although air has a low thermal conductivity ($\kappa = 0.0055$ cal/sec m °C), it sets up circulating flow patterns, so there is considerable heat transfer through the wall by convection. The various fibrous materials placed in the air space as insulation to reduce the heat transfer actually have much larger thermal conductivities than air. Explain why the insulation should not be packed in tightly. Why is there an optimum density of packing?

✶✶ **6. Home Insulation** Roof insulation can be bought in sheets. Usually, the manufacturer quotes an **R-value** to indicate how effective it is. If a 1-in-thick sheet has R-3, then a 2-inch sheet has R-6. Assume that the temperature difference across the sheet is given. The quotient of that temperature difference divided by the heat flow J across the sheet is its R-value.
(a) Show that it is equal to the thickness of the sheet divided by κ, in suitable units.
(b) Show that the R-value of a sandwich of two sheets is the sum of the two individual R-values.

✶✶ **7. Home Insulation** One house has a wall made of 6 inches of brick, 2 inches of saw-dust insulation, plus plasterboard on the inside. Another house has half-inch wooden siding, 4 inches of sawdust insulation, and plasterboard inside. Which house loses heat faster through the wall, the brick or the wood? Why did you not have to know anything about the plasterboard? Can you guess why the brick house might be cooler in the summer?

✶✶ **8. Stegosaurus** The stegosaurus was a dinosaur with a sawtooth profile (see photo). For decades, paleontologists wondered about the function of those triangular protuberances on its back. A fashionable hypothesis is that they were to keep the animal cool during muscular exertion, much like the vanes of a radiator.
(a) Argue that this might imply that the animal was warm-blooded.



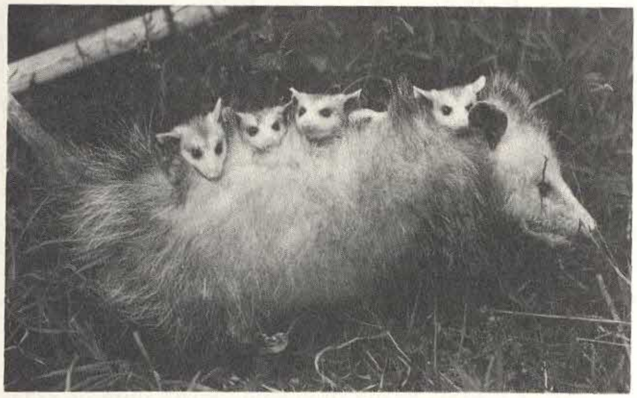
Stegosaurus, as the paleontologists have reconstructed him from fossil evidence. Current theory is that those crenellations helped him cool off.

(NOTE: Warm-blooded really means the animal has an internal mechanism to keep its temperature constant regardless of the temperature of its surroundings.)

(b) Argue that a large animal would have greater need for such a temperature-control mechanism than a much smaller animal. Your argument might use the term *surface-to-volume ratio.*
(c) Do you think he lived among ice and snow? Argue.

✶✶ **9. Surface-to-Volume Ratio. Marsupials** Like the kangaroo, the female oppossum keeps her young in a pouch for a few weeks after birth.
(a) If the surface-to-volume ratio of a mature (5-kg) oppossum is 0.3/cm, estimate the surface-to-volume ratio of the newborn. It weighs about 0.5 grams.
(b) Estimate the surface-to-volume ratio of an adult human.
(c) Estimate the surface-to-volume ratio of a baby.

NOTE: An incubator for premature babies is kept at 37° C. So is a marsupial's pouch.



An opposum with her young in pouch.

✶✶ **10. Metabolism** Here are some estimates of daily metabolism for various species. (From D'Arcy Thompson, *On Growth and Form.*)

	Body Weight (kg)	**kcal per kg of Body Weight per Day**
Guinea pig	0.7	223
Rabbit	2	58
Man	70	33
Horse	600	22
Elephant	4000	13
Whale	1.5×10^5	1.7

Estimate the ratio of daily food intake (kg) to body weight for each species. It is about 1:50 for man. Assume they eat more or less the same food.

HINTS: Metabolism means the rate of burning of nutrients, essentially "power in." The right-hand column gives power divided by body weight. "Same food" means the calories per ounce of food are the same for the various species.

✱✱ **11. Jogging** Running burns up a lot of energy. With every step we raise the body about 5 cm and let it fall back, to bounce again. But there is no steel spring in the ankle joint. The work we do is gaining those 5 cm is all converted into heat.
(a) Calculate the work to raise a 75-kg jogger up 0.05 m.
(b) A jogging pace might be three steps per second. Calculate the average power going into heat through the up-and-down motion. Note that this is only a fraction of the energy burned in jogging. Suppose that it is roughly one-third and use that to calculate how long it takes to jog away a 100-Calorie (= 10^5-calorie) candy bar.

✱✱ **12. Radiation** The Stefan–Boltzmann Law gives the rate at which energy is radiated from a surface at absolute temperature T:

$$\frac{\text{power radiated}}{\text{area}} = \epsilon a T^4$$

The universal constant $a = 5.67 \times 10^{-8}$ watts/m^2 K^4 is called the Stefan–Boltzmann constant. The parameter ϵ is called the *emissivity* and varies between 0 and 1 for different materials and different wavelengths. For the surface of human skin, it is 0.97 for infrared radiation.
(a) Calculate the rate of heat loss per cm^2 from the naked body. Assume that the skin is at 28°C (= 301 K), that is, just a few degrees below the blood temperature of 37°C. The skin is a good thermal insulator!
(b) When you are out in the hot sun, are you getting more heat by radiation from the sun or losing more by radiation from the skin? The solar constant is given as 2.0 calories/cm^2 per minute here on earth. Suppose that you are lying down at the beach.
(c) Why is it different standing up?

✱ **13 Mixtures**
(a) 1 kg of water at 40°C is mixed with 1 kg of water at 20°C. What is the temperature of the mixture? What assumptions did you make?
(b) 2 kg of water of 40°C is mixed with 1 kg of water at 20°C. What is the temperature of the mixture?
(c) 1 kg of sand at 20°C is mixed with 1 kg of water at 40°C. What is the temperature of the mixture (HINT: Use Table 8.2.)
(d) 2 kg of sand at 20°C is mixed with 1 kg of water at 40°C. What is the temperature of the mixture?

✱✱ **14. Ice Water** 100 grams of ice at 0°C is put in 1 kg of water at 20°C.
(a) Does all the ice melt?
(b) What is the equilibrium temperature?
(c) Same problem with 300 grams of ice.

✱ **15. Mountain Climbing** The energy efficiency of the human body as a machine for climbing mountains might be as high as 50%. This means that for each joule of work done to raise the body against gravity, one joule of heat is generated in the body. In mountain climbing, an average altitude gain of 300 meters per hour can be kept up all day. At what rate must the body lose heat at this rate of climb? Answer in watts. Use either your own weight or 75 kg.

✱ **16. Low-Calorie Diet** Much more of the energy we take in as food goes into warming us up than into external work. A low-calorie diet restricts daily food intake to 1500 Calories (= 1500 kilocalories).
(a) Calculate the average rate of heat loss (in watts) on the assumption that external work can be completely neglected.
(b) Contrast your results with the power needed by the heart to pump blood (Problem 7.23). Remember, all that ends up as heat too.

✱ **17. Perspiration** Sweating is the way the body can still lose heat when the ambient temperature is close to body temperature. The heat of vaporization of water is 580 cal/gm at body temperature.
(a) How many calories does the body lose by evaporating 200 grams (a cupful) of perspiration?
(b) How many degrees would this cool a 75-kg body? (HINT: Heat capacity?)
(c) The body regulates its temperature to better than 0.1°C. Why does an athlete put on warm clothing immediately after strenuous exertion?

✱ **18. Heat Loss Due to Breathing** If you are outdoors in freezing temperatures, you might wonder whether the heat you put into the air you exhale is an appreciable part of the energy you spend keeping warm. Suppose that you take in 10 breaths per minute at half a liter per breath. That averages to 3.7×10^{-3} moles of air per second.

(a) Verify.

(b) If the specific heat of air is 5 calories per degree per mole, warming that air by 37°C from freezing to body temperature requires about 0.02 calories per second. Our resting metabolism (see Problem 16, "Low-Calorie Diet") is about 1500 kilocalories per 24-hour day, or ________ calories per second. What percentage of the body's energy goes into warming the air you breathe?

(c) The air we exhale is very moist. Assume that you lose about 50 grams of water per hour when you are out in the cold. Knowing that the heat of vaporization of water is 580 calories per gram at 37°C, calculate how much heat goes into evaporating that water. Answer in calories per second and in watts. Note that this is close to half your metabolic rate! If you want to measure that water loss, weigh yourself accurately before going to sleep and again after you get up.

✱✱ 19. **Law of Dulong and Petit** This law states that the heat capacity of solids is 6 cal/°C per gram atom. It turns out to be a good approximation for many solids.

(a) Verify that the specific heat of iron (atomic weight = 56) should be 0.11 cal/gm°C.

(b) What does the law predict for the specific heat of gold (atomic weight = 197)?

(c) How about ice? The measured value just below the melting point of ice is 0.50 cal/gm°C. The theoretical basis for the law is that each atom has three vibrational degrees of freedom in the solid. Each of these degrees of freedom has an average energy kT, according to the Equipartition Theorem. That is twice as much as a translational degree of freedom, because the vibrational degrees of freedom have both kinetic and potential energy. The reason the specific heat of ice is actually less than the theoretical (Dulong-Petit) value is that not all the internal vibrations of the H_2O molecule can be excited. That "freezing in" of internal degrees of freedom is explained by the quantum theory.

✱ 20. **Primitive Air Conditioner** In hot, dry climates, room air is cooled by blowing it through a wet filter. Explain how this works.

✱✱ 21. **How Reptiles Keep Cool** It is reported that some reptiles can maintain their body temperature 10 degrees below the air temperature by panting. Explain.

✱✱ 22. **Cooling by Pumping** The specific heat of liquid helium is 1.0 cal/gm K just below its boiling point, 4.2K. Its latent heat of vaporization is 4.95 cal/gm.

(a) How much can the temperature of 1 liter (= 178 grams) of liquid helium be lowered by pumping off one-tenth of it?

(b) Is there a contradiction in being able to cool below the "boiling point" even while it is boiling? Explain. (HINT: Pumping means lowering the pressure.)

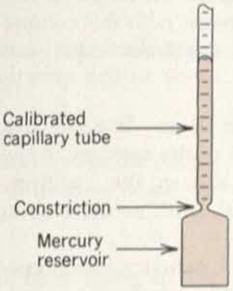
✱ 23. **Ideal Gas Law** Calculate the volume of 1 kg of water vapor at 373 K and 1 atm. (HINT: 1 mole of ideal gas at 273 K and 1 atmosphere occupies 22.4 liters.)

✱ 24. **Perpetual-Motion Machine** An "invention" proposes to use steam to drive a turbine. The turbine is to turn the wheels of a car. Friction in the mechanism heats the boiler and makes more steam, so the car needs no fuel. What is wrong?

✱ 25. **Fever Thermometer** A clinical thermometer reads the highest temperature reached. It differs from a laboratory thermometer in having a small constriction at the bottom of the capillary tube (see figure). When the thermometer cools, the mercury column breaks at the constriction. Its surface tension is not sufficient to overcome friction against the sides of the capillary. The thermometer has to be shaken down before reusing.

(a) The time for a fever thermometer to come to thermal equilibrium in the mouth is about 3 minutes. If it is removed for reading and reinserted, is there reason to expect equilibrium to be reached more rapidly the second time? (HINT: The mouth cools when it is opened.)

(b) If the thermometer is removed and reinserted, must it be shaken down before reinsertion?



PROBLEM 25. Fever thermometer, showing constriction where the mercury column breaks on cooling so that the reading does not change.

✱✱ 26. **Nonuniform Thermal Expansion** The volume expansion of ethyl alcohol is given by the formula

$$V = V_o(1 + \beta t + \gamma t^2),$$

where V_o is the volume at 0°C, t is temperature in °C, $\beta = 1.01 \times 10^{-3}/°\mathrm{C}$, and $\gamma = 2.2 \times 10^{-6}/(°\mathrm{C})^2$.

(a) How much does the volume of 0.1 cm^3 of alcohol increase between 20°C and 21°C?
(b) At 0°C, the parameter β is the coefficient of thermal expansion. How much does the coefficient change in 20 degrees? Answer in percent.

✶✶ **27. Linear Expansion** Almost all solids expand on heating. Over a sufficiently narrow range of temperatures, the increase in length of a solid object on heating is proportional to the temperature rise:

$$\Delta\ell = \alpha\,\Delta T$$

For aluminum, the linear coefficient of thermal expansion α is 24×10^{-6}/°C. This means that an aluminum rod 1 meter long at 20°C will be 1.000024 m long at 21°C.
(a) Suppose that the aluminum rod has a diameter of 1 cm at 20°C. What is its diameter at 21°C?
(b) Express this increase in percent.
(c) Calculate its cross-sectional area at 20°C and 21°C. Express this increase in percent.
(d) Calculate the rod's volume at 20° and 21°C. Express this increase in percent. Note the ratios of the percentage increases of linear dimensions, area, and volume.
(e)If the length is represented by the formula

$$\ell = \ell_0\,(1 + \alpha\,\Delta T)$$

argue that the radius is given by $r = r_o\,(1 + \alpha T)$
(f) The volume $V = \pi r^2 \ell$ is given by $V = V_o\,(1 + \beta\,\Delta T)$. What is the relation between α and β?
(g) Why were you able to neglect terms in $(\Delta T)^2$ and $(\Delta T)^3$ in arriving at the answer $\beta = 3\alpha$?
(h) In a mercury thermometer, why does the length of the mercury column in the capillary increase with the volume expansion coefficient β rather than the linear expansion coefficient α?

✶ **28. Thermal Expansion of the Glass** In a mercury thermometer, the thermal expansion of the mercury in the bulb makes the mercury column rise in the capillary. However, the glass bulb also expands with an increase in temperature.
(a) Argue from your experience with mercury thermometers that the thermal expansion coefficient of glass is less than that of mercury.
(b) The volume coefficient of thermal expansion of glass is around 3×10^{-5}/°C. In Exercise 25, in which you calculated the rise in the mercury column for a 1-degree temperature rise, how much error (%) did you make by neglecting the expansion of the glass? (REMINDER: The inner diameter of the bulb increases in proportion to its length.)

✶✶ **29. Mercury Thermometer** A clinical thermometer has a scale 5 cm long reading from 34°C to 42°C
(a) Find the ratio of the volume of the bulb to the volume of the capillary.
(b) If the capillary has a diameter of 0.1 mm, how much mercury is in the bulb?

✶✶ **30. Linear Expansion of the Thermometer Scale** The volume coefficient of thermal expansion of glass is given as 3×10^{-5}/°C.
(a) Show that in a one-degree temperature rise, the length of the thermometer would increase only one part in 10^5. (HINT: See Problem 27, Linear Expansion.)
(b) For the clinical thermometer of Problem 29 (5-cm scale for 8 degrees), show that the linear expansion of the scale could not be observed by the naked eye.

✶ **31. Pressure Cooker** In a pressure cooker, water boils at temperatures well above 100°C. The vapor-pressure curve (*P vs T*) of water has a positive/negative slope. Explain.

✶✶✶ **32. Newton's Law of Cooling** [Hard] One of the "generalizations" of Sec. 8.1 is that the rate of heat loss is proportional to the temperature difference. If the heat capacity does not change with the temperature, this implies that an object without a source of heat would cool at a rate proportional to the temperature difference.
(a) Prove.
(b) Call ΔT the temperature difference between the object and its surroundings. Then the equation reads

$$\text{rate of cooling} = \text{constant} \times \Delta\text{T}$$

What are the dimensions of the "constant"? Do not confuse temperature difference ΔT with temperature gradient ΔT.
(c) The ambient temperature is 21°C. The object cools from 31°C to 30°C in 5 minutes. How long will it take to cool to 29°C? Please do not settle for the answer "5 minutes." Keep more accuracy!
(d) How long to cool to 28°C?
(e) To 27°C?
(f) To 26°C? That is halfway. Prove that it will take just as long to cool from 26°C to 23.5°C.
(g) How long will it take to cool from 23.5°C to 22.25°C? (HINT: Newton's law of cooling is described as a *decaying exponential*, that is, it is characterized by a half-life.)
(h) Plot temperature T versus time t for the first two hours.

✱✱✱ **33 Inverse-Square Law** [Hard] A 100-watt heat source (light bulb?) is imbedded in a homogeneous material of thermal conductivity $\kappa = 2.0$ watts/m degree. The bulb is a sphere of radius 0.02 m. The outside surface of the conducting (insulating) material is a sphere of radius 0.2 m. It is kept at room temperature = 21°C.

(a) Remembering that 100 watts of heat power flows out through that outside surface, verify that the heat flow there is 199 watts/m^2. (HINT: Area of a sphere = $4\pi r^2$.)

(b) Verify that the heat flow 1 millimeter below that surface is 200.95 W/m^2. That is, when the radius r is decreased by 0.5%, the heat flow J is increased by 1%. The heat flow obeys an *inverse-square* law: $J(r) \propto 1/r^2$.

(c) Calculate the heat flow J at the bulb, $r = 0.02$ m. It is _______ times as great as at $r = 0.2$ m, the outside surface.

(d) Using the heat-conduction equation, $J(r) = -\kappa\, \nabla T(r)$, find the temperature gradient ∇T just below the surface, $r = 0.2$ m. Express in degrees per meter.

(e) According to part (d), the temperature at $r = 0.199$ m should be 21.1°C. But the temperature gradient there is 1% greater than at the surface. So the temperature rise in the second millimeter below the surface will be 1% greater than in the first millimeter. What is the temperature 2 millimeters below the surface?

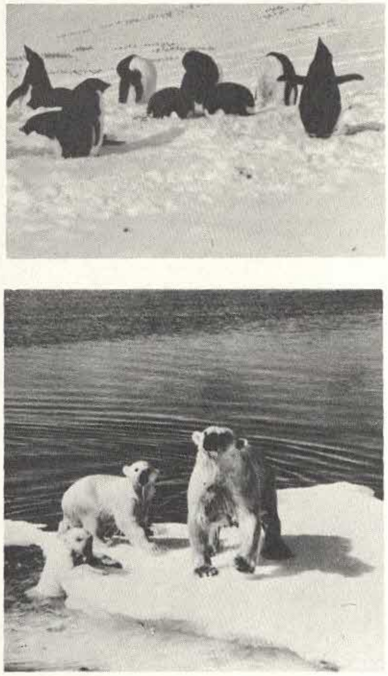
(f) In part (e) we describe the beginning of a numerical integration. An analytical integration yields the formula

$$\frac{\text{temperature}}{^\circ\text{C}} = 21 + 3.98\text{ m}\left(\frac{1}{r} - \frac{1}{0.2\text{ m}}\right)$$

Use the formula to find the temperature at the bulb, $r = 0.02$ m.

(g) How much does the temperature drop in the first millimeter beyond the bulb?

☐ QUESTIONS FOR FURTHER THOUGHT



Penguins, polar bears

α. **Regulating Body Temperature** One of the tricks our body's thermostat uses for keeping the blood warm when the air is cold is to shrink blood vessels near the surface of the skin. This makes the skin even colder, of course.

(a) Using what you know about heat conduction, argue that the blood now loses heat more slowly.

(b) Argue that this mechanism makes us ready victims of frostbite in fingers and toes but may save our lives in prolonged exposure.

β. **Goose pimples** When we get cold, the skin may develop small projections some people call goose bumps. This may well be a vestigial response, useful for a long-haired mammal whose fur would stick out at the bumps. Argue how this response would lower the animal's rate of heat loss.

γ. **Bergmann's Law (1847)** The Arctic regions are the home of large birds (penguins) and large mammals (polar bears). The smallest dolphins live in warm seas. There are no small sea mammals at all.

(a) Explain in terms of surface-to-volume ratio and heat loss.

(b) The "law" is hardly universal. Small weasels and lemmings live in the Arctic, and elephants in the tropics. What specific physiological features adapt the large land animals well to warm climate?

δ. **Metabolic Rate and Body Size**

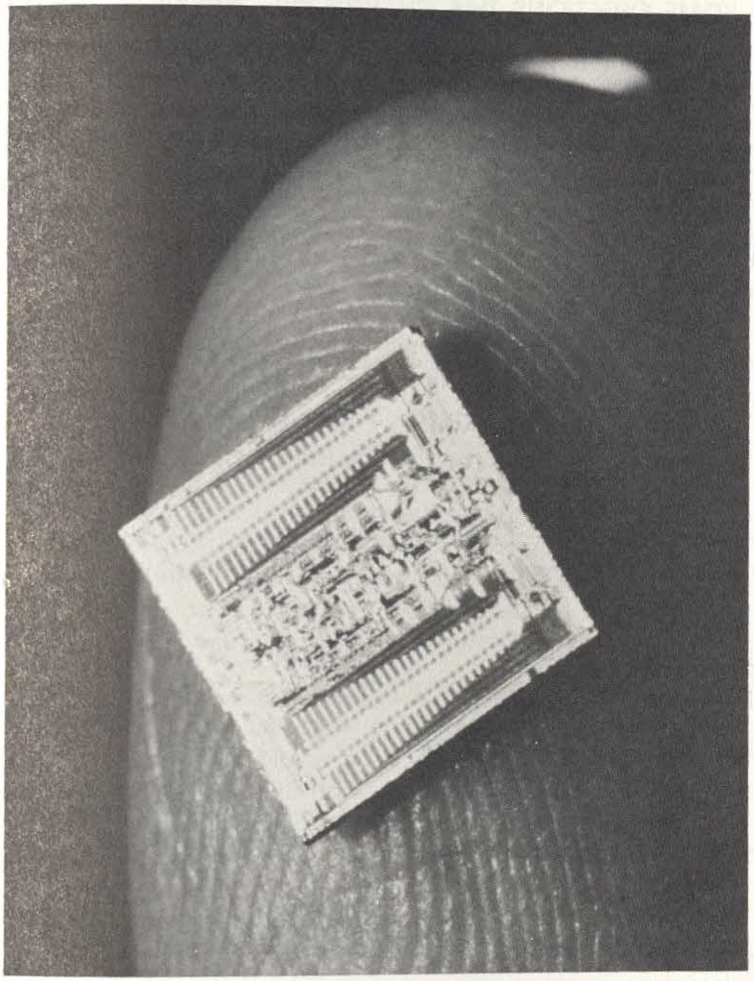
(a) Criticize the following argument: Mammals of various sizes living in the same climate have roughly the same body temperatures. Hence their rate of heat loss is proportional to their surface area. Since they have the same density and comparable shapes, surface area is proportional to the $\frac{2}{3}$ power of the weight W. Hence the rate of heat loss for various species should go as $W^{2/3}$. But heat loss has to be roughly equal to the metabolic rate R (calories per day):

$R \propto W^{2/3}$

(b) In Problem 10, Metabolism, is given a table of R/W versus W, that is kcal per kg per day versus body weight. According to the argument above, we should find $R/W \propto W^{-1/3}$, that is, the kcalories per kg of body weight per day should go as the inverse one-third power of the weight. Plot the data on a log-log scale to check how well this is verified. Do not expect a textbook straight line!

ϵ. Alcohol has a thermal expansion coefficient more than 5 times greater than mercury. Why do you think anyone bothers making mercury thermometers? (HINT: There may be more than one good answer.)

Data:	Surface tension at 20°C—	mercury	484 dynes/cm
		ethyl alcohol	23 dynes/cm
	Vapor pressure at 20°C—	mercury	0.0024 mm Hg
		ethyl alcohol	41.0 mm Hg
	Boiling point—	mercury	357°C
		ethyl alcohol	79°C



Computer chip on a fingertip. One chip can store and/or process thousands of bits of information.

9

INFORMATION AND ENTROPY[1]

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[1] This chapter is somewhat different in style from those that precede and follow it. It is more mathematical and is intended to be a bit provocative.

SOME QUESTIONS THIS CHAPTER WILL ANSWER

Why is the number of letters in a message a reasonable measure of its information content?

Amount of information is often expressed in *bits*. What is a bit?

Boltzmann's constant has units of joules per Kelvin. Why? Is that an appropriate entropy unit?

When a mole of ideal gas doubles its volume irreversibly, argue that its entropy increases by $Nk \log_e 2$.

Argue that the air conditioner removes less heat from the room than it pumps into the outside air.

If the air conditioner were a Carnot engine, would the entropy it removes from the room be equal to the entropy it pumps into the outside air?

How much heat could an ideal (Carnot) refrigerator pump per hour from 273 K (the ice cream) to 293 K (the room) with a lossless 100-watt motor?

9.1 INFORMATION

Suppose you were asked to define what you mean by the amount of information contained in a message. You would probably say that a long message has more information than a short one. You might feel that the *length* of the message is an appropriate measure of its information content. So your first try at a definition of **information content** might be the number of letters in the message.

Some Examples That Argue in Favor of This Definition

1. Telephone book. Two pages of the telephone book contain twice as much information as one page.
2. Class list. Suppose that you want to know the sex of everybody in the class. Next to every student's name on the class list there is either a ♂ or a ♀. If you double the length of the list (the number of students), you double the amount of information.
3. Location of the molecules, left and right. A closed container has in it 100 labeled (i.e., different) molecules. The left (L) half and the right (R) half of the container are separated by a valve. Next to the name of each molecule is written the letter L or R, telling which half that molecule is in. So the message is a list 100 letters long. If there were 200 molecules, the list would be twice as long and would contain twice as much information.

Critique of the Definition

Then you might get critical. How about redundancy? Does the message "HELP, HELP" convey twice as much information as just plain "HELP"? Actually, even that brief message is somewhat redundant (see Fig. 9.1). The message "H_LP" with the second letter unspecified is almost unambiguous if you know that it is an English word.

FIGURE 9.1 Redundant message.

EXERCISE 1 What if the first letter were left unspecified? Is "_ELP" unambiguous?

But redundancy is useful. A language needs some **redundancy** in order to be easily understood. There is always some noise that introduces ambiguity. The word *noise* is used in a very general sense here. For example, a misshapen letter in a written message can be considered noise. So can a spelling error. If you saw the word HULP, you would know there was a mistake, and chances are, you would interpret the U as an E. Computer languages are designed with redundancy in order to catch mistakes. This is done not only for word messages (instructions) but also for numbers (data).

EXERCISE 2 One way to error-proof a number message is to write all the numbers twice. A more economical but less safe way is to put an extra digit at the end that is the sum of the previous digits. Thus 51 would be written 51 6, and 104 would be written 104 5. Why would 76 be written 76 3?

EXERCISE 3 If you know that only one digit can have a mistake in it, is the error-proofing method of the previous exercise 100% foolproof? Argue.

If you accept the usefulness of redundancy, you may be satisfied with our first measure of information: Call the number of letters (digits? symbols?) in the message a measure of its information content. You will criticize this definition as unsophisticated, because there are more and less economical ways of being redundant. There are efficient languages and wasteful languages. But let's stick with our definition for the purposes of this chapter.

The Unit of Information Content: The Bit

We still have to define the unit. If we are allowed a choice of 26 letters, A through Z, then each symbol conveys a lot more information than if we are allowed a choice of only 10 digits, 0 through 9. A particularly simple unit is defined in a language that has only two symbols. You might call the symbols + and −, or YES and NO, or ↑ and ↓. In an electronic computer, it is convenient to call them "open" and "shut." Conventionally, they are called 1 and 0. Actually, a two-symbol (binary) alphabet is just as expressive, just as rich as a longer one. It only takes more letters to say things in it. Clearly, it is the shortest possible alphabet. Each time you write down a symbol you make a binary choice. The length of the message is the number of such choices. We say that a three-letter message contains three binary digits, or three **bits** of information.

Binary Arithmetic

In an electronic computer, each binary digit is a switch (a diode), which is either open or shut. One way to read a computer register is via a bank of lights. Represent "light on" with the symbol ↑, and "light off" with the symbol ↓. The number *one* can be written ↑. Zero is written ↓. Two is ↑↓. That is ↑ + ↑. Of course! Each light only has two states, on or off, so we have to carry to the next digit to the left, to the next light:

Two = ↑ + ↑ = ↑↓

It works just like the addition you learned with 10 symbols. Let's go on. Three is two + one = ↑↑. All we had to do that time was add the ↑ to the ↓ in the "ones" digit. When we get to four (= three + one), we have to carry again:

Four = ↑ ↓ ↓

And so forth:

Five = ↑ ↓ ↑
Six = ↑ ↑ ↓
Seven = ↑ ↑ ↑

You have probably seen this binary notation before, with ↑ written 1 and ↓ written 0. That notation is conventional. With three bits we are able to write eight numbers:

0 0 0
0 0 1
0 1 0
0 1 1
1 0 0
1 0 1
1 1 0
1 1 1

EXERCISE 4 Practice addition in "binary."

(a) 1 0 0 + 0 0 1 = 1 0 1 is pronounced four + one = five.

1 0 1 + 0 0 1 = $\boxed{\qquad}$

Is it easier if you write it vertically?

$$\begin{array}{r} 1\ \ 0\ \ 1 \\ +\ \underline{0\ \ 0\ \ 1} \\ 1\ \ 1\ \ 0 \end{array}$$

(b) How about four + two?

$$\begin{array}{r} 1\ \ 0\ \ 0 \\ +\ \underline{0\ \ 1\ \ 0} \\ \boxed{\qquad} \end{array}$$

(c) Pronounce: 1 0 0 + 0 1 1 = 1 1 1

(d) Why does 1 1 1 + 0 0 1 require four bits?

EXERCISE 5 With three decimal digits we talk about the ones digit, the tens digit, and the hundreds digit.

(a) Show that with three decimal digits you can write a thousand different numbers.

(b) The number three can be written

three = 1 × two + 1 × one

Similarly,

four = 1 × four + 0 twos + 0 ones
five = 1 four + 0 twos + 1 one
six = 1 four + 1 two + 0 ones

Notice that

one = 2^0
two = 2^1
four = 2^2

How many different numbers can you write with four binary digits?

(c) With five?

You may find it strange to talk about a two-symbol *alphabet*, when all we have written is numbers. But it only takes a little imagination to invent a code for translating any written language into "binary." Whatever the code might be, it is nothing but a set of conventions. Once two people agree on a code, they can use it to communicate. This interchangeability of numbers and words is at the heart of computer design. A given memory location in

DNA CODING—A FOUR-LETTER ALPHABET

Genetic information is what a living cell uses in order to reproduce. Such information is stored in complicated molecules called DNA. Problem 14 concerns the genetic code. This code has four letters. In order to learn how to measure information content in *bits* when the alphabet has more than two symbols, we shall use a four-letter alphabet for practice.

How many bits of information does each digit of a four-symbol language convey? Try to make a translator's dictionary from a four-symbol (quaternary) to a two-symbol (binary) code. It can just be done with two binary digits:

0 0, 0 1, 1 0, 1 1

A three-letter word written in a four-symbol language would require a six-letter word in binary.

EXERCISE 6 Call the four letters ↑, →, ←, ↓.

(a) How many bits of information are contained in the "message" → ↑ ←?

(b) In the message ↑ ↑ ↓ →?

(c) Make a dictionary, and write these two messages in a binary code. Obviously, you have a lot of choice in how to make the dictionary.

(d) Show that you could make 24 different dictionaries. [Hard]

EXERCISE 7

(a) Show that with an eight-symbol alphabet each letter conveys three bits.

(b) How many binary digits are required to write a three-letter word of an eight-symbol language?

a computer can be used either to store numerical data or to store an instruction, a part of the computer program.

Practice in Measuring Information. Logarithms

The exercises in the box "DNA coding" may convince you how quickly the arithmetic can get difficult. Let's continue in binary: How many different words can we write with three bits? Eight. With 4 bits we can write 16 different words.

EXERCISE 8

(a) With five bits we can write ______ different words.
(b) With six bits we can write ______ different words.
(c) With seven bits we can write ______ different words.
(d) With eight bits we can write ______ different words.
(e) With nine bits we can write ______ different words.
(f) With ten bits we can write ______ different words.
Repeat: A thousand-word language can be encompassed in ten bits.

EXERICSE 9 Show that an eleven-bit message allows a choice of more than two-thousand different alternatives.

If you can't do this last exercise, go back and try the one before. (HINT: With five bits we can write thirty-two different messages.) Now fill in the table:

Number of bits	0	1	2	3	4	5	6	7	8	9	10
Number of messages	1	2	4	8						512	1024

That is not higher mathematics, to be sure. Nevertheless, a lot of people have trouble formalizing it. If you say that two bits gives you a choice of $2^2 = 4$ different messages, nobody is going to argue with you. If you say that three bits gives you a choice of $2^3 = 8$ messages, that still sounds pretty trivial. The moment you say that p bits gives you a choice of 2^p messages, somebody will shout "Prove it!"

EXERCISE 10 Prove it.

Your proof may have gone something like this. The information content of the message is p bits. That means there are p slots to put symbols in. But you are only allowed a choice of two different symbols in each slot. Call N the number of different messages you can write this way. Repeat: p is the number of bits in the message; N is the number of possible messages. Consider the function N_p (pronounced N of p). If you close your eyes to the symbol in the first bit, there are N_{p-1} possible messages spelled out by the other $p-1$ bits. The first bit has two possibilities, so

$$N_p = 2N_{p-1}$$

You also know that $N_1 = 2$, that is, with one bit, there are two possible messages. You might recognize this as a "proof by induction." There is only one way to solve this. The solution is $N_p = 2^p$. Look back at the table and notice how quickly N gets large. An exponential function is a very rapidly varying function.

Now, how to turn that formula around? Given N as a function of p, how do you find p as a function of N? If we write the table

p	0	1	2							9	10
$N = 2^p$	1	2	4	8	16	32	64	128	256	512	1024

and ask you to fill in the blanks, you have no difficulty. What do you call what you did? If N is a function of p written $N(p) = 2^p$, what do you call the inverse function $p(N)$? [Answer: It is called a logarithm. It is written

$$p = \log_2 N$$

In words: The logarithm of N to the base 2 is the power to which 2 is raised to get N]

Somewhere in your schooling you may have learned about **logarithms** to the base 10. You probably knew that the logarithm of 10 to the base 10 is 1: One decimal digit gives 10 different messages. The $\log_{10}$ of 100 is 2: Two decimal digits are enough for a hundred different messages. Now we do the same thing with binary digits, that is, with base 2.

EXERCISE 11

$\log_2 8 = 3$
$\log_2 32 = 5$
$\log_2(8 \times 32) =$ ______

Repeat: To multiply two numbers, add their logarithms. This looks easier done in base 10:

$\log_{10} 10^5 = 5$; $\log_{10} 10^3 = 3$; $\log_{10} 10^8 = 8$
Five + three = eight

EXERCISE 12

$\log_2 1024 =$ ______ (HINT: Use the table.)
$\log_2 128 =$ ______
$\log_2 \left(\frac{1024}{128}\right) =$ ______

Repeat: To divide two numbers, subtract their logarithms. Thus $2^{10}/2^7 = 2^3$.

All these exercises are intended to make the language of exponents and logarithms more familiar. If you ask a question whose answer is a choice between two alternatives (a yes/no question), the answer gives you one bit (binary digit) of information. For a question whose answer is a choice between 8 different alternatives, the answer contains 3 bits: $\log_2 8 = 3$. If there are a million possible answers to your question, then the answer contains a little less than 20 bits of information. You may know how to calculate $\log_2 10^6$ and get 19.9. But you may have trouble interpreting a fractional bit of information. What is nine-tenths of a digit? To learn about a pitfall, see the box.

With these words of warning in the box, you will not misuse the formula for the information content of a message, when the number of possible messages is N:

$$\text{information content} = \log_2 N$$

Choosing 2 for the base of the logarithms gives the information content in bits. Evidently, a different base for the logarithms just changes the units in

CHANGE OF BASE: FRACTIONAL BITS

EXERCISE 13 This kind of discussion is necessary any time you convert from one size alphabet to another. How many bits in a decimal digit? More than three but less than four. Again, the answer 3.32 (= $\log_2 10$) is not very satisfying. But it is useful if the next question is "How many bits in six decimal digits?" [Answer: $6 \times 3.32 = 19.9$] Oh, yes, that is what we just said. Six decimal digits means a million different words. Suppose instead of using that 3.32 we had rounded upward and said, it takes four bits to write one decimal digit. True. Then we might have been tempted to say it takes 24 bits to write six decimal digits. False. That is 4 bits too high.

which information content is measured. If you choose base 10 instead and write the formula $\log_{10} N$, then the information content is measured in "decimal digits." This is a larger unit, so the answer comes out a smaller number. The ratio of the units is 1 to 3.32.

EXERCISE 14 $\text{Log}_2 10 = 3.32$. What is $\log_{10} 2$? How are those two numbers related?

For historical and practical reasons, a base is often used that is not even an integer:

$e = 2.7183$

The number e is called the base of the *natural* logarithms. To change a logarithm from base 2 to base e, just multiply it times 0.693.

EXERCISE 15

(a) $\log_e 2 = 0.693$. Which of the following is correct,

$\log_e N = 0.693 \log_2 N$

or

$\log_2 N = 0.693 \log_e N$?

(b) What is $\log_2 e$?

EXERCISE 16

(a) $\log_e 10 = 2.3026$. What is $\log_{10} e$?

(b) $\log_{10} 1{,}000{,}000 = 6$. What is $\log_e 1{,}000{,}000$?

In section 9.4, Work Needed to Create Order, the number $0.693 = \log_e 2$ will come up in the definition of entropy. You will know where it came from. It just has to do with the choice of units for measuring information.

9.2 MIXTURE OF TWO GASES. SECOND LAW OF THERMODYNAMICS

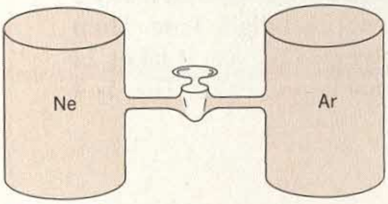

FIGURE 9.2 How much information is lost if the stopcock is opened between the cylinders?

One mole of neon and one mole of argon are each stored in its own cylinder. How much information is lost if the two cylinders are connected and the gases are allowed to mix? (See Fig. 9.2.)

From Chapter 6 you remember that if the two cylinders have the same volume and are at the same temperature, they have the same pressure of gas and contain the same number of molecules. In this case the molecules are atoms, since neon and argon are noble gases. At room temperature, they don't even stick to others of their own species. They are chemically quite inert. There are 6×10^{23} neon atoms and 6×10^{23} argon atoms. (Remember Avogadro's Number?)

Before you open the stopcock in the connecting tube, you have a lot of information. You know that all 6×10^{23} of the neon atoms are in the Ne cylinder, and all 6×10^{23} argon atoms are in the Ar cylinder. All that information is lost when the stopcock is opened. A short time later, the probability of any particular neon atom being in the Ne cylinder is only 50%. The distribution of the two gases in the two cylinders becomes random. The amount of information lost is 6×10^{23} bits for the neon and 6×10^{23} bits for the argon, or 12×10^{23} bits.

EXERCISE 17 If all the atoms had shirts with numbers, like ball players, there would be $2^{6 \times 10^{23}}$ ways of distributing the neon atoms and an equal number of ways of distributing the argon atoms. Show that this makes for a total of $2^{12 \times 10^{23}}$ different possible configurations, that is, ways of distributing the atoms between the two cylinders. (HINT: The neon atoms don't care where the argon atoms are, and vice versa.)

NOTE: The shirts-with-numbers business is just a way of saying the atoms are distinguishable. According to the quantum theory, the atoms are, in fact, indistinguishable so that our "classical" counting of configurations gave too large an answer.

When the probability of finding a particular atom in a particular cylinder changes from 100% to 50%, we say that one bit of information about that atom has been lost. The knowledge that atom number so-and-so is in the left-hand cylinder rather than the right-hand cylinder constitutes one *bit* of information. It is the answer to a question with only two possible answers—right or left—a binary choice. When the stopcock is opened, that one bit is lost. There is something arbitrary in that statement, even though you accepted it readily. Suppose that the Ne cylinder had been three times as big as the Ar cylinder. Then opening the stopcock would have changed the probability of finding a particular argon atom in the Ar cylinder from 100% to 25%. You might want to say you lose more information about that particular atom then. Two bits ($= \log_2 4$)? And that you lose less information about a particular neon atom. Its probability of being in the Ne cylinder is reduced from 100% to 75% by opening the stopcock. Is the information loss $\log_2(100/75)$? We are not going to go into these problems. But it may be useful for you to know that the problems exist.

We now make a statement you will not find surprising: If you close the stopcock again and measure how many atoms of each gas are in each cylinder, even though you repeat this experiment many times, the original ordered state is not restored. The atoms won't go back home into their original cylinders by themselves. The old order will not be restored spontaneously.

Why don't you find the statement surprising? Because you have had experience in allowing things to get mixed up, and they never got straightened out without a lot of effort. That empirical law has the imposing title **"Second Law of Thermodynamics."** For our example the proof is simple: Only one observation out of $2^{12 \times 10^{23}}$ would give the original ordered configuration. That is so improbable that we might as well say it is impossible.

Statistical Laws

So far in your physics course all the laws were stated as completely certain.[2] There was no "maybe" in the predictions. The Second Law of Thermodynamics is different. It is a statistical law, whose validity depends on the large number of particles in the systems we study. It is the large size of **Avogadro's Number,** that 6×10^{23}, that makes statistical predictions so accurate. With only three or four particles, statistical laws have much greater uncertainties. Fluctuation phenomena are more evident with small systems than with large ones. But even things we think of as microscopic really consist of many particles. A bacterium has billions of atoms. Even for a microbe, the law that order is not spontaneously created out of chaos can be stated with a high degree of certainty.

9.3 IRREVERSIBILITY. IS THERE A PARADOX?

A collision between two atoms is a reversible event. If you were to take a movie of a collision and play it backward, it would look perfectly natural. Certainly, the backward version would violate no laws. The laws of mechanics are reversible. Making time run backward, that is, changing the sign of the time variable in all the equations, is an allowed symmetry operation.

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[2] There was one exception: In Sec. 6.4 Gas Laws, we stated that the average translational kinetic energy per molecule is $\frac{3}{2}kT$. With a gas of few molecules that average also would show statistical fluctuations.

The physicist would say, "The laws of mechanics are time-reversal invariant."

Now comes the apparent paradox: If each individual event in the history of those atoms is reversible, why isn't the history of the entire gas reversible? What is wrong with taking a movie of the mixing of the two gases and playing it backward? It would look unnatural. How do we explain the irreversibility?

You know the answer: Seeing the entire movie played backward indeed violates no law of mechanics. Unmixing the gases is like unscrambling the scrambled eggs. It is just very, very unlikely. There is no paradox. Out of the huge number of different distributions of the gas molecules, only *one* corresponds to the original segregated state.

Mixing different species of atoms is only one example of an **irreversible process.** Mixing hot and cold is another. Whenever a hot object cools down, there is irreversible heat flow. The word *irreversible* does not mean that the reverse process cannot happen. It just means that it won't happen by itself. All processes involving friction are irreversible. Work is done in friction and all that energy is converted to heat, which flows away. Energy previously stored in an orderly fashion ends up randomly distributed. This goes for any viscous flow, as well as for all electrical conduction. If you look around you, there is friction in just about all the natural processes you see. They all degrade energy from an available form to a less available form. We talk about **dissipating** energy. Of course the energy does not disappear; still it is lost. It becomes unavailable for doing work.

We started to do energy bookkeeping back in Chapter 4. When we introduced the concept of potential energy, you were warned that not all work that is done results in energy being stored. Energy becomes unavailable whenever a system goes from order to disorder, from an improbable to a more probable state. In the next section we do a new kind of bookkeeping. We shall ask how much energy is needed to restore order. How much does it cost to reverse an irreversible process?

9.4 WORK NEEDED TO CREATE ORDER

How much work would it take to get all the neon atoms back into the Ne cylinder and all the argon atoms back into the Ar cylinder? We learned how to do that kind of calculation in Chapter 6. Since the two gases don't interact, it is just the work to compress each gas back to its original volume. The work to unmix is the work to recompress. We won't worry about fine points here, like how to invent a piston that compresses one gas while allowing the other to pass through unobstructed—a semipermeable membrane. Or how to do the work **isothermally,** that is, slowly enough so that the temperature remains constant.

The neon pressure was originally 1 atmosphere. After the mixing, it is 0.5 atm. The volume of neon was originally 22.4 liters. That is the volume of 1 mole of ideal gas at standard temperature and pressure. After the mixing, the neon volume is 44.8 liters. The recompression is begun at 0.5 atm and finished back at 1 atm. You can say right away that the work done on the neon will be less than 22.4 **liter atmospheres** (= 22.4 liters × 1 atm), but more than 11.2 liter atm (= 22.4 liters × 0.5 atm). Problem 27 describes how to do the averaging. How much less than 22.4 liter atm.? The factor turns out to be 0.693 (= $\log_e 2$):

work to recompress 1 mole of neon = 0.693 × 22.4 liter atm

Similarly for the argon,

work to recompress 1 mole of argon = 0.693 × 22.4 liter atm

So the work to reestablish the well-ordered initial state, to get both gases back into their original cylinders, is 0.693 × 44.8 liter atmospheres.

Don't worry about the unfamiliar energy units. The numbers are not what is of interest here. The purpose of the exercise is to relate the difference in information content between the initial and final states to the work needed to get from the less ordered to the more ordered one. Using an **ideal gas** for the calculation makes it easy. The result will turn out to be general. To compress an ideal gas from pressure p and volume V to pressure $2p$ and volume $\frac{1}{2}V$ requires an amount of work $pV \log_e 2$. The assumption is that the work is done reversibly, meaning without friction or turbulence. Also the temperature is assumed to be constant. If you remember the ideal gas law

$$pV = NkT$$

you see that this work is proportional to the absolute temperature T.

Where does the work go that is done in compressing the gas? What happens to the energy? It must go into heat! Since the work is done isothermally, there is no irreversible heat flow. It has to be in good "thermal contact" with something really big that is all at the same temperature, something like a big piece of copper. To stay at the same temperature, the gas gives up to its surroundings an amount of heat equal to the work done. This amount of heat has been shown to be proportional to the absolute temperature:

amount of heat ΔQ given up during compression = $NkT \log_e 2$

We now proceed to interpret the proportionality constant, the $Nk \log_e 2$. Of course, there is nothing special about doubling the volume. A more general case—increasing the volume by a small fraction—is treated in Problem 13.

Entropy

The proportionality constant has a name:

$$\frac{\Delta Q}{T} = \text{difference in } \textbf{entropy} \text{ between the initial and the final state}$$
$$= \Delta S = Nk\log_e 2$$
$$= k\log_e 2^N$$

You have no trouble recognizing 2^N as the number of possible distributions of those N atoms in the two containers. The number $\log 2^N$ is the amount of information lost in the expression to double volume. Or you might say that it is the information regained in the recompression back to the original volume.

Equation 1 for the entropy of mixing is for the special case of equal initial volumes and equal numbers of the two different gas atoms. That is how the "2" got into the formula; the volume doubles in the mixing. The argument would be similar for less symmetrical situations. The result is that the entropy change is Boltzmann's k times the information lost.

Let's try to get a feeling for what an entropy increase means. An entropy increase is an increase in disorder, or a decrease in order. Before the free mixing, we had the neon and argon atoms in their labeled compartments. Afterwards, that order was destroyed. The entropy increase is a measure of the information lost in the mixing. Similarly, when a warm fluid is mixed with a cold fluid, order is destroyed and the total entropy is increased. The entropy increase is associated with the irreversibility. When we recompress our gas reversibly, entropy is removed from the gas, but that same entropy is added to the surroundings (the big piece of copper. . .). Remember, work was done on the gas and heat was removed from the gas in order for the gas to remain at the same temperature. In a reversible process, no net entropy is produced. In an irreversible process, there is net production of entropy. One of the ways to state the Second Law of Thermodynamics is to say that the total entropy of a closed system never decreases (see Section 5).

Entropy Units

What is the base of the logarithm in Eq. 1? Does it matter? Yes, it matters in defining the size of the units of entropy. That logarithm is a logarithm to base e. When we first defined a unit of information content, we chose the bit. That meant that in the formula

$$\text{information} = \log 2^N$$

the logarithm was to base 2. Choosing the base e means a change in the size of the units by a factor 0.693 ($= \log_e 2$). But the entropy defined in Eq. 1 also has a factor k. In Chapter 6, k was called Boltzmann's constant. Its units depend on the units of T, the absolute temperature. If T is measured in Kelvins, then k can be measured in joules per Kelvin. That is the conventional unit for entropy:

$$k = 1.3805 \times 10^{-23}\,\frac{\text{J}}{\text{K}}$$

EXERCISE 18

(a) Calculate the change in entropy of 1 mole of neon at $T = 273$ K in being compressed from 44.8 to 22.4 liters. Express in joules per Kelvin. SOLU-

TION: We already know that work done in the isothermal expansion is 22.4 liter atmospheres × 0.693. Therefore, that is the heat given off. The entropy difference is $Q/T =$ ______. To convert units, remember that 1 liter $= 10^{-3}$ m^3 and 1 atm $= 1.013 \times 10^5$ newtons/m^2.

(b) Why does this entropy come out a reasonable size, while k is so small? (HINT: Avogadro's number.)

Work Needed to Run a Refrigerator

It is tempting to summarize the results of this section using our new terminology: The work needed to get from a disordered to an ordered state is the entropy difference between the two states multiplied by the absolute temperature.

$$\text{work needed} = T\,\Delta S \qquad \text{(Eq. 1')}$$

That is almost right. But we should be more careful. We should say that the *minimum* amount of work needed is $T\,\Delta S$. If the work is not done reversibly, it costs more. Setting the work done equal to the heat given off, we can be more accurate by writing an inequality:

$$\Delta Q = \text{heat given off} \geq T\,\Delta S \qquad \text{(Eq. 1'')}$$

The = sign holds when there is no friction and the temperature is uniform.

The purpose of a refrigerator is to take heat out of a cold object and discharge it into a warmer reservoir, usually the room. That is quite the opposite of a natural process. In a spontaneous process, heat flows "downhill," from hot to cold. The refrigerator is a device for pumping energy "uphill," from cold to hot. This is the reverse of the natural (irreversible) process. Of course, the refrigerator must do work in order to do that. That is why you have to plug it in.

How much work? Again, our rule relating the entropy change to the work needed gives us only an inequality. It tells us only the minimum amount of work needed. But that is a useful thing to know. We now calculate it for a typical household refrigerator. Don't expect your real refrigerator to be quite as efficient as our ideal one. Our calculation will assume that there are no irreversible steps, no places in the mechanism where entropy can increase. Note that the action of the refrigerator is to decrease the entropy of the object being refrigerated.

Suppose that our **ideal refrigerator** is pumping heat from 278 to 294 K.

EXERCISE 19 Is your home refrigerator colder or hotter than 278 K? Is your kitchen colder or hotter than 294 K? Express the respective temperatures in Celsius or Fahrenheit degrees, whichever your body intuits more readily.

Suppose we happen to know that the "coils" outside the refrigerator are putting 100 joules of heat into the room per second, on the average. How much electric power does it take to run the refrigerator? Assume that heat leaks keep its "cold" compartment at 278 K.

The entropy being pumped into the room by those heat-exchanger coils is

$$\Delta S_{\text{hot}} = \frac{100\text{ joules at }294\text{ K}}{294\text{ K}}$$

If our refrigerator is to be ideal, that is also the amount of entropy being removed from the cold object. Remember, no friction, no irreversible heat flows. So we can calculate the heat removed from the cold object per second, by simple proportions:

$$\begin{gathered} \Delta S_{\text{hot}} = \Delta S_{\text{cold}} \\ \frac{100\text{ joules}}{294\text{ K}} = \frac{\text{heat removed at }278\text{ K}}{278\text{ K}} \end{gathered}$$

Solve:

$$\begin{aligned} \text{heat removed at }278\text{ K} &= 100\text{ joules} \times \left(\frac{278}{294}\right) \\ &= 94.6\text{ joules} \end{aligned}$$

Where do the extra 5.4 joules come from (= 100 J − 94.6 J)? That is work done by the refrigerator. That is what we are trying to calculate.

$$\text{work needed to pump 100 J from 278 to 294 K} = 5.4\text{ J}$$

To pump 100 joules per second into the room requires an expenditure of 5.4 joules per second. The motor has to use 5.4 watts of power on the average to pump 100 watts of heat. Are you surprised that so little power is required to pump so much heat?

Let's go over those numbers again, just to keep the concepts straight. Every second, 94.6 joules of heat are removed from the food stored in the refrigerator. Assume that there is a lot of it, so it stays at 278 K. Every second, 100 joules of heat are pumped from inside to warm the coils outside the refrigerator, which are in contact with the room air. The difference, the 5.4 joules per second, is work done by the motor. The fact that the motor may get warm too is neglected here; the motor is assumed lossless, 100% efficient.

EXERCISE 20 How much power is required to pump 100 joules of heat into the room per second from 286 to 294 K?

The kind of reasoning we have been going through is called **thermodynamics.** Note that the only principle we needed for our thermodynamics calculation was the equation

$$\Delta S = \frac{\Delta Q}{T} \qquad \text{(Eq. 1)}$$

We set ΔS_{cold}, the entropy removed from the cold object, equal to ΔS_{hot}, the entropy pumped into the warm room:

$$\frac{\Delta Q_{\text{cold}}}{T_{\text{cold}}} = \frac{\Delta Q_{\text{hot}}}{T_{\text{hot}}}$$

Having done the bookkeeping on entropy, we turned to bookkeeping on energy. We knew how to account for the difference between the heat taken from the cold object and the heat pumped into the warm room:

$$\Delta Q_{\text{hot}} = \Delta Q_{\text{cold}} + \text{work done} \qquad \text{(Eq. 2)}$$

Then we solved for the work done to pump all that heat:

$$\begin{aligned} \text{work done} &= \Delta Q_{\text{hot}} - \Delta Q_{\text{cold}} \\ &= \Delta Q_{\text{hot}} \left(1 - \frac{T_{\text{cold}}}{T_{\text{hot}}}\right) \\ &= \Delta Q_{\text{hot}} \left(\frac{T_{\text{hot}} - T_{\text{cold}}}{T_{\text{hot}}}\right) \qquad \text{(Eq. 3)} \end{aligned}$$

EXERCISE 21 Do the algebra. Take Eq. 1 and substitute it into Eq. 2.

EXERCISE 22 If the temperature difference is halved while T_{hot} is kept fixed, what happens to the ratio work done/ΔQ_{hot}? Does this exercise look familiar?

Carnot Formula

That ratio

$$\frac{T_{\text{hot}} - T_{\text{cold}}}{T_{\text{hot}}}$$

is a good formula to memorize. You may object. A memorized formula, you may say, is a shortcut to thought. Granted. Here we use the shortcut for some quick conclusions:

1. If the temperature difference is small, a lot of heat can be pumped with little work.
2. For a given temperature difference, the hotter the hot reservoir, the less work it takes to pump a given amount of heat into it.
3. For a given temperature of the hot reservoir, the colder the cold object, the less heat is pumped out of it per unit of work done.

This gives a useful interpretation to the absolute zero of temperature. At $T_{\text{cold}} = 0$, no heat can be pumped out of the cold object. You may have expected that result. If affirms that the absolute zero is really the lowest temperature that can ever be reached. It also helps us to see why absolute zero is so difficult to even approach. The closer you get, the more work it takes to get even a tiny fraction closer.

The science of thermodynamics is not restricted to ideal refrigerators. It is used in all branches of chemistry and biology. Historically, it was the brainchild of nineteenth century engineers. They were not thinking about pumping heat uphill. They were concerned with the reverse process: getting work out of a device that takes energy in hot and discharges it cold. A heat engine is a heat pump run backward. Using our ideal reversible heat pump as an engine, we can apply all the same reasoning. Now ΔQ_{hot} is the heat taken in at temperature T_{hot}. A smaller amount of heat ΔQ_{cold} is discharged at temperature T_{cold}, smaller by the amount of work the device has performed. Again

$$\frac{\text{work performed by the engine}}{\text{heat taken in at } T_{\text{hot}}} = \frac{\Delta Q_{\text{hot}} - \Delta Q_{\text{cold}}}{\Delta Q_{\text{hot}}}$$

$$= \frac{T_{\text{hot}} - T_{\text{cold}}}{T_{\text{hot}}}$$

The ratio of work out to heat in is called the efficiency of the engine. The efficiency

$$\frac{T_{\text{hot}} - T_{\text{cold}}}{T_{\text{hot}}} = 1 - \frac{T_{\text{cold}}}{T_{\text{hot}}} = \textbf{Carnot efficiency}$$

is the highest efficiency any heat engine can have operating between given temperatures T_{hot} and T_{cold}. Such an ideal reversible engine is called a Carnot[3] engine. A real engine, having friction and irreversible heat flows, will have a lower efficiency. Some of the heat taken in will be wasted right in the machine. The Carnot formula gives an upper limit on the efficiency of any heat engine. Steam engines, gasoline engines, diesel engines, all have this ceiling on the efficiency they can possibly attain. Notice that 100% efficiency is effectively prohibited by the Carnot formula. It would require that all the heat energy taken in be converted into work. That would be possible only by having $T_{\text{cold}} = 0$, so no heat would be discharged at the cold temperature—absolute zero. We don't have any zero-temperature reservoirs sitting around waiting for a heat engine to discharge zero heat into them reversibly.

For practical engines, T_{cold} is usually room temperature or whatever outdoor temperature the weather might bring. T_{hot} is limited by materials. The engine parts must not melt. The lubricating oil should not burn up. In a steam engine, T_{cold} is usually 100°C (= 373 K), the boiling point of water. If T_{hot} is 200°C (= 473 K), the Carnot efficiency is only

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[3] Pronounced "car no." Sadi Carnot, a French engineer, was interested in how much work could be got out of a steam engine.

$$\frac{T_{\text{hot}} - T_{\text{cold}}}{T_{\text{hot}}} = \frac{473 - 373}{473} = 21\%$$

Internal combustion engines can have Carnot efficiencies over 60%. Still, the fuel efficiency of automobile engines is usually around 22%. Most of the energy liberated by burning gasoline goes out the exhaust pipe.

9.5 VARIOUS STATEMENTS OF THE SECOND LAW

The Second Law of Thermodynamics is stated in Sec. 9.2: "A system containing many particles will not go spontaneously from a disordered to an ordered state." Section 9.4 makes a number of statements that were derived from this rather bland formulation. It may be interesting to point out that some of these statements are logically equivalent. Each statement turns out to be a necessary and sufficient condition for another statement that looks quite different. No proofs are to be presented here. In reading the literature in modern biology, however, you will come across the term "Second Law of Thermodynamics" referring to various alternative formulations. Here are a few:

1. "Heat does not flow uphill." More carefully stated: There exists no device that does nothing to its surroundings except to take energy out of a colder object and put it into a hotter object. Popular statement: Refrigeration costs.
2. "No heat engine is 100% efficient." More careful statement: Define a cyclic engine as one that does work and returns to its original state. Then the statement is that no cyclic engine can take in heat and convert it entirely into work. It also has to discharge some heat.
3. "$\Delta S \leq \Delta Q/T$." More careful statement: The work one must do on a system to get from one state to another is not uniquely specified. The heat one must add to a system to get from the one state to the other is not uniquely specified. But the entropy difference between the states *is* uniquely specified.
4. "The entropy of a closed system can never decrease." Consequence: Thermal equilibrium is the state of maximum entropy.

9.6 ENERGY CRISIS. A DIGRESSION ON ECONOMICS

Everybody talks about the energy crisis. We are told that we are using up the earth's energy resources. Actually, the total energy of the earth is not decreasing. Furthermore, the heat energy of the air, the oceans, and the earth's hot core is enormously greater than the energy of all the fuel burned in the history of mankind. The worry is over energy easily available for turning wheels and heating houses.

Here are some figures in order to give an idea of the orders of magnitude involved. An electric clothes drier uses about 3 kilowatts of electric power. The electric generating capacity of the United States is about a hundred million times that—about 3×10^{11} watts. The entire population of the earth "consumes" energy at the rate of about 10^{13} watts. This refers to the rate at which energy is made unavailable for doing work. It comes to about 3×10^{20} joules per year.

The world's oceans are an enormous energy reservoir. Let us focus on a small part of it. The Gulf Stream is about 20°C warmer than the surrounding water. In principle, each cubic meter of Gulf Stream water could deliver 20×10^3 kilocalories of energy.

EXERCISE 23 Convert that to joules. $1\ \text{kcal} = 4.18 \times 10^3\ \text{J}$.

Its rate of flow is $2.5 \times 10^7\ \text{m}^3/\text{sec}$. The power that could be extracted from the Gulf Stream by running a heat engine between it and the surrounding colder water is thus around

$$2.5 \times 10^7 \frac{\text{m}^3}{\text{sec}} \times 20 \times 10^3 \frac{\text{kcal}}{\text{m}^3} \times 4.18 \times 10^3 \frac{J}{\text{kcal}}$$
$$= 2 \times 10^{15}\ \text{watts}$$

Why is there an energy crisis, with all that power available just off the coast of Florida?

You know the answer. But it has little to do with physics. The problem is economic. The capital cost of building large marine installations to tap temperature differences in the ocean is simply too great. At the current price of electric power, people are not even designing pilot plants. The Gulf Stream idea is just one of a large number of impractical schemes for tapping the enormous energy resources of the earth. As the price of energy rises, schemes that are unthinkable today may become more attractive to practical technologists. Lots of good inventions are needed to make energy conversion cheaper and better. For example, the sun beams more than 10^{17} watts of solar power at the earth. Perhaps one part in 10,000 of that is used in photosynthesis by trees and other plants. The rest of that nice, clean energy is wasted today. Why? Because solar collectors are expensive. They don't necessarily have to become cheaper before people start to use them on a large scale. All that has to happen is for the price of oil to rise some more. Similar things can be said about any number of presently "impractical" energy sources, from nuclear and **geothermal energy** to the blindfolded ox on a treadmill.

9.7 DOES LIFE VIOLATE THE SECOND LAW?

Living things are constantly creating order out of chaos. But isn't that exactly what the Second Law forbids?

In growing and reproducing, living things take in air and water and a few simple nutrients and build highly organized structures. Warm-blooded animals seem to fly right in the face of the classical statement of the Second Law. They create and maintain a temperature difference between themselves and their environment. Even on the molecular level, the Second Law appears to be flagrantly violated: How could as complex a structure as a protein molecule ever have happened spontaneously?

Let's sort out the rhetoric from the physics here. Yes, living things can decrease their own entropy. But they eat and they excrete. The net result is an increase in the entropy of the organism *plus* surroundings. The organism is not a closed system. Its entropy can decrease without violating the Second Law.

Living things are not unique in their ability to create order. There is a class of nonliving systems that progress in time into more and more highly organized states. Such systems are called **dissipative structures.** Fluid flow offers some beautiful examples. One you can observe in the bathtub. With the drain open, stir the water so that you get a number of vertical vortices. As you watch, you can often see those vortices rolling up into one big vortex over the drain. Here is an example of a chaotic situation developing spontaneously into an orderly one. Of course, there had to be a source of energy. The stirring was essential.

The phenomenon of biological evolution is also a case where Nature has progressed into increasingly organized patterns. Evidently, the entire biosphere is a dissipative structure. Again, since it is not a closed system, there is no violation of the Second Law.

SUMMARY

Definition of information content

Redundancy to combat noise

Binary digits and computer arithmetic

Codes with more than two symbols

Exponents and logarithms, base 2

Changing base of logarithms means changing unit of information

Information lost when two nonreacting gases mix

They don't unmix spontaneously: Second Law

Why the time-reversal invariance of individual events does not contradict the irreversibility of large systems

Friction and heat flow as dissipation

Calculate work needed to get the gases back into their own cylinders

Definition of entropy

The joule per Kelvin as an entropy unit; log to base *e* is understood

Relation between heat and entropy—an inequality

Work needed to pump heat from cold to hot

Work needed is proportional to temperature difference

Work needed is inversely proportional to T_{hot}

Consequences of the Carnot formula

Reversible heat engine has the Carnot efficiency, an upper limit

The various ways of stating the Second Law, all logically equivalent

What the politician means by the term "energy crisis"

10^{13} watts out, 10^{17} watts in

Mining the Gulf Stream's energy. Why not?

Life → increasing organization → local decrease of entropy

Definition of dissipative structure

KEY TERMS

Information content of a message Logarithm of the number of possible messages.

Unit of information content Depends on the base of the logarithms. For base 2, the unit is the *bit*.

Bit Binary digit. Unit of information content. If the only possible messages are Yes and No, the message delivers one bit of information.

Redundancy Saying something more than once.

Logarithm The inverse of the exponential function. If $x = 10^y$, then $y = \log_{10}x$. $\text{Log}_{10} 1 = 0$; $\log_{10} 10 = 1$; $\log_{10} 100 = 2$.

e Base of the natural logarithms = 2.71828183. This base has the property that $\log_e(1 + x) \cong x$ for small values of x.

Avogadro's number 6.02×10^{23}, the number of molecules in a gram mole.

First Law of Thermodynamics The heat energy of a closed system remains constant unless it has a way of doing work on its environment. In that case, the heat energy is decreased by the amount of work the system does. This work can also be negative, that is, the heat can be increased by the amount of work done *on* the system. (Discussed in Chapter 8.)

Second Law of Thermodynamics Heat flows from hot to cold. To transfer heat from cold to hot requires work. For alternate statements, see Section 5.

Reversible process No entropy is created. There is no diffusion, no heat conduction, no friction. Reversible work is work done slowly enough that temperature differences and friction are negligible.

Irreversible process A process in which energy is dissipated; that is, entropy is produced. This may involve conduction of heat or electricity, viscous flows or other friction, or diffusion.

Dissipation of energy Any irreversible energy conversion. Any process in which entropy is produced.

Entropy The thermodynamic measure of disorder. In any change in which information is lost, entropy increase = $k \times$ information loss. For a reversible transfer of heat ΔQ at absolute temperature T, the entropy transferred is $\Delta S = \Delta Q/T$.

Thermodynamics The science relating heat and work.

Ideal gas Satisfies the ideal gas law $pV = NkT$. See Chapter 6.

Atmosphere Unit of pressure = 1.013×10^5 newtons/m^2.

Liter atmosphere 1.013×10^2 joules.

Kilocalorie 4.18×10^3 joules, the amount of heat needed to raise the temperature of 1 kg of water by 1°C.

Extensive variable A variable proportional to the amount of substance that the variable characterizes. Examples: mass, volume, energy, entropy. *Intensive* variables, on the other hand, are independent of how much is present: pressure, temperature, density. (Not discussed.)

Isothermal At constant temperature.

Ideal refrigerator A reversible heat pump.

Carnot cycle A cycle of a reversible heat engine consisting of an isothermal expansion, an adiabatic expansion, an isothermal compression, and an adiabatic compression. More generally, a cycle of two isothermal and two adiabatic steps. [Not discussd in the chapter]

Adiabatic Without gain or loss of heat.

Carnot efficiency $(T_{\text{hot}} - T_{\text{cold}})/T_{\text{hot}}$ = the efficiency of a reversible heat engine operating between those two temperatures.

Geothermal energy Energy derived from steam coming from deep in the earth, where ground water is heated by rock heated by radioactive decay long ago still deeper in the earth.

Dissipative structure A system able to decrease its entropy, that is, to progress in time to a more orderly state.

REVIEW QUESTIONS

- Define information content of a written message.
- How many different messages can you send with N bits?
- How much longer is a message in a binary code than in a four-symbol code?
- How much longer is a message in a binary code than in a decimal code?
- The function $N(p) = 2^p$ is called the exponential function. What is the inverse function $p = p(N)$ called?

- Define what is meant by the *base* of a logarithm.
- How many different ways are there of distributing N objects in two containers?
- If there are $2^{6\times10^{23}}$ ways of distributing the neon atoms and the same number of ways of distributing the argon atoms, why isn't the total number of distributions the sum of the two?
- Calculate the probability that all the neon and argon atoms will be back in their own cylinders for a particular observation.
- Whenever there is friction, there is irreversibility. Under what other circumstances is energy dissipated?
- Whan a gas is compressed isothermally, where does the energy go that is used to do the work of compressing?
- If there are 2^N configurations of an N-molecule gas, why is information content an extensive variable?
- If we define information content $= \log_e 2^N$, is the unit of information content bigger or smaller than the bit? How much?
- From the formula $pV = NkT$, what are conventional units for k?
- What else is measured in those units?
- When heat flows isothermally, how do you calculate the entropy flow from the heat flow?
- How does one calculate the work needed to pump a quantity of heat Q into a reservoir at T_{hot}, taking heat out of a reservoir at T_{cold}?
- In doing that calculation, we set $\Delta S_{\text{hot}} = \Delta S_{\text{cold}}$. Why was this justified?
- The result of the calculation was that it requires 5.4 joules to pump 100 joules into the room. Where do the "missing" 94.6 joules come from? Do the energy bookkeeping.
- The Carnot formula $(T_h - T_c)/T_h$ is the ratio of two energies. What two energies?
- The Carnot formula represents an upper limit to something. What?
- Why can a heat engine never be 100% efficient? Answer in terms of the Carnot formula. (A glib answer would be, "because the Second Law forbids it.")
- If heat flowed uphill, that is, from cold to hot, would entropy be created or destroyed? Which is forbidden by the Second Law?
- When the sun warms up the Gulf Stream water, is the water's entropy increased?
- Approximately one ten-thousandth of the solar energy reaching the earth is absorbed by plants and used in photosynthesis. Compare the rate of this energy conversion with the world rate of energy consumption.
- When can the entropy of an object decrease without violating the Second Law?

PROBLEMS

✶ **1. Binary Arithmetic**

(a) Write the number nine in binary.

(b) How many different numbers can be represented with four binary digits?

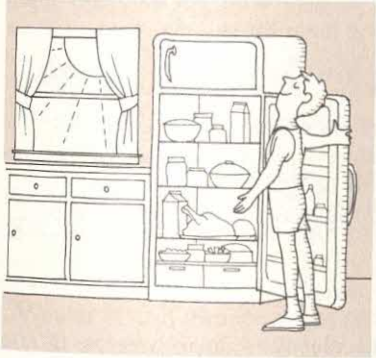
(c) Write in binary notation:

nine + two = eleven
nine + six = fifteen

(d) Write the number one-thousand in binary.

(e) Develop a prescription for converting numbers from decimal to binary.

✶ **2. Cooling the Kitchen** On a hot day you open the refrigerator and a wave of cool air greets you. Comment on the use of the refrigerator for cooling the kitchen.



PROBLEM 2. Does opening the refrigerator cool the kitchen?

✶ **3. Using the Environment as an Energy Source** An inventor applies for a patent on an industrial process for extracting energy from river water. The water is to be taken in at 20°C and discharged into the river at 19°C. Each cubic meter of water would thus deliver 10^3 kilocalories of heat, or 4.2×10^6 joules, which is to be converted into work to run a generator. With a flow-through of 10^2 m^3 per second, the power output of the plant would be 4.2×10^8 watts, minus any frictional losses. If you were the patent examiner, how would you deal with the application? (HINT: Second Law.)

✶ **4. Coin Toss** A coin is tossed 10 times, and a record kept of the result (heads or tails).

(a) Show that there are 1024 different possible records.

(b) How many of those 1024 records have strict head-tail alternation, that is, no head-followed-by-head or tail-followed-by-tail?

✶✶ **5. Labeled "Molecules"** Ten balls carry numbers 1 through 10. Each is to be placed in one of two bins, called ↑ and ↓.

(a) Show that there are 1024 possible distributions.

(b) Suppose you find out that bin ↑ has exactly one ball in it, but you don't know which one. How many of the 1024 possible distributions are eliminated by this knowledge?

✶✶ **6. 16-Bit Computer** A computer is advertised as having "16-bit words." A 16-bit register is to be used for storing numbers, and the first bit is used for the ± sign. Show that the largest integer it can store is 32767. (HINT: Don't forget the zero.)

NOTE: Now you know why 16-bit computers use two registers to store a six-digit number. Actually, that leaves some leftover bits, which can be used to insert a decimal point, that is, to store a "floating-point" number.

A large mainframe computer. The central processor unit (CPU) of a large modern computer can store millions of "words" and do arithmetic operations on many of those words in parallel, each in a few nanoseconds.

✶ **7. Word Size** How much bigger is a 32-bit word than a 16-bit word? Don't just say "16 bits bigger." Say, "It can store ______ times as many numbers." Alternatively, you might say, "It can store ______ times as much information."

✶ **8. Bytes** In computer jargon, 8 bits is called a byte.
(a) Show that one byte of information is a choice among 256 alternatives.

NOTE: You might think you need only 26 alternatives for writing with the English alphabet. But you want capitals and lower case, so that makes 52. You want numbers so that makes 62. You want punctuation marks, and maybe even Greek letters. The so-called ASCII code uses almost half of the 256 and is now rather standard in computer printers.
✶✶(b) Suppose that one bit is redundant. It could, for example, say whether the sum of the other bits is even or odd. Argue that this would permit most hardware errors to be detected.

✶ **9. Kilobytes and Megabytes** When a computer-disk manufacturer talks about "kilobytes," he really means 1024 bytes.
(a) Express that in bits.
(b) When he says "megabytes," he means $(1024)^2$ bytes. How many bits of information can a 1-megabyte disk contain?
(c) A book page might contain 2^{11} bytes (letters, numbers, spaces, etc.). How many book pages can be stored on a 1-megabyte disk?

✶✶ **10. Heat Pump** A certain room has enough insulation in the walls so that a 1000-watt heater will maintain its temperature at 294 K (= 21°C) when the outdoor temperature is 278K (= 5°C).
(a) At 5 cents per kilowatt-hour, what is the monthly heating bill for electric heating?
(b) An ideal heat pump operating between those two temperatures needs only (294 − 278)/294 joules of work for each joule of heat it pumps into the room. Verify that this is what the Carnot formula asserts. How many watts of electric power are needed to pump 1000 watts into the room?
(c) At 5 kWh, what is the monthly electric bill?
(d) At what rate does the heat pump cool the outdoors? That is, how much heat does it withdraw from the cold outside air? Answer in watts.
(e) If a heat pump is so much cheaper to run than an electric heater, why doesn't all home heating utilize heat pumps?

✶✶ **11. Statistics of a Small System** Suppose that there are only two neon atoms in one cylinder and two argon atoms in the other. The stopcock is opened.
(a) What knowledge is lost? How much information is lost?
(b) What is the probability that when the stopcock is closed a long time later, the original configuration will have been restored? (HINT: How many possible ways are there of distributing four distinguishable objects in two containers?)

NOTE: In the quantum theory, all the neon atoms are assumed indistinguishable, as are all the argon atoms. The calculation for indistinguishable particles is different.

✶✶ **12. Calculating Logarithms**
(a) $\log_e 10 = 2.3026$. Write down a prescription for converting $\log_{10} x$ to $\log_e x$.
(b) Look up the following logarithms:

$\log_{10} 1.0001$ $\log_e 1.0001$ $\log_{10} 0.9999$ $\log_e 0.9999$

✶✶ **13. Information Loss on Expansion**
(a) How much information is lost when 1 mole of gas expands from 22,400 cm^3 to 22,500 cm^3?
SOLUTION: The ratio of the new volume to the old is

$$\frac{22500}{22400} = 1 + \frac{1}{224}$$

For N (= 6×10^{23}) molecules the information loss is

$$\log\left(1 + \frac{1}{224}\right)^N \cong \frac{1}{224}N \text{ if the log is to base } e.$$

Verify using a calculator or a table of logarithms.
(b) How much does the entropy increase?
(c) Show that this is equal to the isothermal work done divided by the absolute temperature. (HINT: Yes, the pressure changes as the gas expands. No, it does not change very much: less than one-half percent. Approximate it as constant.)

✶✶✶ **14. Genetic Code** The four "letters" of the genetic code are the four nitrogenous bases, A = adenine, G = guanine, C = cytosine, and T = thymine. They are arranged in a sequence along the strand of a DNA molecule. Word length is not specified, nor is there a fifth symbol one might call a comma or a space. We speak of a commaless code. Yet different genes seem to require different lengths of DNA. Evidently, something in the code has to mean "this is the end"; there has to be a terminator.

A group of three bases is called a *codon*.

(a) How many bits of information does each codon carry?

(b) How many different three-letter codons are there? Sixty-one of these codons code for the 20 different amino acids found in living things. Three of the codons are "nonsense," that is, they do not code for any particular amino acid. They are terminators.

(c) There are six codons for leucine. We say, the code for leucine is six-fold degenerate. The code is six-fold degenerate for three of the amino acids, fourfold degenerate for five of them, two-fold degenerate for nine, and nondegenerate for two. What is the degeneracy of the code for the remaining amino acid (isoleucine)? (HINT: You don't have to know any biology or chemistry to do this problem. It is an exercise in reading and counting.)

(d) Why do you think the code has degeneracies? This question is not intended to have a unique answer. You might ask yourself how else 20 amino acids could be coded by four symbols.

(e) The English language also has degeneracies. What is the difference between HELP and AID? What functions can degeneracy serve in a code?

* **15. Entropy Increase in Irreversible Heat Flow** One joule of heat flows irreversibly from 294 to 293K. How much does the entropy increase?

** **16. Entropy of Irreversible Heat Flow** To pump 100 joules of heat into the room (294 K) from the outdoors (293 K) a reversible heat pump has to do 0.34 joules of work.

(a) Verify.

(b) Calculate the entropy increase of the room.

(c) Calculate the entropy decrease of the outdoor air. (HINT: $Q_{\text{cold}} = 100\text{ J} - 0.34\text{ J}$.)

(d) If 100 joules of heat leaked irreversibly from the room to the outdoors, how much would the total entropy increase?

** **17. Entropy of Fusion** The entropy of melting ice to water is 5.26 calories/mole K at 273 K (= 0°C).

(a) How much heat is needed to melt 18 grams (= 1 mole) of ice?

(b) In Chapter 8 on Heat and Temperatures we say that the latent heat of fusion of H_2O is 80 calories/gm. Does this agree with your answer to part (a)?

(c) Which is the more ordered state of the H_2O molecules, the solid or the liquid?

(d) Where does the energy go when ice melts isothermally?

** **18. Solar Constant and Solar Energy**

(a) The "solar constant," 2 calories/cm^2 minute, is the mean value of the total radiation incident from the sun outside the earth's atmosphere. Suppose that half of that energy were available for melting ice. How much ice would melt in one year per square centimeter of polar ice cap? Answer in gm/cm^2 year. Use data from Problem 17.

(b) Part (a) gives the "solar constant" as 2 cal/cm^2. Section 9.6 states that the sun beams more than 10^{17} watts of solar power at the earth. Reconcile these two figures. You will need to know the dimensions of the earth: Its circumference is 4×10^4 km. 1 cal = 4.18 joules.

** **19. World Energy Balance** Section 9.6, Energy Crisis, states that the total energy of the earth is not decreasing.

(a) Where does the energy go when you burn gas in your furnace and the warm air leaks out through the cracks around the doors and windows of your house?

(b) With all the fossil fuels burned by our industries, why has the mean temperature on earth not increased markedly over the years?

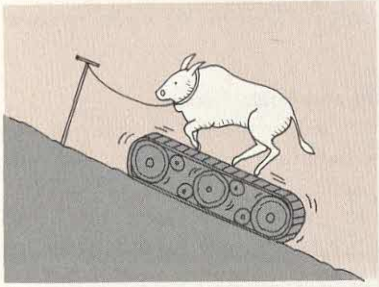
(c) Present evidence that the earth loses energy by invisible radiation into outer space. (HINT: Energy bookkeeping.)

* **20. Solar Power** Suppose that all of the 10^{17} watts of solar power incident on the earth were absorbed by the ocean. At what rate would the ocean temperature rise? Express in degrees per year. Neglect heat loss by the ocean.

DATA: Specific heat of water = 4.18×10^3 joules/kg K
Mass of ocean water = 10^{21} kg

* **21. Energy from the Hot Core** The earth's core is much hotter than the surface. 100 km down, the rock is about 2000°C hotter than at the surface. Calculate the amount of heat energy potentially available for doing work, using that 2000-degrees temperature difference. An estimate of the mass of the core is 5×10^{24} kg, with a specific heat of 4×10^2 joules/kg K.

*** **22. Solar Energy Conversion Versus Draft Animal** A solar collector is used to boil water, which is used to run a steam turbine, which is used to generate electricity, which is used to run an electric motor, which pumps water for irrigation. The entropy of the earth increases in the process.



PROBLEM 22.

(a) Explain how the ox on the treadmill pumping irrigation water increases the entropy of the earth (see figure).
(b) Estimate which produces more entropy per ton of water pumped, the electric pump or the ox. Justify!
(c) The ox must be fed daily; the solar-powered pump runs by itself. Why would anyone use an ox instead of an electric pump?
(d) How is it that the sun is the ultimate energy source for the ox as well?

✶ **23. Entropy of Mixing** Write the number $2^{12 \times 10^{23}}$ in conventional notation, that is, as a decimal number times 10 to an integer power. (HINT: $\log_{10} 2 = ?$)

✶✶ **24. Entropy of Dilution** When N molecules of gas expand from volume V_1 to volume V_2, the information loss is $N \log (V_2/V_1)$, corresponding to an entropy increase of $Nk \log (V_2/V_1)$. When a solution containing N ions is diluted from concentration c_1 to concentration c_2, the information loss is $N \log (c_1/c_2)$.
(a) Concentration means the number of ions per unit volume of solution. Why do we write $\log (V_2/V_1)$, but $\log (c_1/c_2)$? Is one formula upside down?
(b) When a solution containing 1 mole of ions is diluted with an equal quantity of water, how much does its entropy increase?

✶✶✶ **25. Active Transport** The concentration of potassium ion inside a living cell is 0.15 moles per liter. Outside, in the extracellular fluid, the concentration is only 0.005 moles per liter.
(a) Calculate the entropy change per mole of K^+ in being transported across the cell membrane. Use the reasoning of Problem 24.
(b) Where does the potassium have greater entropy, inside or outside the cell? In the phenomenon of active transport, K^+ ions travel in across the cell membrane, going *up* the concentration gradient. This is the opposite of diffusion. Explain why this is not a violation of the Second Law of Thermodynamics. (HINT: The cell takes in nutrients.)

✶✶✶ **26.** [Hard] **Entropy of Mixing**
(a) Why do we say that energy is dissipated when the stopcock between the Ne and Ar cylinders is opened? (HINT: Invent a way in which the mixing of the two gases could be used to do work. Semipermeable membranes?)
(b) Show that the work done by a reversible engine in the thought experiment of part (a) is equal to the entropy of mixing times the absolute temperature.

✶✶✶ **27. Work to Double the Volume** Numerical Integration
(a) A cylinder of length 22.4 cm whose base has an area of 1000 cm^2 has a volume of ________ cm^3.
(b) According to the Ideal Gas Law, it would contain 1 mole of gas at 273 K and 1 atmosphere. How many molecules is that?
(c) The force on the piston that closes the cylinder is 1 atm $\times\ 10^3\ \text{cm}^2$ = ________ newtons. Remember that 1 atm = 1.01×10^5 newtons/m^2.
(d) If the piston slowly moves 1 millimeter in response to this force, the gas does an amount of work,

work = force × distance = ________ joules

Since the volume increase is only 100 cm^3 out of an initial volume of 22,400 cm^3, the pressure does not drop very much during this isothermal expansion. According to the Ideal Gas Law, the pressure is inversely proportional to the volume. If the volume increases by 1 part in 224, the pressure decreases to

$$\frac{1}{1 + \frac{1}{224}} \text{ atm}$$

When the piston moves the next millimeter, the amount of work done by the gas is a little less than the work done in that first millimeter. Before moving the third millimeter, the pressure is

$$\frac{1}{1 + \frac{2}{224}} \text{ atm}$$

To find the work done in moving all the 224 mm needed to double the volume, sum up the work done in each successive millimeter.

$$\text{work} = 1000 \text{ cm}^2 \times 1 \text{ mm} \times \left(1 + \frac{1}{1 + \frac{1}{224}} + \frac{1}{1 + \frac{2}{224}} + \cdots + \frac{1}{1 + \frac{223}{224}} \right) \text{ atm}$$

The sum in parentheses has 224 terms. The answer will be a bit too large, because we did not take account of the drop in pressure *during* each millimeter's movement. We would have done better to take smaller increments. For example, we might have moved only 0.1 mm each time and had 2240 terms in the sum. The result, good to three significant figures, is

$$\text{work to double the volume} = 22400 \text{ cm}^3 \text{ atm} \times 0.693$$

(e) Do the numerical integration for 10 increments of 22.4 mm each.

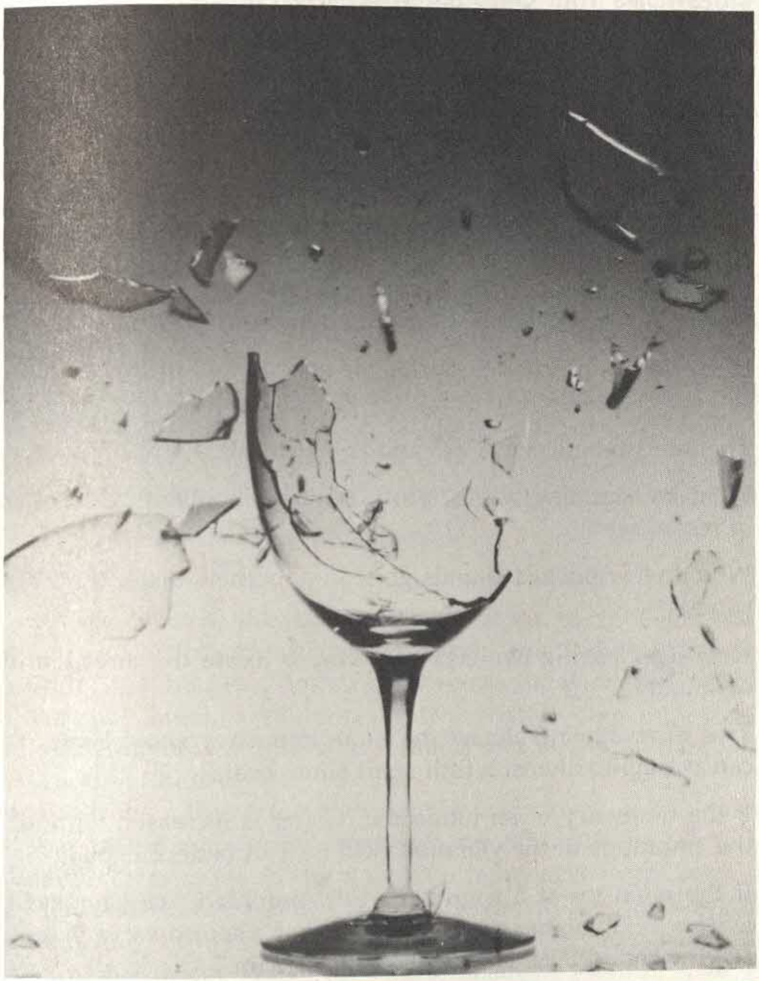
✶ **Supplement to Problem 27.** If you know calculus, do the same calculation using the notation of definite integrals. Let x = piston displacement, A = piston area, so volume $V = Ax$ and force on the piston is $F = pA$. The work in moving from $x = 22.4$ cm to $x = 44.8$ cm is

$$\text{work} = \int_{22.4 \text{ cm}}^{44.8 \text{ cm}} F(x)\,dx = \int_{22{,}400 \text{ cm}^3}^{44{,}800 \text{ cm}^3} p(V)\,dV$$

Now use the Ideal Gas Law, $pV = p_oV_o$:

$$\text{work} = p_oV_o \int_{22{,}400}^{44{,}800} \frac{dV}{V} = p_oV_o \log_e 2$$

Substitute $p_o = 1$ atm and $V_o = 22{,}400$ cm^3 and look up $\log_e 2$. How far off was the numerical integration?



A wineglass being shattered by the sound of Ella Fitzgerald's voice.

10

SOUND AND HEARING

QUESTIONS THIS CHAPTER WILL ANSWER

Sound waves travel in solids, liquids, and gases. Why not in vacuum?

Does sound travel faster in water than in air? In steel?

What kind of materials are good sound absorbers?

What is the relation between pitch and frequency?

What is simple-harmonic motion?

How can sounds of the same frequency have different tone quality? How can tone quality be described quantitatively?

If a temperature change causes the speed of sound in air to increase by 1%, how does the wavelength of A-440 change?

The interval between A-440 and A-880 is called a(n) ________.

What are standing waves? How are they involved in the phenomenon of resonance?

Why do low-pitched sounds go around corners better than high-pitched sounds?

How does having two ears help you to locate the direction of a source of sound?

Two tones having almost the same frequency show beats. Why? How can two tones almost a fifth apart show beats?

If the frequency of an ultrasonic cleaner is increased without changing the amplitude of the vibration, will it clean better? Explain.

If the intensity of a sound wave is tenfolded, we say that the sound level is 10 db higher. If the amplitude of a sound wave is tenfolded, we say the sound level is 20 db higher. Explain.

Outdoors, the intensity of a sound wave falls off inversely as the square of the distance from the source. Why? Why is this not true indoors?

10.1 VIBRATION AND SOUND

A tree falls in the forest. No one is there to hear it. Is there a sound?

Most modern physicists dodge this conundrum. They don't distinguish between sound and vibration; they point out that not all vibrations can be heard. Some vibrations are too slow, too fast, too weak, or have no way to reach the ear.

In general, too slow means less than 15 vibrations per second. We feel slower vibrations, but the sensation is different from what we call hearing. For a teenager, too fast means somewhere above 20,000 vibrations per second. As we get older, we gradually lose the range above 16,000 vibrations per second.[1] What is too weak? The ear is a very sensitive instrument. Just how sensitive it is, you will learn in this chapter. How does sound reach the ear? The usual way is through the air. If you enclose a bell in a jar and start it ringing, and then pump the air out of the jar, you won't hear the bell any more (Fig. 10.1).

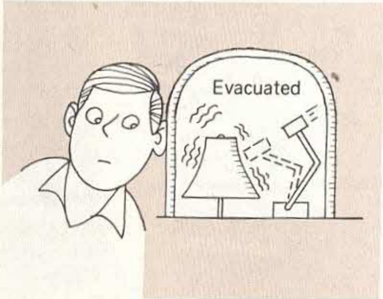


FIGURE 10.1 If a bell rings in a vacuum, you can't hear it outside.

What do we know about sound just from everyday experience? The harder you strike an object, the louder the sound. The farther away you get, the softer the sound (usually). Hard surfaces reflect sound; soft surfaces absorb it. Sound goes around corners but not too well. We have an idea of the speed of sound from listening to echoes. Sound travels faster than most airplanes, but we talk about "supersonic" flight and the noise problems created by **"sonic boom"** (Problem 36).

What do we know about **pitch?** Here our knowledge comes from singing, or playing a musical instrument. The higher tones correspond to faster vibrations. The lower tones usually come out of bigger instruments. The next section is about the relation between pitch and frequency.

10.2 PITCH AND FREQUENCY

When you clap your hands, it is hard to say what the pitch of that sound is. Nonetheless, we talk about a high-pitched clap and a low-pitched clap. (A thud? A thump?) When you push a chair across the floor, that scraping sound also does not have a well-defined pitch. Neither does the rustle of leaves or the splashing of water. What is it that allows us to identify a pitch? What makes a sound "musical"?

That turns out to be a tough question, with a fuzzy answer: If the vibration pattern is repeated more or less regularly, we can discern a pitch. The pitch we identify corresponds to the repetition frequency. Every pitch has its frequency. This one-to-one relation between pitch and frequency was discovered by the ancient Greeks. They founded the science of acoustics, and gave music a mathematical foundation. They made a musical scale and named the intervals.

Most scales go up an **octave.** Can you identify an octave? Sing a sustained tone. Have a friend try to sing the same note. If you and your friend are of opposite sex, chances are the notes are not really the same, but an octave apart. It's just that an octave is such a consonant interval, it sounds almost as good to our ears as a unison. Now try to sing two octaves apart. If you sing together, does the harmony still sound good?

How many octaves wide is your whole singing range? Mine is two octaves on a good day. Early in the morning I can really get down low. The lowest I ever get is the C two octaves below middle C. That has a frequency of 64

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[1] One vibration per second is called one Hertz and written "1 Hz." This unit of frequency is named in honor of the German physicist Heinrich Hertz, who discovered electromagnetic radiation.

Hz. The C one octave higher has twice the frequency, or 128 Hz. Middle C, another octave up, is 256 Hz. Going up an octave means doubling the frequency, regardless of what note you start from.[2]

Going down an octave means halving the frequency. Other musical intervals are also defined in terms of the *ratios* of the frequencies. For example, the **fifth** has a frequency ratio of 3:2. This means that the upper note has a frequency 1.5 times that of the lower note. The intervals in our songs have frequency ratios that are close to ratios of small integers, like 3:2 and 4:3. Section 10.4 on Beats suggests why these intervals are pleasant. The "Three Blind Mice" box names some familiar intervals and their frequency ratios.

Most people cannot remember a pitch but can recognize an interval. The rare person with long-term pitch memory is said to have *absolute pitch*. You can probably sing "Three Blind Mice" starting on any note, but you may not know if you are singing in the same key as yesterday. What you remember is *relative pitch;* you remember the intervals.

FIGURE 10.2 The index finger is on middle C.

EXERCISE 1 The top note of the piano is five octaves above middle C. Calculate its frequency.

NOTE: We called the frequency of middle C 256 Hz. That is actually an old-fashioned standard, which we used here because it divides by 2 easily. The modern standard gives the A above middle C a frequency of 440 Hz, making middle C 261.6 Hz on the so-called "equally tempered" scale. (See Problem 13.)

EXERCISE 2 How many octaves are there in our auditory range, from 15 to 20,000 Hz? Don't bother with the fraction.

Timbre

If you and your friend really sing the same note, really get in unison, you still sound different. We can recognize people's voices, just as we can tell a violin from a trumpet playing the same note. The pitch just has to do with the repetition frequency of the vibrations. But the vibration pattern determines the *quality* of the sound, also called *tone color* or **timbre.** Figure 10.3 shows oscilloscope traces of the waveforms of various instruments playing the same note.

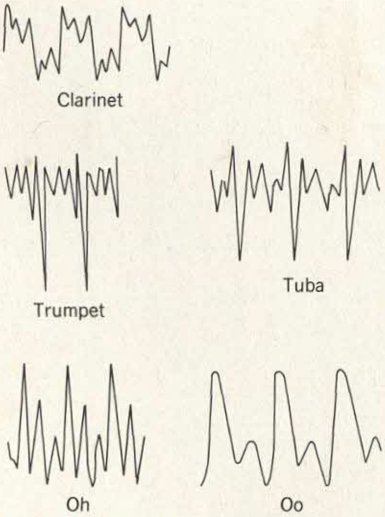

FIGURE 10.3 Waveforms of tones from different instruments, including the human voice (two different vowels).

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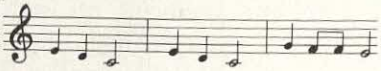
[2] If you don't recognize the octave, try singing "Somewhere over the rainbow." "Some" to "where" is an octave.

THREE BLIND MICE—MUSICAL INTERVALS

Three blind mice three blind mice See
major third fifth
how they run See how they run
minor third
They all run after the farmer's wife. . . .
fourth



"All" is an octave higher than "mice." Frequency ratio 2:1
"See" is a fifth higher than "mice." Frequency ratio 3:2
"All" is a fourth higher than "They." Frequency ratio 4:3
"Three is a major third higher than "mice." Frequency ratio 5:4
"See" is a minor third higher than "run." Frequency ratio 6:5

It may seem remarkable to you that a trumpet heard from far away has the same trumpety quality as a trumpet heard from close up. That seems to say that tone quality does not depend on the intensity of the sound wave. You conclude that scaling the intensity up or down does not change the tone quality. It seems less remarkable when you turn the volume control on your amplifier. Indeed, you say it affects *only* the intensity of the sound. That says something about how our ears are constructed. If you see nothing remarkable in this **linearity** property, see if talking louder doesn't affect the tone quality of your voice. Playing the trumpet louder certainly changes its tone color. You can't just turn up the volume control.

10.3 SOUND WAVES

To have oscillation, there has to be restoring force. When something is displaced from an equilibrium position, it won't vibrate unless it has a tendency to return toward that equilibrium. There has to be **elasticity** in the material. Also there has to be enough mass to overshoot the equilibrium. The elastic potential energy has to be converted to kinetic energy. So there mustn't be too much damping. To have oscillation, there must not be too much friction. Otherwise all the elastic energy goes into heat right away.

But if the vibration is to be heard, there has to be some energy loss. The sound has to get to the ear. Some of the energy of the oscillation has to be radiated, has to go into a propagating sound wave. That traveling wave is an oscillatory excitation of the medium in which it is propagated, so that medium has to fulfill the same requirements: enough elasticity and not too much viscosity. Materials used for acoustic absorption, on the other hand, are soft and yielding.

When we think about sound waves we usually think about air. Its elasticity is due simply to its pressure—to the rapid motion of the air molecules bouncing off each other and off any surface in contact with the air. Not surprisingly, the **speed of sound in air** is comparable to the average speed of the molecules. That is why sound travels faster in hot air than in cold.

EXERCISE 3 Speed of Sound in Air

The Ideal Gas Law was derived in Sec. 6.4: Pressure is proportional to absolute temperature and to the density of the gas molecules: $p = NkT/V$. The energy kT turned out to be a measure of the average kinetic energy per molecule:

$$\tfrac{3}{2}kT = (\tfrac{1}{2}mv^2)_{\text{average}}$$

(a) Solve this equation for the root mean square (r.m.s.) speed of the molecules.

(b) Put in values for nitrogen gas at 273 K (molecular weight = 28).

(c) The speed of sound in a gas is given by the formula (we won't derive it) $c_s = (\gamma kT/m)^{1/2}$, where the constant γ has the value 1.4 for air. Plug in the values for nitrogen gas from part (b). How close do you come to the "textbook value" of 330 meters/sec?

Waves in General

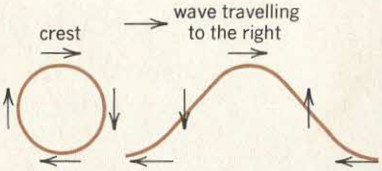


FIGURE 10.4 Wave traveling to the right.

This may be a good place to talk about the wave concept. You hear the word **wave** and you think about water. Look at a leaf floating on the surface as a wave goes by. You see it moving in a little circle (Fig. 10.4), ending up where it was before the wave passed.[3] Another leaf a little further on does the same dance, but at a somewhat later time. We say "its **phase** lags" behind that of the first leaf; the wave gets there later. But the pattern of the motion is the same. The propagation of the pattern is the essence of wave behavior. Notice that the speed of the wave and the speed of the leaf are not the same.

What different kinds of waves have in common is that some kind of pattern propagates in space. The speed at which the pattern travels is called the **phase velocity.** Most waves also involve the transmission of energy and momentum. The **energy velocity** is not always the same as the phase velocity. In fact, standing waves (see Sec. 10.4) don't transmit energy at all. In this book you will meet elastic waves, gravity waves, surface-tension waves, electromagnetic waves, and even matter waves (also called probability waves). Sound waves are elastic waves.

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[3] Try standing in the shallow water on an ocean beach! You can feel the push and pull.

Waves in a Homogeneous Medium

The sound of a hand clap or the explosion of a firecracker is a single pressure pulse, which travels outward from the source. The wave front, the beginning of the region of increased pressure, is essentially spherical, since the pulse travels through the air at the same speed in all directions. With a periodic sound source, like a vibrating tuning fork, the waves of increased pressure, called **condensations,** alternate with regions of decreased pressure called **rarefactions.** (See Fig. 10.5.) As the prong moves back and forth, it pushes on the air going one way and sends out a condensation. When it pulls, going back the other way, it sends out a rarefaction.

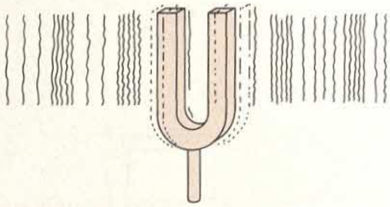

FIGURE 10.5 Tuning fork radiating sound, showing compressions alternating with rarefactions.

In a solid or liquid, sound travels much faster, because the atoms or molecules are "touching" all the time. They don't have to travel from collision to collision. The stiffer, the less compressible the medium, the faster a pressure wave travels. Again, just as with a gas, the more massive the atoms, the slower the speed of sound.

EXERCISE 3 Steel is less compressible than water—its bulk modulus is about 40 times greater—but it is 8 times denser. Guess: Which propagates sound faster, steel or water?

EXERCISE 4 **Speed-of-Sound Formula**

In many materials the speed of sound is given by the formula $c_s = (B/\rho)^{1/2}$, where B is the bulk modulus and ρ the density. The bulk modulus of stainless steel is 26.4×10^{10} newtons/m^2; its density is 7.9 grams per cm^3. (See Sec. 3.2 on Strength of Bone.)

(a) Calculate the speed of sound in steel using the formula.

(b) The speed of sound in water at 25°C is 1493 m/sec. Use the formula to calculate its bulk modulus. Are you surprised that water is more compressible than steel?

Compressional and Shear Waves

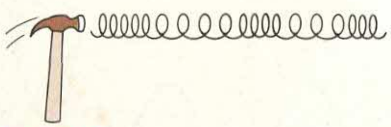

FIGURE 10.6 Transverse waves and longitudinal waves.

If you hit the end of a metal rod with a hammer, a wave of compression—a condensation—travels down the rod toward the other end. But if you hit the side of rod, so it bends a little, there are two waves in the rod. In addition to the wave of compression, a wave of bending also travels down the rod. The motion of the metal atoms due to compression is along the direction of travel of the wave—**longitudinal.** Their motion due to the bending, also called the flexural or **shear** wave, is perpendicular to the wave direction—transverse. A transverse wave is a vector wave: To describe a **transverse** wave, we have to specify a direction, or a plane or vibration. (See Fig. 10.6.) That is called its **polarization.** You will meet that concept again when you read about light waves. In a longitudinal wave, the vibration is only in one dimension, back and forth along the direction of travel. For a transverse wave, two dimensions are needed to describe the vibration, a right-and-left dimension and an up-and-down dimension. Note that the idea of polarization comes up only for transverse waves, not for longitudinal. In gases and liquids, only longitudinal waves can be propagated: pressure waves. Air does not resist bending. There is no restoring force for transverse displacements. The same is true for water. But be careful not to confuse transverse elastic waves with surface waves. Water cannot support transverse elastic waves. Surface waves due to gravity and surface tension, which are transverse waves, are seen in every pond and bathtub.

A transverse wave in one medium can give rise to a longitudinal wave in another. The vibrations of the string of a guitar or violin are transverse, but they couple to vibrations of the body of the instrument and in turn to vibrations of the air surrounding it. That is how the sound of the instrument is radiated. The reciprocal effect can also be observed. Sound incident on the violin can set a string vibrating if its frequency is close to a "natural" frequency of the string. That is *resonance,* which is discussed in the next section.

10.4 WAVELENGTH AND RESONANCE

With a periodic wave, with condensations alternating with rarefactions, the distance from one pressure maximum to the next is the same regardless of how far from the sound source the wave has traveled. That distance is just how far the wave travels in one *period* of the vibration. Suppose that the vibration frequency is 1000 Hz, one-thousand vibrations per second. Then one-thousandth of a second elapses between successive maxima—or successive minima, if you prefer. The period is 0.001 second, 1 millisecond. At a sound speed c_s of 330 meters per second, the distance between maxima is

$$\left(\frac{330\text{ m}}{\text{sec}}\right) \times (0.001\text{ sec}) = 0.330\text{ m}$$

That is called the **wavelength.** So the wavelength λ is the distance the wave travels in one period T:

$$\lambda = c_s T \qquad \text{(Eq. 1)}$$

Obviously, the concept of wavelength has meaning only for a periodic wave. It doesn't make sense to ask what is the wavelength of an explosion—just as you don't ask what is its pitch.

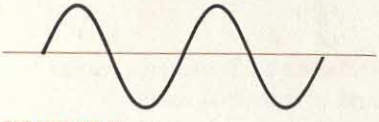
FIGURE 10.7 Sinusoid.

The wavelength in water of our 1000-Hz wave would be much longer than in air, because the speed of sound in water is so much faster: 1500 m/sec. In 0.001 sec a wave going that fast will travel 1.5 meters, compared to the 0.33 meters in air. In steel the speed of sound is about 6000 m/sec, so our 1000-Hz wave would have a wavelength of 6 meters.

EXERCISE 5 Verify.

Simple-Harmonic Motion: Sinusoidal Oscillation

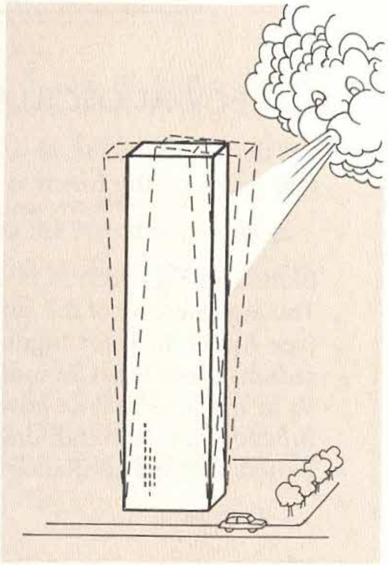

FIGURE 10.8 Skyscraper swaying in the wind. The natural frequency might be a fraction of a hertz, on the order of a few vibrations per minute.

A tuning fork, a guitar string, a chime, a playground swing, a skyscraper swaying in the wind—all such "oscillators" have a favorite frequency. They are easy to set in oscillation at this *natural frequency*. Once started, they keep oscillating for a while. They may also be able to sustain free oscillations at frequencies higher than this natural frequency. For this reason, the natural frequency is also called the *fundamental*.

That mythical oscillator, the "mass point on a massless spring"[4] has free oscillations of only a single frequency. Its motion is called **simple harmonic.** Figure 10.7 is a graph of a simple-harmonic displacement versus time. You may recognize it as *sinusoidal,* that is, the graph of a sine or cosine function. The box "Sinusoidal Oscillation" does the mathematics of simple-harmonic motion. Here we give its definition:

> A point is in simple-harmonic motion if its acceleration toward equilibrium is proportional to its displacement from equilibrium.

Evidently, a mass is in simple-harmonic motion if the restoring force is proportional to the displacement. (HINT: $F = ma$) A spring that has the property that its tension is proportional to how far it has been stretched is called a linear spring, or is said to obey *Hooke's Law.* (See Sect. 3.2.) Most springs obey Hooke's Law well for small enough displacements from equilibrium. That is why simple-harmonic motion is not just a myth. In real structures, the small-amplitude oscillations that are easiest to excite are nearly simple harmonic. To see why, you have to know about **standing waves.**

EXERCISE 6 The spring constant K is defined by Hooke's Law,

force = $K \times$ displacement

Our analysis has shown that $K = m\omega^2$. (See box on next page)

(a) Express the period T of the oscillator in terms of K, m, and A. Are you surprised that the formula does not contain the amplitude A? The *period is independent of amplitude.* That is the hallmark of the simple-harmonic oscillator.

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[4] Have you noticed something funny about the way physicists talk? They start with an idealization. When the oversimplified model has been understood, then comes the time for the complications that make the model more realistic.

SINUSOIDAL OSCILLATION

Let the displacement of a "mass point" at time t be given by the function

$$\text{displacement} = A \sin \omega t$$

plotted in Fig. 10.7. A is called the amplitude. The argument ωt of the sine function is an angle (see Appendix B for trigonometry) measured in radians. Recall that 2π radians is 360°, full circle; so in the time that ωt advances by 2π, the sine function goes through one period. Repeat: The period T of the oscillation is such that

$$\omega T = 2\pi$$

or

$$\text{period} = T = \frac{2\pi}{\omega} \qquad \text{(Eq. 2)}$$

The rate ω (pronounced omega) at which the angle is swept out is called the *angular frequency*. If we call the frequency ν (pronounced nu) the number of periods per unit time, then

$$\omega = 2\pi\,\nu \qquad \text{(Eq. 3)}$$

is the number of radians per unit time. For our 1000-Hz oscillator, the period is 0.001 sec. The angular frequency is

$$\omega = 2\pi \times \left(1000 \frac{\text{periods}}{\text{sec}}\right) = 6283\ \frac{\text{radians}}{\text{sec}}$$

You may have seen a demonstration in which the projection (the shadow) of a point executing uniform circular motion is plotted versus time.* If you think of the light coming from the left and casting a shadow on a screen on the right as shown, the displacement of the shadow from the center of its motion is given by the function $A \sin \omega t$, where A is the radius of the circle (Fig. 10.9).

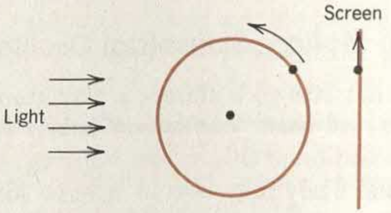

FIGURE 10.9 When the point moves in uniform circular motion, its shadow performs simple harmonic motion.

The velocity of the point on the circle changes direction all the time, of course, but its speed remains constant in uniform circular motion:

$$v = A\omega = \frac{2\pi\,A}{T} \qquad \text{(Eq. 4)}$$

The circumference of the circle is $2\pi A$ and is swept out in one period T. How about the velocity of the projection, the velocity of our simple-harmonic oscillator? It is given by the projection of the velocity vector of the point in uniform circular motion. That is also a periodic function of time, same period, but evidently out of phase with the displacement: When the oscillator is at the center of its motion (displacement = 0), it is going fastest. When it is at the limits of its motion (displacement = $\pm A$), it reverses, so its velocity is instantaneously zero. Evidently, the fastest velocity is ωA, the speed of the point in the circle. The velocity function is†

$$A\omega \cos \omega t$$

Now for the acceleration of our simple-harmonic oscillator. We can use the same argument over again. The acceleration is the time rate of change of the velocity. We therefore want to project the tip of our velocity vector and ask for its velocity. The velocity vector, of magnitude $A\omega$, rotates uniformly, sweeping out a "circumference" $2\pi\,\omega A$ in time T. Its maximum rate of change is thus

$$a_{\max} = \frac{2\pi\omega A}{T} = \omega^2 A$$

The acceleration is given by the function

$$\text{acceleration} = -\omega^2 A \sin \omega t$$

You can see how you get the minus sign: When the displacement is up, the acceleration is down, and vice versa. The acceleration is half a period out of phase with the displacement. Since the acceleration is proportional to the displacement, our definition of simple-harmonic motion is satisfied!

We can interpret all this in terms of our "point mass on a massless spring." Use the law of motion $F = ma$. The restoring force is given by

$$F = ma = -m\omega^2 A \sin \omega t$$

The displacement of the oscillator from equilibrium is

$$\text{displacement} = A \sin \omega t$$

Indeed, the restoring force is proportional to the displacement. The proportionality constant (Hooke's Law constant, spring constant) is $m\omega^2$.

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*PSSC Film No. 0306, "Periodic Motion," by J. N. P. Hume and D. G. Ivey, is excellent.

†If you know how to differentiate the function $A \sin \omega t$ with respect to time t, that comes as no surprise.

(b) Dimensional analysis: amplitude A ~ meters
mass m ~ kilograms
spring constant K ~ kilograms/second^2

How can you combine these three quantities to obtain a *time*, a quantity with the dimensions ~ seconds?

EXERCISE 7 A force of 10^{-3} newtons stretches a linear spring 1 mm.

(a) What is the spring constant $m\omega^2$?

(b) Attached is a mass of 10^{-2} kg, which is much more than the mass of the spring. Find the period for small oscillations.

Standing Waves

Stretch a long clothesline or wire and try to observe a wave running down it. That spiral "retractable" telephone cord is good. You can shake a transverse pulse into it that is slow enough to follow with your eye. An extra-long "slinky" toy, stretched on a smooth floor, makes a dramatic demonstration of transverse waves (Fig. 10.10). You can shake quite a sharp pulse into one end and watch it travel down with little loss of amplitude. What happens when it gets to the clamped end? It is reflected. But a pulse with displacement to the right is reflected as a pulse with displacement to the left. A crest is reflected as a trough. The reflected wave has changed sign.

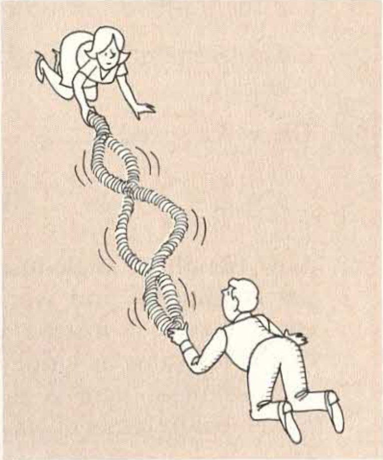


FIGURE 10.10

SUPERPOSITION OF TWO RUNNING WAVES TO MAKE A STANDING WAVE

A sinusoidal running wave on a string is described by a displacement

$$A \sin(\omega t + \text{phase angle})$$

of each point of the string, where the frequency ν and the period T are

$$\nu = \frac{\omega}{2\pi}, \quad T = \frac{1}{\nu} = \frac{2\pi}{\omega}$$

and the phase angle depends on what point x of the string you are watching. By the way, that is called an "angle" just because it is measured in radians. A snapshot of the string gives the displacement as a function of position x (Fig. 10.11):

$$A \sin(kx + \text{phase angle})$$

where the wavelength is

$$\lambda = \frac{2\pi}{k}$$

and the phase angle depends on the time t. We can put all this into one equation, using our knowledge that the displacement has to be a function of $x - vt$ for waves traveling at speed v in the positive x-direction:

$$\text{displacement} = A \sin(kx - \omega t + \text{phase angle})$$

The wave speed is

$$v = \frac{\lambda}{T} = \frac{\omega}{k} = \lambda\nu \qquad \text{(see Eq. 1.)}$$

Now the phase angle just depends on the initial conditions, and we might as well set it equal to zero. If, instead, you set it equal to $\frac{1}{2}\pi$, that is the same as writing $A \cos(kx - \omega t)$. For completeness, here is the running-wave displacement in terms of wavelength λ and period T:

$$\text{displacement} = A \sin\left(\frac{2\pi}{\lambda} x - \frac{2\pi}{T} t\right)$$

A running wave in the negative x-direction is

$$\text{displacement} = A \sin(kx + \omega t)$$

Now to add them up. The superposition of the two running waves gives a standing wave:

$$A \sin(kx - \omega t) + A \sin(kx + \omega t) = 2A \sin kx \cos \omega t$$

EXERCISE 8 Prove this trigonometric identity, starting from $\sin(A + B) = \sin A \cos B + \cos A \sin B$.

If you look at the motion of one point x, the time dependence of the displacement is given by the term $\cos \omega t$, and its amplitude is $2A \sin kx$. This is maximum ($= 2A$) at the midpoint $x = \frac{1}{2}\pi/k = \frac{1}{4}\lambda$.

If you look at a snapshot of the whole string, it has a shape proportional to the term $\sin kx$, with amplitude $2A \cos \omega t$. To see that this is a standing wave, note that the time dependence and the x-dependence are in separate terms.

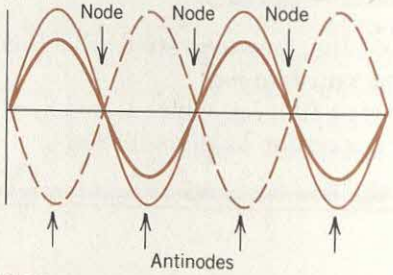


FIGURE 10.11 Three "snapshots" of a standing wave. The points of no-displacement are the nodes. The points of maximum displacement are the antinodes.

If while the reflected pulse is coming back you shake in another pulse, you can watch the two "annihilate" each other when they meet. They seem to pass right through each other. At one instant they cancel each other out. What happens if, instead of sending down sharp pulses, you move your end of the rope in simple-harmonic motion? There is a natural frequency. That is the frequency at which the wave makes the round trip in exactly one period of the motion. At this frequency you no longer see one wave running down and a reflection coming back. You see the whole rope (or slinky) oscillating in phase. Every point is moving to the right at the same time; then the whole thing moves to the left, and so on. We call that a *standing wave.* You know it is made up of the superposition of one running wave going down and another one coming back; but the superposition is such that you are no longer aware of anything traveling.

Notice that standing waves are possible only if the waves running both ways are sinusoidal. It is instructive to see this mathematically, using the properties of the trigonometric functions. We take two running waves going in opposite directions and add them and—presto—we get a standing wave. This is done in the box labeled "Superposition."

Not surprisingly, this "standing-wave" motion is exactly the natural-frequency oscillation of the rope (or slinky). Now you can see why this mode of excitation builds up easily, why it is easy to excite oscillations at the fundamental frequency. A wave crest going down is reflected as a trough coming back. But just as that trough gets halfway back, the next trough is halfway down, and the two add up to make a doubly deep trough. A trough going down is reflected as a crest and gets to the middle just as the next crest does, making a crest of double height. Crest meets crest exactly in the middle; trough meets trough exactly in the middle. The wave going down and the wave coming back are said to *interfere constructively* at the midpoint. At the two ends, they interfere *destructively;* there crest meets trough and trough meets crest and they cancel. The points on a standing wave where the interference is constantly destructive are called **nodes.** The points of constant constructive interference are **antinodes.** For oscillation at the fundamental frequency, the nodes are at the two ends, and there is an antinode in the middle.

Think about what happens at frequencies near the fundamental, say just below it. A crest is reflected from the end as a trough, which gets back to the middle just ahead of the next trough going down. Trough meets trough in a different spot following each reflection. No standing wave results. If you are shaking waves in by moving your hand, the motion of your hand feeds energy into the wave one moment and takes energy out the next. When you were shaking in waves at the fundamental frequency, your hand's motion was in phase with the standing wave and fed energy into the standing wave all the time.

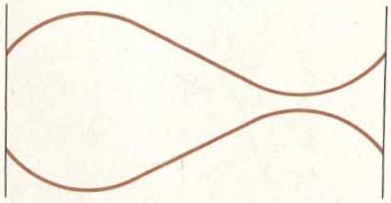


FIGURE 10.12 A sinusoidal wave traveling to the right and a similar one traveling to the left superpose (interfere) to form a standing wave.

Now shake in waves having a frequency twice the fundamental (Fig. 10.12). This time two complete wiggles occur in one round trip. By the time a crest is reflected from the end, the trough following it has got to the midpoint. That crest is reflected as a trough, and when that trough gets to the midpoint, it meets a crest. The midpoint is now a node: Trough meets crest, and crest meets trough. But the two points one-quarter of the way from each end are antinodes. There crest meets crest and trough meets trough. We again have a standing wave. All the points in the near half of the rope (slinky) are oscillating in phase; all the points in the far half are in phase. But the two halves are exactly out of phase: When one half is moving to the right, the other half is moving to the left.

A little thought will convince you that standing waves will also result at a frequency 3 times the fundamental, 4 times the fundamental, and so on. Integer multiples of the fundamental frequency make a standing-wave pattern, and no other frequency does. If you have a long slinky, it is quite a spectacular demonstration to set it vibrating in its fundamental mode, then speed up the frequency and see the standing wave give way to chaos. When the frequency an octave higher (i.e., double the fundamental) is reached, standing waves reappear, this time with a node in the middle. Going to higher frequencies, that regular pattern disappears and chaos reigns until you get to 3 times the fundamental, where you get standing waves with a node one-third of the way from each end, and three antinodes. Higher and higher frequencies take you through standing-wave patterns with an ever-increasing number of nodes. You can verify that the frequency that gives standing waves with, say, six antinodes is exactly 6 times the fundamental frequency.

FIGURE 10.13 If the cello string is lightly touched exactly at its midpoint, it vibrates at its second-harmonic frequency, that is, an octave above its fundamental.

EXERCISE 9 On Harmonics of a String

Let the lowest string of the cello (the C string) be tuned to 64 Hz. This means that the tension is such that the fundamental frequency is 64 Hz. If a finger tip is placed very lightly at its midpoint, that fundamental mode is suppressed. The midpoint becomes a node. The string can still oscillate in the mode called the second harmonic, the next-highest mode. (See Fig. 10.13.)

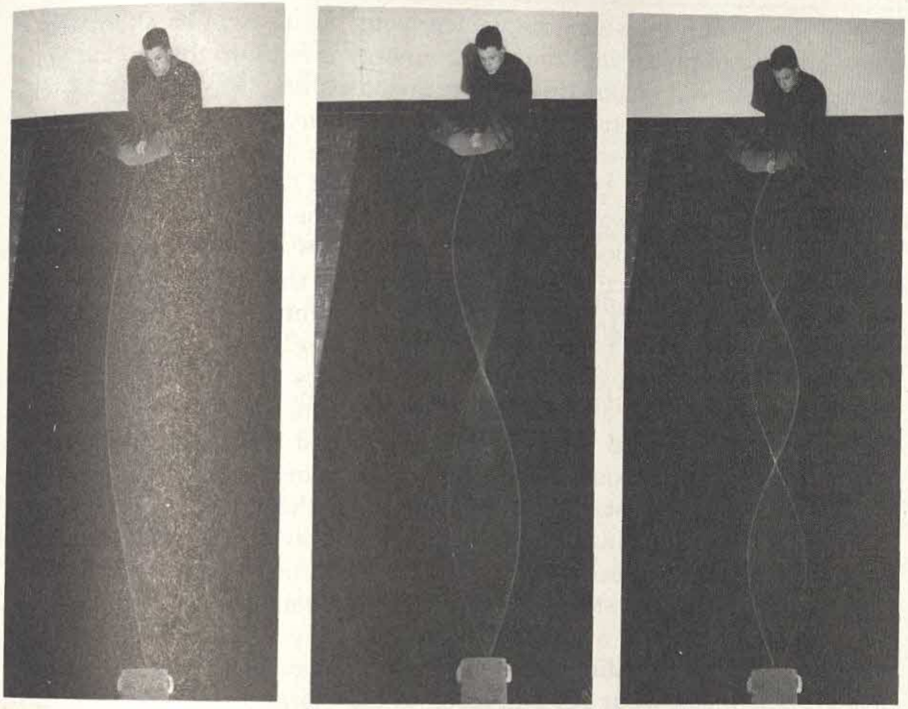
(a) What is its frequency? Is that exactly one octave above the fundamental? Is it called C?

(b) If a finger tip is touched instead to a point one-third of the way from the bridge, the string can oscillate in three equal parts. Find the frequency of this mode. The note is called G, the G below middle C.

(c) What is the ratio of this "third-harmonic" to the "second-harmonic" frequency of part (b)? Is the interval a fifth? Check with the "Three-Blind-Mice" box if in doubt.

EXERCISE 10 On Standing Waves

When the slinky is oscillating at its fundamental frequency, the wavelength λ_1 is twice the distance L between the ends: The running wave goes down and back in one period. In the next-higher mode, with two antinodes, the wavelength λ_2 is equal to the length L.



Slinky toy (triple-length) standing-wave demonstration. *(a)* One antinode. *(b)* Two antinodes. *(c)* Three antinodes.

(a) The next standing-wave mode has three antinodes. Relate its wavelength λ_3 to the length L.

(b) For a standing wave with n antinodes, relate the wavelength λ_n to the length L.

The speed at which a wave crest or trough travels down a slinky depends on the tension in the slinky. The tighter the spring, the faster the wave. But the wave speed does not depend on the frequency. Waves of different frequencies travel at the same speed. We say that the medium is *nondispersive*. The description of standing waves in the text tacitly assumes this lack of dispersion when it talks about doubling the frequency to get from the fundamental to the next standing-wave mode.

(c) What is the frequency of the standing wave with n antinodes? Express in terms of the fundamental frequency.

Resonance

We can now start to understand resonance. If you are pushing a child on a playground swing, you can build up the vibration amplitude fastest by pushing at the natural frequency (Fig. 10.14). If your pushing frequency is

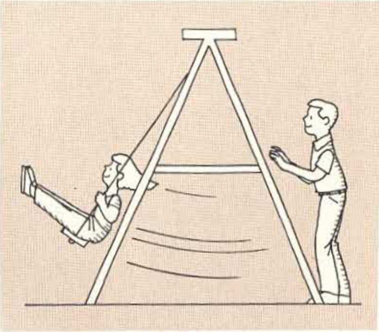


FIGURE 10.14 The amplitude builds up if the natural period of the swing is in resonance with the period of the pushing.

"out of tune" with the swing frequency, your pushes will be alternately in phase and out of phase with the swing. Energy is fed into the motion of the swing and then out again. How about a violin string? A trumpet? A wine glass? A room? A structure can absorb energy easily at its standing-wave frequencies. If a sound wave having one of these special frequencies is incident on the structure, the vibrational motion of the structure can stay in phase with the sound wave. Remember, if the force on an object is in the same direction as its velocity, work is done on the object. The force transmits energy *to* the object. If, on the other hand, the force on the object is antiparallel to its velocity, the work done by the force is negative; the force then takes energy away from the object. At incident sound frequencies well away from standing-wave frequencies, the pushes and pulls of the sound wave will sometimes be *with* the motion of the structure, sometimes *against*. Energy is alternately fed from the incident sound wave *into* the structure and back *out* into the sound wave. At the standing-wave frequencies, the pushes and pulls can stay with the motion of the structure. This allows energy to flow predominantly from the sound wave to the structure. This ability of a structure to select certain sound frequencies and to convert incident sound energy into standing-wave energy is called *resonance*.

If you have access to a piano, hold down a key to lift the damper, and sing right into the sounding board to look for resonances. After singing at a constant pitch, stop your voice to listen to the vibrations you have excited. You might even be able to feel them by touching the string gently. That kills them, of course. How many different harmonics can you excite in one string? The fundamental, of course. Twice it? Three times it? Four? Five? The higher ones will probably be weaker.

How sharply tuned is the resonance? If the note you sing is a semitone away from (out of tune with) the note whose key you are holding down, do you excite any noticeable vibrations?

Ringing and Resonance

In describing resonance, we have quite forgotten to talk about the dissipation of vibrational energy in our oscillating structures. We have neglected damping. Yet it is a simple fact of experience that some free oscillations ring for a long time, others damp out in a few periods. That is the difference between a clang and a click, between a ding and a thud. A somewhat more subtle generalization from experience relates this ringing time to the frequency selectivity of the oscillator: The longer its ring, the more selective the oscillator. An oscillator having little damping rejects incident sound if its frequency is slightly out of tune with its natural frequency. The more damped the oscillator, that is, the shorter its ringing time, the wider the band of frequencies over which the oscillator can absorb sound. We can talk about the "sharpness of the resonance." The sharper the resonance for forced oscillations, the longer the ringing for free oscillations.

BANDWIDTH THEOREM

The following is an intuitive argument, not a mathematical proof.

A sound has to start at some time, and it stops at some time. So we can talk about a wave train. But a wave train doesn't have a sharp beginning or end. If you try to count the number N of wiggles in a wave train, you probably can't do much better than ±1. If the wave train passes (the sound lasts) for a time t, then its frequency ν is

$$\nu = \frac{N}{t} = \text{number per unit time}$$

Since the uncertainty* in N (call it ΔN) is ±1, the uncertainty in the frequency ν is

$$\Delta\nu = \pm\frac{1}{t}$$

Statement: The statistical uncertainty (error) in the frequency of a train of waves is inversely proportional to its duration. The proportionality constant is of order unity.

The relative error $\Delta\nu/\nu$ in the frequency is $\pm 1/\nu t$, or

$$\frac{\Delta\nu}{\nu} = \pm\frac{1}{N}$$

EXAMPLE The frequency of a wave train 100 periods long can be measured to ±1%.

EXAMPLE To measure a frequency to an accuracy of one part in a million, one needs a wave train at least a million periods long.

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*To say that the error in counting waves is ±1 is surely too conservative. Actually, we never specified a precise statistical meaning for ΔN. A mathematically more supportable claim is that we can count to $\pm 1/2\pi$, that is, to the nearest radian. Then the Bandwidth Theorem is written $\Delta\nu/\nu = \pm(1/2\pi)N$.

This generalization about physical systems is usually approached through a mathematical technique called Fourier analysis (or harmonic analysis). The theorem called the **Bandwidth Theorem** is presented in the box.

Harmonic Content

A sound wave whose vibration pattern is sinusoidal, that is, simple harmonic, is called a **pure tone.** A low-pitched whistle through pursed lips comes close. So does a tuning fork, or an audio-oscillator. Among musical instruments, only the flute in its middle register is reasonably "pure," and only if it is played softly enough.

Suppose that two pure tones an octave apart are sounded together. Figure 10.15*a* shows the resulting vibration pattern. The repetition frequency is that of the fundamental. We shall call the lower tone the fundamental, and the tone with twice the frequency the second harmonic. It is just as if they were both generated by the same uniform string or pipe. The addition of the second harmonic changes the shape of the vibration pattern of the fundamental but not its frequency. So the ear does not hear two pitches. It hears a tone having the pitch of the fundamental but a different tone quality. Indeed the tone quality depends on the relative intensity of the two components.

Figure 10.15*b* shows a superposition of fundamental and third harmonic—an interval called a twelfth. Again, the perceived pitch is that of the fundamental, but the tone color is different. Figure 10.15*c* has a mixture of fundamental, second harmonic, and third harmonic. You can imagine what it might be like to add fourth harmonic, fifth, and so on. The higher the harmonic, the spikier the pattern. Now for a mathematical theorem:

> Any repeated pattern can be represented as the superposition of a fundamental and its integer harmonics, and the recipe is unique.

Each superposition of harmonic components gives a different vibration pattern. So each vibration pattern has a unique harmonic decomposition.

This theorem is due to the French mathematician J. B. J. Fourier. The harmonic decomposition is also called a Fourier series or **Fourier decomposition.** There are instruments called harmonic analyzers that can measure the relative strength of the various harmonics making up a vibration pattern. In recent years, harmonic *synthesizers* have been produced commercially and have been used in popular music.

Armed with **Fourier's Theorem,** you can make sense out of a statement like this: "The tone of the double bass is characterized by a large third-harmonic content. In fact, there is more third harmonic than fundamental." This means that the vibrations of the string are a combination of standing waves. Remember, a superposition of standing waves is not a standing wave. You won't see the two nodes corresponding to the third harmonic because there is some fundamental and some second harmonic, as well as higher harmonics. Why can't the double bass put out a pure sine wave? Why can't the string oscillate in its fundamental mode? It can. If it is excited by a sinusoidal sound wave at the fundamental frequency, it will. It will resonate at the fundamental. But it won't sound like a double bass. Normally, the double bass is bowed or plucked, and then the various harmonics are produced in the proportions that give it its characteristic double-bass tone color.

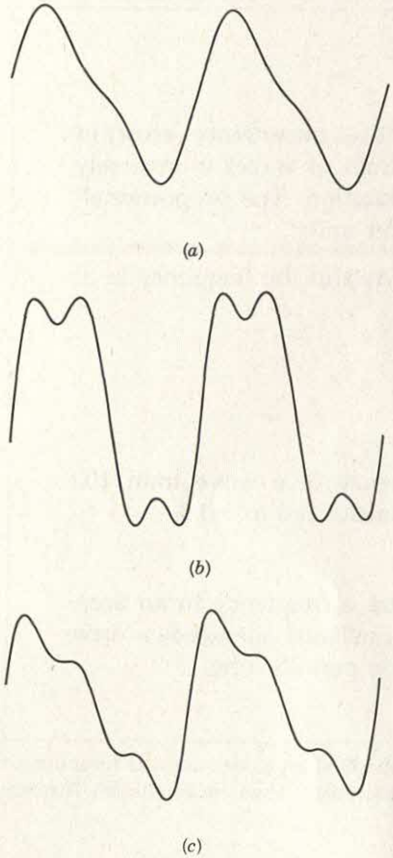

FIGURE 10.15 *(a)* Fundamental plus second harmonic; *(b)* fundamental plus third harmonic; *(c)* fundamental plus second and third harmonic.

Beats

You will need your friend again, or else a musical instrument that can play a sustained tone while you sing. Sing the same note together. Really try for unison. You can hum or sing "la" or any vowel. Then, while your friend sustains his or her note, make yours slightly flat or sharp. Listen to the "harmony." It will sound bad. If you listen carefully, you will hear the sound pulsate—loud, soft, loud, soft. . . . The closer your two pitches are to unison, the slower the beats. As you widen the interval between your two pitches, the beats speed up. Gradually, they become imperceptible as beats. Finally, you just hear the harmony.

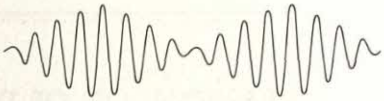

FIGURE 10.16 Superposition of two sinusoids whose frequencies differ by roughly 10%.

Figure 10.16 shows how beats arise from the superposition of two sinusoidal vibrations whose frequencies are close. The two start in phase, so they interfere constructively. After a while they are exactly out of phase and interfere destructively. Slowly, they get back in phase. The cycle repeats.

Let's do the arithmetic. Superpose a 100-Hz signal and a 101-Hz signal of the same amplitude. Look at Fig. 10.16. The two signals start in phase. The first crest has double amplitude, the sum of the two. Half a second later, the 100-Hz signal is putting out its fiftieth crest, but the 101-Hz signal is on crest number $50\frac{1}{2}$ (= 101 periods/s × 0.5 s). Nonsense; that's a trough. Crest meets trough, and they cancel. The superposition signal is at a minimum. Not until a full second has elapsed are the two sinusoidal signals back in phase, and the superposition signal at maximum again. What you hear is one beat every second. Once every second the sound gets loud. Is its frequency 100 or 101 Hz? It is hard to tell by listening to the pitch. The wobbliness in intensity is perceived as an uncertainty in pitch.

If we had superposed a 100-Hz signal and a 102-Hz signal, the **beat frequency** would have been 2 per second. Evidently, the beat frequency is the difference between the two "beating" sinusoidal frequencies. The box shows the algebra.

Harmony

Now you can appreciate why our ears are so good at pitch matching. Two sounds heard together will have "zero beat" if they have the same frequency. If they are out of tune, we can actually hear the beat frequency, and we find it unpleasant. If someone is sounding A-440 and you are playing the A-string of a violin that is a bit sharp—say it is tuned to 444 Hz—you can feel that 4-Hz beat. Is your pitch discrimination really good to 1%? It is, if the two tones are sounded together.

In the air in which the sound waves travel, nothing is actually vibrating at this 4-Hz beat frequency. Air is a "linear" medium, and the two sinusoidal signals simply superpose.[5] The ear is not so linear. It actually manufactures new frequency components. It generates harmonics, as well as difference frequencies. It generates a 4-Hz vibration.

Now we can make good on our promise to explain why some intervals sound more pleasant than others. Why does A-440 sound good when played together with A-220, an interval of an octave? The difference frequency, the frequency of the beats, is 220 Hz. The ear actually generates that difference frequency, and it has zero beat with the incident 220-Hz signal. The ear also generates the harmonics of 220 Hz and the harmonics of 440 Hz, and their difference frequencies. If you do the arithmetic, you'll find

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[4] All alliteration allegedly accidental.

SUPERPOSITION OF TWO SINUSOIDS OF DIFFERENT FREQUENCY

$$\sin \omega_1 t + \sin \omega_2 t =$$
$$2 \sin\left(\frac{\omega_1 + \omega_2}{2}\right) t \cos\left(\frac{\omega_1 - \omega_2}{2}\right) t$$

To prove this, write

$$\omega_1 = \frac{\omega_1 + \omega_2}{2} + \frac{\omega_1 - \omega_2}{2},$$
$$\omega_2 = \frac{\omega_1 + \omega_2}{2} - \frac{\omega_1 - \omega_2}{2}$$

and use the trigonometric identity $\sin(A + B) = \sin A \cos B + \cos A \sin B$. The term $\sin \frac{1}{2}(\omega_1 + \omega_2)t$ on the right has a fast frequency that is the average of the two sinusoidal frequencies. The (slow) term $\cos \frac{1}{2}(\omega_1 - \omega_2)t$ gives the "envelope." It goes from +1 to 0 to −1 with a frequency that is the *difference* between the two sinusoidal frequencies (*not* half of the difference. See Fig. 10.15). This difference frequency is the slow frequency, the frequency at which the "fast" signal is modulated. Your ear can't tell whether the cosine is positive or negative. That just affects the *phase* of the fast signal. What you want to know is how fast its *amplitude* is modulated. The difference frequency is the beat frequency.

no slow beats! But suppose you sound 444 Hz together with the 220 Hz. The difference frequency generated by the ear would be 224 Hz, which beats with the incident 220-Hz signal at 4 beats per second. That sounds ugly. Also, the 444 Hz beats with the second harmonic of the 220 Hz. Again, 4 beats per second. The second harmonic with the fourth: 888 Hz with 880 Hz gives 8 beats per second. You can see why we like our octaves clean.

Try an interval of a major third. The fourth harmonic of A-220 is A-880; the fifth harmonic is C♯ −1100. The frequency ratio 1100:880 (= 5:4) is a major third. The difference frequency is 220 Hz. That beats with the upper frequency at (1100 − 220 =) 880 Hz, which matches the lower frequency: Zero beat. You see why the major third is pleasant.

EXERCISE 11 The 220 Hz also beats with the 880 Hz. Do you expect to find their difference frequency in the ear? (HINT: It is a harmonic of. . . .)

The fifth, the fourth, the minor third, all those intervals we identified in "Three Blind Mice" are characterized by frequency ratios that are close to ratios of small integers. That fact makes for no slow beats between their harmonics and the various difference frequencies. Traditional harmony is based on the consonance of those intervals.

10.5 LOUDNESS AND INTENSITY

In the quiet of the night you can hear a pin drop 10 meters away. That is how sensitive your ears are. Let's put in numbers. We want to know how much sound power has to hit, say, each square millimeter of eardrum surface in order to be audible. The answer depends on frequency, of course. In the middle of the audible range, where the ear is most sensitive, the answer is about 10^{-18} watts. Translate that into sound power per square meter: In consistent units, that energy flux is 10^{-12} watts/m^2. The ear is very sensitive. Another measure of its sensitivity is the fraction by which the air pressure has to change to be perceptible. What is the pressure amplitude of a sound wave at the threshold of hearing? In the same frequency range, the answer is about 3 parts in 10 billion.

EXERCISE 12 Multiply that out. Atmospheric pressure is 10^5 newtons/m^2. The least perceptible pressure amplitude is 3×10^{-10} of that, or ______ newtons/m^2.

Not only is the ear sensitive; it also has an enormous dynamic range. The threshold of pain is a **sound intensity** (= power per unit area) 10^{12} times greater than the threshold of hearing. A dynamic range of a trillion! Is there any instrument in your physics lab that can measure over a range as wide as that?

Find an instrument that is good for measuring the very small as well as the very large. You will notice that it has a "range switch." Usually, each click of that knob changes the sensitivity by a factor 10. Two clicks to the left, and the instrument is a hundred times more sensitive. Three clicks to the right, and the sensitivity is one-thousandth of what it was. The ear has no such knob. It has a number of mechanisms that make its response nonlinear, that attenuate intense sounds more than weak ones. As a consequence, we measure sound level on a nonlinear scale. For historical reasons, we chose a logarithmic scale.

Decibels

The telephone engineers have given us the terminology here: A tenfold increase in intensity is called an increase of 10 **decibels.**[6] (Abbreviation: 10 db.) Reducing the intensity to one-tenth is called a decrease of 10 db. A hundredfold increase in intensity is called an increase of 20 db.

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[6] Named in honor of Alexander Graham Bell, the inventor of the telephone. Ten decibels used to be called 1 bel, but no one talks that way any more.

EXERCISE 13 A thousandfold increase in intensity is called an increase of ______ db.

EXERCISE 14 The threshold of pain corresponds to a sound intensity that is 10^{12} times that at the threshold of hearing. It is therefore ______ db higher.

DEFINITION The difference in *level* of two sounds, measured in decibels, is equal to 10 times the logarithm of their intensity ratio.

$$\text{level difference in db} = 10 \log_{10}\left(\frac{\text{intensity}_1}{\text{intensity}_2}\right)$$

That is a definition based on measurable physical quantities. Intensity is energy per unit time per unit area. It is difficult to measure, but there is no ambiguity about what is to be measured. *Loudness* is quite another matter. That is a psychological variable. Its definition depends on asking people how they perceive different sounds. In the nineteenth century, scientists thought that the ear had logarithmic response, that the loudness scale paralleled the decibel scale of sound level. That turned out to be a bad approximation. For a modern loudness scale, see Problem 25.

EXERCISE 15 The logarithm to the base 10 of 2 is 0.301. If one sound is twice as intense as another, their level difference is 3 db. (Pardon the roundoff.) If one sound is 4 times as intense as another, their level difference is ______ db.

Inverse-Square Law

If a sound source radiates 1 watt of sound power equally in all directions—we say isotropically—then 1 meter away the sound intensity is

$$\text{intensity} = \frac{1 \text{ watt}}{\text{area of a 1-meter sphere}} = \frac{1 \text{ watt}}{4\pi\ (1 \text{ m})^2}$$

Two meters away, the sound intensity is one-quarter of that. Three meters away, it is one-ninth of that. Ten meters away, it is one-hundredth of that. Why? The area through which the energy has to flow when it has gone 2 meters is 4 times the area is passed through after 1 meter of travel. The wavefront is a sphere centered at the source of sound. The area of a sphere is proportional to the square of its radius, and that area is in the denominator here:

$$\text{intensity} = \frac{\text{power}}{4\pi r^2}$$

The intensity goes as the inverse square of the distance from a point source. We could just say: Doubling the distance reduces the sound level by 6 db.

How good is the inverse-square law in real life? It neglects (1) reflection, (2) absorption, and (3) scattering. Don't expect it to hold in a room. Reflection from the walls accounts for most of the sound we hear indoors. When is the inverse-square law a reasonable guide? Answer: Outdoors, on a flat lawn without trees, with the sound source less than a mile away. Outdoors, because there are no obstacles to scatter sound; on the grass, because it does not reflect much sound; sound source not too distant, so that absorption by the air can be neglected.

10.6 DIFFRACTION

You hear footsteps coming down the hall long before you see the person appearing in the doorway. Sound goes around corners much better than light does. Light casts rather sharp shadows. What kind of "shadows" does sound cast? Obviously, they are fuzzy. To look at an object, you have to turn your eyes toward it. To listen to a sound, your ears can be turned in any direction. The sound waves can bend around the head and get to the ear. You may have noticed that low-frequency sound turns corners more readily than higher frequencies. Walk in front of a loudspeaker: The high pitches are beamed in the direction the speaker faces; the lows go out in all directions.

Water waves bend around an obstacle. Sit on a pier and watch the waves as they hit the supporting piles. Or lie in the bathtub and watch the ripples "break" on your toes. Because we can see water waves, they are easier to experiment with than sound waves. If straight waves impinge on a large obstacle, the obstacle casts a shadow. For a smaller obstacle, the waves bend right around it. It is not that the smaller obstacle casts a smaller shadow. The word *shadow* does not describe the wave pattern behind the smaller obstacle. We say that the wave is *diffracted* by the obstacle.

What determines the scale of lengths? It is the wavelength of the ripples. Long waves, with wavelengths longer than the obstacle is wide, bend around the obstacle. Short waves go more in a straight line. Shorter wavelength means higher frequency. This shows what we observed with the sound waves from the loudspeaker: The higher frequencies are more directional.

The ripple tank (Figures 10.17–10.19) is a good device for studying waves. In Figure 10.17, straight waves are incident on a wide opening. We would call that a *collimator*, because it delimits the waves to a narrow beam. You can barely tell that the beam is spreading. The shadow of the edge is quite sharp. The wavelength sets the scale: The opening is several wavelengths wide. In Figure 10.19 the opening is just about one wavelength wide, and the directionality of the waves is all but lost. Beyond the opening, the wave pattern looks almost like the circular wavefronts emitted by a point source.

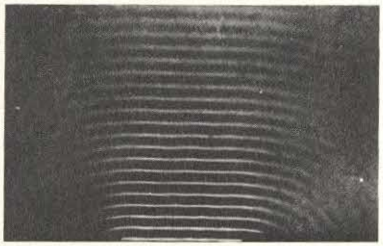


FIG. 10.17

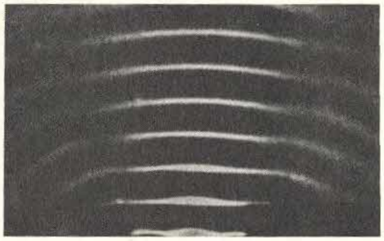


FIG. 10-18

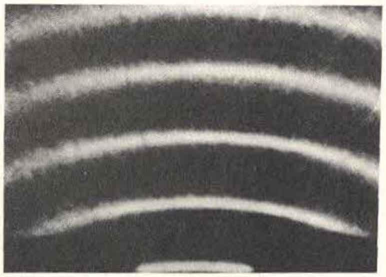


FIG. 10.19
Ripples of three different wavelengths incident on the same opening. The longer the wavelength, the less directional the beam.

We shall discuss this situation more quantitatively for light waves in Chapter 12, under "Diffraction by a Single Slit." The wavelengths of visible light are around 5×10^{-7} meters. We therefore expect to notice little deviation from straight-line propagation until we start looking at very small objects or fine details.

Ultrasound

In applications where we want to use sound to locate objects, the wavelengths have to be small compared with the size of the objects. In echo location of submarines (sonar), that is not much of a limitation. But the same technique can be used to locate foreign bodies in the eye. If the foreign body is a splinter one-tenth of a millimeter in size, the wavelength of the sound in the medium in which it is traveling—in the vitreous humor—should be of the order of one-tenth of a millimeter or less. Estimating the speed of sound to be 1500 meters per second, as in water, we are talking about a frequency (Eq. 1)

$$\nu = \frac{c_s}{\lambda} = \frac{1500 \text{ m/sec}}{10^{-4} \text{ m}} = 1.5 \times 10^7 \text{ Hz}$$

That is almost a thousand times higher than the highest frequency we can hear. Sound of frequencies higher than the audible range is called *ultrasound*.

The use of ultrasonic techniques for "imaging," for seeing through human tissue much as we do with X rays, is a rapidly developing technology. Other applications of ultrasound have become quite standard. For example, ultrasonic cleaners are used to wash dirt from glassware. Here the point is just

Ultrasonic imaging is a noninvasive and risk-free technique for visualizing the fetus. Unlike x rays, ultrasound is not ionizing. Unlike ultrasonic cleaning, the intensities needed to make good pictures on the computer screen are sufficiently small to avoid cavitation.

to set the cleaning solution flowing rapidly back and forth past the surface to be cleaned (Problem 45).

How We Know Where the Sound is Coming From

The highest frequencies we can hear have wavelengths around 2 centimeters.

EXERCISE 16 Check that. The top of the audible range might be 20,000 Hz. The speed of sound in air is 330 m/sec. Wavelength = ______. (HINT: Eq. 1)

For such sound waves the human head casts a real shadow, but the ear funnels them right in. You can imagine how the ears could tell the direction of such a high-frequency sound wave: The ear facing the sound source hears it louder. You turn your head until the sound is equally loud in both ears. (See Problem 28.)

What do our ears do with long waves? Wavelengths comparable to the size of the head diffract around it. Differential loudness discrimination by the two ears could not be the clue. Yet we are very good at turning the head accurately toward the direction of the sound, even for low-frequency sounds. How does it work?

The ears can discriminate very small differences in the arrival times of sound at the two ears! It is easy to see how that would work for a bang or a click. How does it work for a long, steady tone? It turns out that the ears are sensitive to differences in the *phase* of the two arriving waves. Exercise 17 shows how acute this ability to measure phase difference is.

EXERCISE 17 Suppose that a source of 64-Hz sound is very far away so that the waves incident on the two ears come essentially from the same direction. The head is turned 10° to the right of this direction. (HINT: $\sin 10° = 0.174$.) The ears are 17 cm apart.

(a) How much closer is the left ear to the sound source than the right?

(b) How much later does the right ear receive a given wave crest than the left?

(c) What fraction of a wave period is this?

NOTE: Even smaller angles than this can be discriminated by the ear.

10.7 THE EAR AS A TRANSDUCER

A device that transforms acoustic energy into electrical energy and/or vice versa is called a *transducer*. That is the generic term for microphones and loudspeakers. Sound energy comes into the ear and sets the eardrum (tympanic membrane) vibrating. Electric impulses go from the ear to the brain along the auditory nerve. The coding of these electric signals—the language of the auditory nerve—is not yet well understood. They travel in some 30,000 separate channels (neurons). The actual conversion from acoustic to electrical energy takes place in the inner ear in the cochlea, an organ somewhat analogous to a microphone. The pathway of the vibrations from the eardrum to the membrane covering the entrance to the inner ear is via the middle ear, which is a sort of mechanical amplifier.

Don't misunderstand. There is not more vibrational energy entering the middle ear than hits the eardrum. There is much less. The middle ear is an attenuator, attenuating strong signals more than weak. But it is a system of levers, which amplifies the sound pressure as much as 60 times. The three little bones of the middle ear, called ossicles, are so arranged that a vibration of the eardrum is coupled to the hammer (malleus), then to the anvil (incus), and finally to the stirrup (stapes). This is attached to the membrane. The mechanical advantage of this lever system is about 3. So the oscillating force the membrane applies to the liquid in the inner ear is 3 times the oscillating force on the eardrum. The area of the membrane, however, is only one-twentieth that of the eardrum, so the oscillating pressure, force per unit area, is multiplied by a factor $3 \times 20 = 60$. These numbers are typical; they vary as much as a factor 2 among individuals.

Although the sound pressure is magnified by such a large factor in the middle ear, the term *mechanical amplifier* may seem misleading. Perhaps we should call the middle ear a mechanical transformer. For weak sounds, it serves to get as much of the sound energy into the inner ear as possible. The organs of the inner ear are suspended in liquid. If the sound had to pass from air to liquid directly, most of it would be reflected. (How well do you hear speech with your head under water?) The middle ear "matches the impedances" of the air outside and the liquid inside, so less sound is reflected back and more gets in. (See Fig. 10.20.)

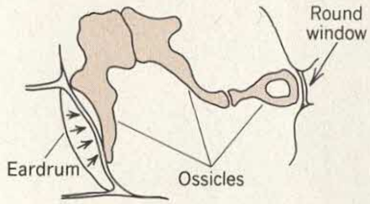


FIGURE 10.20 The three bones of the middle ear: hammer, anvil, and stirrup, showing how motion of the eardrum is amplified.

10.8 SOUND AND HEAT

Heating something does not appear to make it move around. Nevertheless you know that its atoms are in more violent random motion than before. We say that its internal degrees of freedom are more highly excited. There is more vibration. Heat is disorganized sound.

You will come across the term *thermal noise*. That refers to noise due *only* to this random motion, to the existence of a temperature. The child screaming next door, the pneumatic hammer digging up the street, they are noise too. Unwanted sound. It may seem disorganized to you. It is not. It is far

SIGNAL-TO-NOISE RATIO FOR THE EAR

Here we calculate the thermal noise power in a band 7 Hz wide. Why 7 Hz? That is just an estimate of the best we can do in distinguishing frequencies around 1000 Hz, roughly in the middle of the auditory range. It corresponds to 6 cents sharp or flat. A cent is 1/100 of a semitone. This is all very rough. Perhaps we can do 10 times better.

The formula is $kT\,\Delta\nu$, with $\Delta\nu = 7$ Hz. The absolute temperature of the ear is $273 + 37 = 310$ K. The Boltzmann constant k (Sect. 6.4) is 1.4×10^{-23} joules/Kelvin. Multiply it out:

$$\begin{aligned} kT\,\Delta\nu &= 1.4 \times 10^{-23}\,\frac{\text{joules}}{\text{K}} \times 310\ \text{K} \times 7\ \text{Hz} \\ &= 3.5 \times 10^{-20}\ \text{watts} \end{aligned}$$

That is the thermal noise power. In Sect. 10.5 we gave the threshold intensity as 10^{-18} watts per square millimeter of eardrum. If the eardrum has an area of 10^2 mm^2, that is a signal power of 10^{-16} watts. It is 35 db above the 3.5×10^{-20} watts we just calculated for the thermal noise power.

EXERCISE 18 Verify. Then repeat the calculation for a bandwidth of 1 Hz.

from random. In thermal noise all frequencies are equally represented, statistically, and so are all phases. That is why it is called *white noise,* by analogy to light: White light is a mixture of all colors, all light-wave frequencies. White noise sounds like "sh," a little bit like a waterfall.

Signal-to-Noise Ratio

To be distinctly heard, a sound signal has to be strong enough to stand out well above this thermal noise. Otherwise it will be drowned out and will not be intelligible. Let's say that more quantitatively. A gas molecule has an average kinetic energy of $\frac{3}{2}kT$ when the gas is at temperature T (Sect. 6.4). That is $\frac{1}{2}kT$ along each spatial coordinate x, y, and z. A sound signal traveling in the x-direction has to impart to the average gas molecule an additional kinetic energy comparable to $\frac{1}{2}kT$ in order not be drowned out by thermal noise.

It is more conventional to talk about the noise power in a given band of frequencies. The thermal noise power in a band from frequency ν to $\nu + \Delta\nu$ is equal to $kT\,\Delta\nu$. (This theorem is stated here without proof.) To be able to tell the signal from the noise, we want a signal-to-noise ratio greater than 1. This means that the signal power in that frequency band has to be greater than $kT\,\Delta\nu$. The box "signal-to-noise ratio for the ear" gives a sample calculation. It shows that our threshold of hearing is perhaps 35 db higher than the lower limit set by thermal noise. If our ears were much more sensitive, we would not hear a lot better. We would start to hear the thermal motion of the atoms our ears are made of.

SUMMARY

What vibrations are audible

What sounds have a pitch

Intervals; definition of the octave

Tone color is vibration pattern

Conditions for oscillation

Sound wave is a pressure wave; condensations and rarefactions

Speed of sound

What travels is a pattern: $f(x - vt)$

Scalar and vector waves, longitudinal and transverse

Waves in liquids and solids

When a wave has a wavelength

Relation between frequency and wavelength

Definition of simple-harmonic motion, sinusoidal displacement, Hooke's Law

Superposition of running waves to make a standing wave

Fundamental and higher harmonics

Bandwidth theorem

Fourier decomposition

Beats between tones whose frequencies are close

Beats between harmonics generated in the ear

Threshold of hearing

Dynamic range

Logarithmic scale of sound level: db

Derivation of the $1/r^2$ – law; when it fails

Long waves bend around obstacles, spread out from openings

Echo location requires short waves

Binaural location for low frequencies involves phase difference

Conversion of mechanical energy to electrical in the inner ear

The middle ear as an impedance-matching transformer

Thermal noise is white

Noise-imposed limit to sensitivity of the ear

KEY TERMS

Hertz (abbreviation: Hz) Cycles per second, or vibrations per second. The international unit of frequency.

Auditory range The frequencies of sound we can hear; roughly from 15 Hz to 20 KHz.

KHz Kilohertz = 10^3 Hz.

MHz Megahertz = 10^6 Hz.

GHz Gigahertz = 10^9 Hz.

Sonic boom The shock wave generated by an object "crossing the sound barrier," that is, moving faster than the speed of sound. The noise heard on the ground below supersonic jet airplanes.

Pitch How "high" or "low" a musical tone sounds. Nonmusical sounds can have an approximate pitch. "White" noise has no pitch. Each musical pitch has a unique frequency.

Pitch standard The A above middle-C has a frequency of 440 Hz, by international agreement.

Octave Musical interval between two pitches whose frequencies are in the ratio 2:1.

Fifth Interval corresponding to frequency ratio 3:2.

Relative pitch The interval between two musical tones. Also, the ability to recognize such intervals.

Absolute pitch The ability to recognize a pitch and name the note to which it corresponds. Long-term pitch memory.

Timbre Tone color, or tone quality. Depends on the wave shape or harmonic content.

Linear system The Principle of Superposition holds: If you double the amplitude of the input, that doubles the amplitude of the output.

Speed of sound in air 331.45 m/sec at 0°C, increasing 0.59 m/sec for each °C temperature increase.

Speed of sound in water 1498 m/sec at 25°C, increasing 2.4 meters per second per degree C temperature increase.

Speed of sound in seawater 1531 m/sec.

Elasticity The ability of a material to return to its original shape after a small deformation. The more a medium resists deformation, that is, the larger the *elastic moduli*, the faster sound travels in it.

Wave A pattern changing in space and time in such a way that the same spatial pattern appears at a later time displaced in space. The displacement, as well as possible changes in amplitude and shape of the pattern, are continuous functions of the time.

Phase velocity The speed at which the pattern is displaced. Usually just called wave speed, the term *phase* distinguishes this speed from the *group* velocity and the *energy* velocity (not discussed in the chapter), which may be different in dispersive media.

Group velocity The speed of propagation of a wave packet. Mathematical definition: $v_{\text{group}} = d\omega/dk$, where $\omega = 2\pi\nu$ and $k = 2\pi/\lambda$.

Energy velocity The speed at which energy is propagated in a wave. Mathematical definition: v_{energy} = energy flux ÷ energy density.

Dispersive medium A material in which the speed of sound depends on the frequency.

Condensation Region of increased density in a sound wave.

Rarefaction Region of decreased density in a sound wave.

Shear wave A wave of bending or flexing, a flexural wave, a transverse elastic wave.

Longitudinal wave A wave in which the displacement is parallel to the direction of travel of the wave.

Transverse wave A wave in which the displacement is perpendicular to the direction of travel of the wave.

Wavelength The distance in which the wave pattern repeats. The distance the wave travels in one period. Only a periodic wave has a wavelength.

Simple-harmonic oscillation Motion in which the acceleration toward equilibrium is always proportional to the displacement from equilibrium. Sinusoidal oscillation.

Phase Two sinusoidal waves having the same frequency are said to be in phase if their crests and troughs coincide. If crests of one are coincident with troughs of the other, they are said to be 180° ($= \pi$ radians) out of phase.

Fundamental The lowest natural frequency at which an oscillator performs simple harmonic motion.

Harmonics (also "higher harmonics") Integer multiples of a fundamental frequency, corresponding to standing waves.

Standing wave Superposition of sinusoidal waves traveling forward and backward, characterized by stationary nodes. All parts of a standing wave are in phase. Mathematical definition: Solution of the wave equation (not discussed in the chapter) represented as a product of a function of space times a function of time. In one dimension, the standing-wave frequencies are integer multiples of the fundamental.

Node A point on a standing wave where the amplitude is zero. Distinguish the displacement node and pressure node.

Antinode A point on a standing wave where the amplitude is maximum. Halfway between two nodes, in a homogeneous medium.

Resonance frequency One of a set of special frequencies at which standing waves are possible. A frequency at which a system can absorb and store energy much better than at neighboring frequencies.

Bandwidth The measure of how sharply tuned a resonance is. The range of frequencies over which a structure responds to harmonic (sinusoidal) excitation.

Bandwidth theorem An oscillator with a sharply tuned resonance displays a long damping time for free oscillation at its resonance frequency. The damping time (length of the wave train) is roughly equal to the reciprocal of the bandwidth.

Pure tone A sinusoidal sound wave. Approximated by the flute in its lower register.

Fourier's theorem Any periodic signal can be considered a superposition of sinusoids with frequencies that are integer multiples of the fundamental frequency.

Fourier analysis Also Fourier decomposition, or harmonic decomposition. Finding the recipe, the relative strength of each harmonic.

Beat frequency The difference between the frequencies of two pure tones sounded together.

Sound intensity Sound power per unit area, usually measured in watts per square meter. Not to be confused with loudness, which is a subjective property of a sound: Two listeners may disagree on the relative loudness of two sounds.

Decibel A logarithmic measure of the relative intensity of two signals. Sound level (in db) = 10 log (intensity) = 20 log (sound pressure). The reference level (0 db) is a sound pressure amplitude of 2×10^{-5} newtons/m^2. The popular myth is that 1 db is the smallest discernible difference for most people's ears.

Sone Modern unit of loudness, defined so that the loudness of two simultaneous sounds measured in sones are approximately additive.

Attenuation Loss in sound intensity. This can be due to spreading or to dissipation, that is, absorption and scattering of sound.

Diffraction The bending of waves around obstacles. One cause for the fuzzy edge of shadows.

Binaural Using both ears.

Transducer A device for converting electric signals into acoustic signals or vice versa. A loudspeaker or microphone. Also called electromechanical or electroacoustic transducer.

Malleus hammer **Incus** anvil **Stapes** stirrup	The three bones of the middle ear, which couple the ear drum (tympanic membrane) to the cochlear window, the membrane covering the inner ear.

Signal-to-noise ratio Ratio of signal sound power in a given frequency band to the noise power in the same band. When this ratio drops much below unity, the signal becomes unintelligible.

Acoustic impedance The ratio of the pressure amplitude to the velocity amplitude of a sound wave. Given by the formula ρc_s, where ρ is the density of the medium and c_s is the speed of sound in it.

REVIEW QUESTIONS

- The text says, "Hard surfaces reflect sound; soft surfaces absorb it." Draw on your own experience to defend this statement.
- Most scales go up an octave. Define *octave*. Define *unison*.
- "See" is a fifth higher than "mice." "All" is a fourth higher than "See." Therefore, "All" is ______________ (what interval?) higher than "mice." (HINT: $(\frac{3}{2}) \times (\frac{4}{3}) = ?$)
- When you play a record or tape of a human voice, you have a volume-control knob on your amplifier. How can you tell whether the singer was singing loud or softly?
- What is meant by phase lag?
- Water waves of long wavelength travel faster than short ones. This is not true of sound waves. Their speed is independent of wavelength. Give evidence for this last statement.
- Show that a function $f(x + vt)$ represents a wave traveling in the negative x-direction at speed v.
- Suppose a function $f(x - vt)$ is periodic, so that when x is increased by an amount λ, the function is unchanged. By how much must t be changed (increased or

decreased) in order to arrive at the same value of the function? Give your answer in terms of v and λ. What would you call the quantity v/λ?

- What is a rarefaction? Why is it usually preceded or followed by a condensation?
- Give an example of a longitudinal running wave giving rise to (transmitting its energy to) a transverse standing wave.
- Define simple-harmonic motion.
- Define linear spring. Why does a mass oscillating on a linear spring execute simple-harmonic motion?
- If the amplitude of a simple-harmonic oscillator is doubled, the frequency is multiplied by ________.
- To double the frequency of a simple-harmonic oscillator, the spring constant must be multiplied by ________.
- When a violin string has a node at its midpoint so that it vibrates in two parts, the frequency is ________ times the fundamental, an interval of a(n) ________ above it.
- An oscillator with a very sharply tuned resonance frequency has more / less damping than one with a broad resonance peak.
- If you push a child on a playground swing at the natural frequency of the swing, you can feed energy into the oscillation, that is, increase its amplitude. Is this true for very weak pushes? Explain.
- There are stories of a certain note played on the violin being able to shatter a thin wine glass. What phenomenon are these based on? Could they be true? (See p. 267)
- An A supposedly tuned to 440 Hz sounds for only 2 seconds. How precisely can its frequency be measured? Give answer in Hz and in percent.
- To a tone consisting of a fundamental frequency plus second harmonic is added some third harmonic. This changes its: pitch / timbre.
- A flute in its middle register has a rather sinusoidal / spiky pattern, quite rich / lacking in higher harmonics.
- The closer two musical tones are in frequency, the faster / slower the beats between them.
- If one sound has twice the intensity (sound power per unit area) of another, we say that its level is 3 db / 6 db higher.
- If one audioscillator puts out twice as much power as another at the same frequency, at what relative distances will they sound equally loud? (HINT: The answer is *not* 2 : 1.)
- When short waves go through a narrow hole, they spread out more / less than long waves.
- Low-frequency sounds coming from the left are heard only by the left ear / by both ears.

- The inner ear, which contains the cochlea, is filled with: bone / liquid / air.
- The sound pressure going into the inner ear is greater than that incident on the ear drum. The sound power going into it is also greater / less.
- Explain the function of the middle ear in the mechanism of hearing.

☐ QUESTIONS FOR FURTHER THOUGHT

α. The pressure amplitude of the heart beat is on the order of one-tenth of an atmosphere, or 10^4 newtons/m^2. The clinician would talk about the differences between systolic and diastolic blood pressure. Why don't we hear it? (HINT: There is more than one reason.) After strenuous exercise we do sometimes hear the pulse. What is going on? When you hold a large seashell to your ear, you hear a faint roaring noise (Fig. 10.21). Some people naively talk about hearing the sound of the sea. What are you really hearing?

FIGURE 10.21 Does a seashell allow you to hear the sound of the sea?

β. The pressure of the air on the eardrum is due to the random impacts of the air molecules. It therefore has statistical fluctuations. Why don't we hear this fluctuating pressure as noise? Would we, if the ear were more sensitive?

γ. An expensive electronic sound system has the properties of (1) flat response over a wide spectrum of frequencies and (2) linearity over a wide dynamic range. Discuss the consequences of too cheap a system for reproducing music and speech: (i) Peaks in the frequency response, with high and low frequencies cut off; (ii) nonlinear response, with loud sounds not as well reproduced as soft sounds.

δ. The telephone has a bandwidth of only 2000 Hz or so. How well can you distinguish plosives (b's and p's) and sibilants (s's and sh's)?

ε. A certain nonsinusoidal sound wave consisting of several Fourier components travels in a dispersive medium: The low-frequency components travel faster than the highs. Discuss the difference between the emitted and received sounds. Distortion?

ζ. A puncture in the eardrum (tympanic membrane) results in a higher threshold for low frequencies, little change in the ability to hear high frequencies. Is this what you would have predicted? Try an explanation.

η. Directional microphones are usually quite large. Small microphones are nondirectional, that is, equally sensitive to sound coming from all directions. Explain. When would you want a directional microphone?

θ. When we double the frequency of a tone and then we double it again, we say that the two pitch intervals are the same. Each is an octave, and we feel that they are the same. The scale of pitches is logarithmic. It is therefore tempting to think that we respond to sound intensity in the same way: If we double the intensity of a sound, and then we double it again, does the ear judge the loudness to have increased by the same amount both times? Psychologists used to think so. Can you see the "historical reason" why the decibel scale of sound level was adopted?

NOTE: Recent psychoacoustic measurements show that the sound level (decibel) and loudness scales do not correspond well. (See Problem 25, Loudness.)

☐ FOR THE LAB: Ripple Tank

If you don't have one and can't borrow one, a window will do fine—the larger the better. A 5-millimeter depth of water is good to start with. Later you will want to experiment with varying depths. What happens to the ripple speed in very shallow water? Leveling takes some skill!

The patterns are confusing unless you eliminate reflections from the sides. Bent strips of window screen draped with cheese cloth make absorbing beaches. You can make ripples with a finger tip (point source), with a ruler (line source), or with a little motor suspended from rubber bands

To "stop" the wave pattern, you need a stroboscope. The quick way is to wave your outstretched fingers back and forth in front of your eye. Investigate the relation between frequency and wavelength. Is this a good way to measure the speed of the ripples?

Try barriers of various sizes for diffraction experiments. Cakes of canning wax have flat edges, so they stand up. They also cut easily.

Emptying the water is harder than you think. An attempt to carry across the room is bound to result in a wet floor. A siphon into a pail is one way. If you leave tap water in overnight, some water will evaporate and hard water will leave a film on the glass.

☐ FOR THE LAB: Oscilloscope

Connect a microphone to the "Vertical Input" of a cathode-ray oscilloscope and watch the trace as you sing or play into the microphone. Compare different vowels sung on the same note, male and female voices, whistling, and so on. If you want the trace to stand still, you will need to adjust the "sweep" control until the sweep frequency matches the sound frequency, to see one vibration period. If the sweep frequency is one-half the sound frequency, you see two periods; if one-third, you see three, and so on.

PROBLEMS

✶ **1. Frequency and Wavelength** In 1 millisecond, sound waves travel 0.33 meters. In 1 millisecond, sound waves of frequency 1000 Hz travel one wavelength. Verify the relation between frequency and wavelength for these waves.

✶ **2. Speed of Sound** On a hot day, the wavelength of a 1000-Hz sound wave is measured to be 0.350 meters.

(a) What is the speed of sound?

(b) Why is the wavelength longer than on a cold day?

✶✶ **3. Sound Velocity and Temperature** At 273 K, the speed of sound in air is 331 m/sec; at 300 K, it is 347 m/sec. Repeat: A 10% increase in absolute temperature gives a 5% increase in sound speed.

(a) Is sound speed proportional to temperature?

(b) What relation between sound speed and temperature would explain the data? Sketch a rough graph.

✶ **4. Period of Sound Waves** For A-440, what is the time between the passage of a condensation and the succeeding rarefaction? How far apart are they as they travel?

_______ meters

✶ **5. Wavelength in . . .** Calculate the wavelength of a 440-Hz sound wave

(a) In air
(b) In water
(c) In steel. Do you need to specify the temperature?

* **6.** A 500-Hz tone has a period of 1/500 of a second. That is a consequence of the definitions of *frequency* and *period*.
(a) Traveling in air at 330 m/sec, how far does the wave travel in one period?
(b) Argue that this distance is the wavelength.
(c) If the frequency were doubled, the wavelength would be multiplied times ________. This assumes that the speed of sound is independent of frequency, which is a very good approximation over many octaves.

* **7. Wavelength in Air and Water**
(a) If the wavelength of a sound signal is 1 cm in air, what is it in water?
(b) What is its frequency?

* **8. Wavelength in Air and Water**
(a) If the wavelength of a sound signal is 1 cm in water, what is it in air?
(b) What is its frequency?

** **9. Trick Question** If the frequency of a sound signal is 10 kHz in air, what is it in water?

** **10. Phonograph Record** A turntable frequency of 33⅓ revolutions per minute is now standard for "classical" discs. Most of these discs are 12 inches in diameter. How far apart are the crests and troughs in the record groove for a 1000-Hz tone
(a) At the edge of the record?
(b) At the end of the record, only 2 inches from the center?

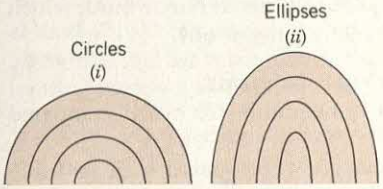
** **11. Effect of Temperature on Sound Speed** Data on the speed of sound in seawater are given at two temperatures:

t	c_s
5°C	1461.0 m/sec
20°C	1513.2 m/sec

(a) Find the slope of the straight line drawn through these two points, and express it in meters per second per degree Celsius.
(b) The display of an echo locator is calibrated at 20°C. What it measures is the time between the signal and the echo. What it reads out is the distance of the reflecting object from the instrument. How many percent off is it at 37°C? What assumption did you have to make to answer this question?

NOTE: The jargon word is *extrapolation*. 37°C is the temperature of the human body. Body fluids have ion concentrations roughly comparable with seawater.

** **12. Refraction of Sound** With the air at uniform temperature, a wavefront from a small source is an expanding sphere. Figure (*i*) shows spherical wavefronts at intervals of 10 milliseconds. The source is pictured close to the ground.
(a) Verify that successive wavefronts should be 3.3 meters apart. Radial lines might be called "sound rays" by analogy with light rays. They intersect the wavefronts at right angles.
(b) If the temperature of the air increases with height, as, for example, over a cool lake, sound travels faster the higher it gets. The wavefronts will look like Fig. (*ii*). Sketch in the rays intersecting these wavefronts normally and note that they curve down. Have you ever noticed how much better you can hear a distant conversation across a calm lake?



PROBLEM 12. Have you noticed how much better you can hear distant sounds across a calm lake?

** **13. Even-Tempered Scale** A musical scale constructed out of intervals defined in terms of ratios of small integers is suitable for simple tunes. (See the box "Three Blind Mice.") It fails when we want to modulate into a different key. For this reason "fixed-pitch" instruments like pianos are tuned in "equal temperament": The octave is divided into 12 equal *semitones*. The frequency ratio r corresponding to such a semitone has to be such that 12 of them make an octave. An octave is a frequency ratio of 2:1. Therefore,

$$r^{12} = 2 \qquad r = 2^{1/12}$$

Repeat: A semitone corresponds to a frequency ratio that is the twelfth root of 2, or 1.05946.
(a) A fifth is equal to seven semitones. An equally tempered fifth then corresponds to a frequency ratio $r^5 = 2^{7/12}$ = ________. How far off is this from the "perfect" fifth we defined as 3:2? Express your answer in percent.
(b) The E that is a fifth above A-440 would have a frequency on the equally tempered scale of 659.3Hz. Verify.
(c) How fast would this frequency beat with the frequency exactly 3/2 times 440 Hz? Do you think you could feel a

beat that slow? If your answer is yes, then you could tell the difference between the two tunings. Piano tuners can.

✶✶ **14. Major Third** Using the definition of Problem 13, compare the major third (frequency ratio 5 : 4) with the equally tempered third (4 semitones).

✶✶ **15. Minor Third** Same as Problem 14: Test the minor third (frequency ratio 6 : 5) against the equally tempered minor third (3 semitones).

✶✶✶ **16. Violin Tuning** (Compare Problem 15) Violin strings are tuned in fifths: E, A, D, and G, going from high to low. Suppose that the A is tuned with the piano and the fifths are "perfect," that is, have frequency ratios 3:2.

(a) What is now the frequency ratio between the G two octaves above the G string and the open E string?

(b) You have been taught (perhaps) that from G down to E is a minor third; so you expected a frequency ratio of 6:5. What you calculated is much smaller. How many percent smaller?

(c) Compare with the equally tempered minor third, which comes out 1.1892. How many percent off?

✶ **17. Helium and Voice Formants**

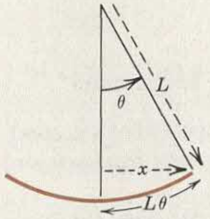
(a) The speed of sound in helium is 970 m/sec, compared with 330 m/sec in air.

(b) Helium has atomic weight 4, compared with air's molecular weight of 29. Why would you expect the speed of sound to decrease with increasing molecular weight?

(c) If a person inhales helium* and then talks, his voice is unnaturally high. The natural resonances of the cavities that "form" the voice are all raised in frequency. The resonant wavelengths are all the same, but the sound speed is almost 3 times as great with helium. By what factor are the frequencies increased? Is this more than an octave?

✶✶ **18. Pendulum and Simple Harmonic Motion** We talked about a playground swing as an oscillator.

(a) Show that a simple pendulum oscillating with a small amplitude is simple harmonic. (HINTS: Here is one way. Remember, you want to show that the restoring force is proportional to the displacement. The displacement is along the arc of a circle of radius L (= string length). When the string makes an angle θ with the vertical, the mass point is displaced through an arc of length $L\theta$. The restoring force is the tangential component of the weight, or $mg \sin \theta$. You may object that $\sin \theta$ is not exactly proportional to the displacement, but $\sin \theta \cong \theta$ is a good approximation for small displacements.)



PROBLEM 18.

(b) Use the argument in the box "Simple Harmonic Motion" to derive the formula for the period of the simple pendulum, $2\pi\,(L/g)^{1/2}$.

✶ **19. Standing Waves** A tube 0.66 m long supports standing waves of 1000-Hz frequency.

(a) How many wavelengths long is the tube?

(b) If there are nodes at the two ends, how many antinodes are there?

NOTE: For a tube closed at both ends, there are displacement nodes and velocity nodes at the ends, but pressure antinodes. For a tube open at both ends, it is vice versa.

(c) At what lower frequencies could this tube resonate (support standing waves)?

20. Beats

✶(a) The piano has two or three strings for each note, which sound together. Suppose that one A-string is tuned to 220 Hz, another to 223 Hz. How many beats per second will you hear?

✶✶(b) The B♭ above it is supposed to be tuned to 233.1 Hz. What fraction of a semitone out of tune was that 223-Hz string?

HINT: Semitones are conventionally divided into 100 *cents*. Our string would be called 23 cents sharp relative to the A.

✶ **21. Decibels** Twenty violins playing together produce 20 times the sound intensity of one violin. How many decibels higher is the sound level in the auditorium?

NOTE: Orchestra sound levels range from 20 db (very soft) to 120 db (painfully loud) above the reference level. The conventional reference level corresponds to a pressure amplitude of 2.82×10^{-5} newtons/m^2, which is approximately the threshold of hearing near 1000 Hz.

✶✶ **22. Inverse-Square Law** When the helicopter is 100 feet high directly overhead, the sound level is 110 db

---

*_**Not**_ without danger!

above the reference level. When it is at the same height 100 feet further south, what is the sound level here?

23. Audiometry An audiometer is a medical instrument used to test the patient's hearing. Tones of various frequencies are presented through earphones in pulses at decreasing levels. The patient signals when he can no longer hear the tone.

∗(a) A patient is retested after five years. His hearing threshold at 1000 Hz is the same as last time. His threshold at 16,000 Hz is 10 db higher. For what age patient would this amount of hearing loss be perfectly normal?

∗∗∗(b) An audiometer can also be used to test bone conduction of sound to the inner ear. A patient has no hearing loss for bone conduction, but a 30 db higher threshold for sound coming normally through the air. What part of the ear might be impaired? Explain.

∗∗ **24. Vibration of the Eardrum** For soft sounds, the amplitude of vibration of the eardrum is tiny. Estimate its magnitude for a sound wave whose pressure amplitude is 2.82×10^{-5} newtons/m^2 at a frequency of 1000 Hz. That is often taken as the reference level we have called 0 db, and is roughly the threshold of hearing. Compare Problem 21. This standard is not universal. Use the formula (see Problem 45 "Ultrasonic Cleaner")

$$p_{\max} = v_{\max}\, \rho c_s$$

to calculate the velocity amplitude. Then use the equations in the box "Simple-Harmonic Motion" to calculate the vibration amplitude A. The assumption is that the velocity of the eardrum is the same as the velocity of the air next to it.

HINT: ρc_s for air is 428 kg/m^2 sec. Air has molecular weight 29. This means that 29 grams of it occupy 22.4 liters at 273 K. The speed of sound is 330 m/sec. How did we get that acoustic impedance of 428?

∗∗ **25. Loudness** The sone, a unit of loudness, has been defined by the acoustical engineers such that an intensity 40 db above the reference level has a loudness of 1 sone. An empirical formula relating loudness and intensity for sound levels over 40 db (loudness over 1 sone) is

$$\text{loudness} = \text{constant} \times (\text{intensity})^{0.3}$$

Show that this formula has the property that a 10 db increase in intensity doubles the loudness.

∗∗ **26. Sones** How many sones loud is a sound whose level is 100 db above the reference level?

∗∗ **27. Sonar** If you shout short syllables at a cliff and want to hear the faint echo, your repetition frequency should be slow enough so that you have not started your next shout before the echo comes back. Otherwise you drown out the echo.

(a) What is a suitable repetition frequency for the pulses of a ship's sonar looking for echoes from submarines within a 3-kilometer radius? The speed of sound in water is 1500 m/sec.

(b) If the water is shallow, does a distant submarine have a better chance of escaping detection? Explain.

∗∗ **28. Binaural Location** If your head is facing in the direction of the sound source, the two ears experience the same sound intensity as well as zero phase difference.

(a) How do you know whether the sound source is straight in front of you or straight in back?

(b) How can you tell if the sound source is directly above you?

∗ **29. Echo Location** Bats use ultrasonic sonar to locate their insect prey in the dark. Their brief chirp starts around 25,000 Hz and slides up an octave to 50,000 Hz. Show that the wavelengths in this signal are suitable for locating insects 6 to 12 millimeters in size.

∗∗∗ **30. Whispering Gallery** Sound usually spreads out, so that the farther away you are from the source, the weaker the sound. (Inverse-square law. . . .) But reflection from curved walls can focus sound. An *ellipse* has the property that a ray from one focus is reflected so as to pass through the other focus. (Guess why they call those two points foci.) An elliptical room is therefore the ideal whispering gallery. A conversation at one focus can be clearly heard at the other.

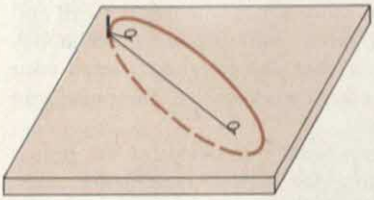
(a) Construct an ellipse (instructions below) and two or three rays from one focus. Remember equal-angle reflection, even though we haven't proved it yet.

(b) Where does a ray go if it suffers two reflections?

(c) Parabolic reflector: A parabola reflects any ray coming in parallel to the axis so that it goes through the *focus*. Describe how a parabolic reflector could be used to eavesdrop on a distant conversation.

(d) The "whispering gallery" principle is used in a stethoscope. Conjecture why tygon (plastic) tubing made a better stethoscope than rubber.

Construction of an ellipse, using a loop of string and two pins: Stick the pins where you want the foci. Slip the string over the pins. Keep it tight with the point of your pencil. Move the pencil to trace out the "locus of points



PROBLEM 30.
Construction of an ellipse.

the sum of whose distances from two fixed points is constant."

✶ **31. Diffraction** The directionality pattern of a loudspeaker is mathematically quite complicated. But if the size of the diaphragm is more than a few wavelengths, a rough rule of thumb can be formulated: The angular spread of the sound beam, measured in radians, is approximately equal to the ratio of the wavelength to the diaphragm diameter. (Chapter 12 gives the underlying reasoning, under "Diffraction from a Single Slit.")
(a) What is the angular width (in degrees) of the sound beam from a sonar hydrophone 0.75 m in diameter emitting ultrasound at 5×10^4 Hz? The speed of sound in water is 1500 m/sec.
(b) Can you use the rule of thumb for a 0.12-m loudspeaker emitting 440-Hz sound in air? What kind of directionality pattern do you expect?

✶ **32. Ultrasonic Splinter Location** An ultrasonic probe is designed to locate foreign objects in the eye. As a rough approximation, assume the speed of sound in the various layers of the eye to be uniform and equal to that in water, 1500 meters per second.
(a) Calculate the time between transmission of a sound pulse and reception of the echo from an object 1 cm below the surface of the eye.
(b) It may be desirable for the signal transmission to be over when the echo comes back. If the frequency of the ultrasound is 1 MHz (= 10^6 Hz), how many cycles long can a pulse be?
(c) Diffraction effects make it difficult to localize an echo if the reflecting object's size is less than one wavelength. What is the smallest sliver the 1 MHz wave can "see" clearly?

✶✶✶ **33. Power in a Sound Wave** A sound wave transmits energy. The rate at which energy is transported is the power in the wave. The power per unit area of wavefront is given by

$$\frac{\text{power}}{\text{area}} = \frac{\text{force}}{\text{area}} \times \frac{\text{displacement}}{\text{time}} = pv$$

the product of the sound pressure times the streaming velocity of the medium, that is, the air. These are "instantaneous" values. In a periodic wave, both p and v oscillate about the value zero. Pressure goes above and below ambient; velocity goes forward and back.

Pressure and streaming velocity are pretty well in phase in a running wave. Indeed they are proportional: $p = (\rho c_s)v$. The proportionality constant ρc_s is called the *acoustic impedance* of the transmitting medium.
(a) When the streaming velocity of the air at a condensation is $v_{\max} = 0.01$ m/sec, the sound pressure is $p_{\max} = 4.27$ newtons/m^2. (HINT: $\rho c_s = 427$ kg/m^2 sec.) The instantaneous energy flux is 0.0427 watts/m^2. Verify.

NOTE: This is a loud sound.

(b) One-quarter of a period later, the sound pressure and the streaming velocity are both back to zero. At this instant the power is also zero. But half a period later, both pressure and velocity are again maximum, this time negative, as the rarefaction passes. For a sinusoidal wave, the average energy flux is half the maximum value. [HINT: $(\sin^2 \omega t)_{\text{average}} = \frac{1}{2}$.] Verify the statement made in the text that a pressure amplitude of 3×10^{-5} newtons/m^2 (= threshold of hearing) corresponds to a sound intensity of 10^{-12} watts/m^2. How far off is it? ________
(c) Verify that the sound level in part (a) is 103 db above that in part (b).

✶✶✶ **34. Standing Wave Has No Power** A standing wave has energy, but it transports none. Referring to Problem 33, we would say that the sound pressure and the streaming velocity are one-quarter period out of phase: When the pressure is maximum, the velocity is zero, and vice versa. Present an argument to show that the average power in the standing wave is zero. Note that the instantaneous flux alternates between positive and negative values.

✶✶ **35. Power Out of a Trumpet**
(a) The pressure in a trumpet player's mouth might be one-twentieth of an atmosphere (5×10^3 newtons/m^2) above ambient. If he blows air into his instrument at an average rate of 20 cm^3/sec, you can calculate the rate at which his blowing does work, just by multiplying. (Work = excess pressure × volume) (HINT: Convert the volume per unit time into m^3/sec. Your answer should come out in watts.)
(b) The sound level 1 meter away is perhaps 90 db above the threshold of hearing (reference level), giving an intensity (sound power per unit area) of 10^{-3} watts/m^2. You

know the area of a sphere 1 meter in radius. How much sound power does the trumpet produce?

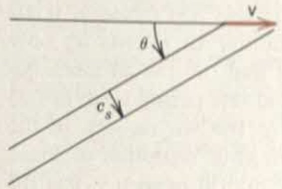
(c) What fraction of the work done by the player goes into sound? That fraction is called the efficiency of his instrument. What happens to the energy that does not go into sound?

★★★ **36. Sonic Boom** When an airplane "crosses the sound barrier"—starts to go faster than the speed of sound—it sets up a shock wave heard on the ground. We call it "sonic boom." You can understand its origin if you think about the airplane compressing the air it moves into. At subsonic speeds, that pressure wave can move ahead of the craft. At supersonic speeds, the craft catches up and overtakes the pressure wave it generates. The shock front spreads out in the shape of a cone. (See photo.)

We now calculate the angle of that cone. Suppose that the aircraft is moving at 400 m/sec. The wavefront advances at the speed of sound in air, 330 m/sec. In one second, the front advances 330 meters, while the craft moves 400 meters. On the sketch, θ is the half-angle of the cone. From the geometry we get

$$\sin\theta = \tfrac{330}{400}, \qquad \text{giving } \theta = 56°$$

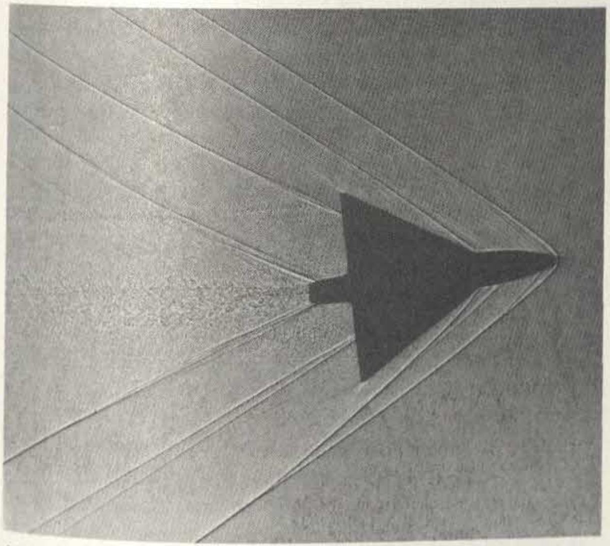
(a) Derive a formula for the halfangle θ as a function of aircraft speed v and sound speed c_s.



PROBLEM 36.

(b) What is the half-angle of the cone for Mach 2, an aircraft speed twice the speed of sound?

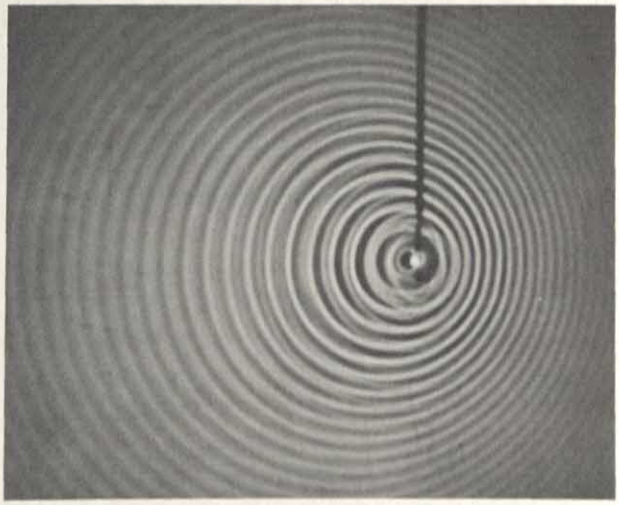
(c) For level flight at speed $v > c_s$ at what speed does the sonic boom advance across the level ground below (see figure)? Is there something wrong with a "sound" signal advancing faster than the speed of sound? (HINT: Is the signal generated on the ground?)



Airplane breaking the sound barrier. The photograph uses a technique called Schlieren photography to "see" large-amplitude sound waves in air.

★★ **37. Doppler Effect** If a source of sound is moving toward the listener, the received frequency is higher than the vibration frequency of the source. You can hear this effect when a car approaches you with its horn blowing; as soon as it passes you and starts receding, the pitch of the horn is lowered. The approaching source is chasing the wave it emits so that the wavelength of the sound is shorter than that of the wave emitted by a stationary source vibrating at the same frequency.

(a) Consider a horn vibrating at 500 Hz in a car approaching at 20 m/sec (= 45 miles per hour). In one period (= 2



Problems 37 and 38 A moving point source of waves in the ripple tank demonstrates the Doppler effect: The wavelength is shortened in the forward direction and lengthened in the backward direction.

milliseconds) the sound wave travels 330 m/sec × 0.002 sec = 0.66 m toward you, while the car catches up only 20 m/sec × 0.002 = 0.04 m of that. So the distance between points on the wave emitted one period apart is 0.66 m − 0.04 m = 0.62 m. That is the wavelength of the sound in air. The time between your reception of those two points of the wave is 0.62 m/(330 m/sec) = 0.00188 sec, corresponding to a frequency of ______ Hz. You hear a 532-Hz sound, although a 500-Hz vibration produced it.
(b) What fraction (percent) is the car's speed (20 m/sec) of the speed of sound? By what fraction is the wavelength reduced by the motion?

★★★ **38. Doppler Effect—Source Approaching** In one period T, while the source emits one wavelength, the wave travels a distance c_sT. The source moves a distance vT. The wavelength $\lambda' = c_sT - vT$ is the distance between the wavefront emitted at the beginning of the period and the wavefront emitted at the end of the period of the emitted sound.
(a) Remembering that $c_s = \lambda'\nu'$ (Eq. 1), show that the received frequency ν' is given in terms of the emitted frequency $\nu = 1/T$ by

$$\nu' = \frac{c_s}{c_s - v}\,\nu$$

(b) If a car is approaching at 20 m/sec (= 45 miles/hr), show that the received frequency is 6% higher than the emitted frequency. This is approximately one semitone.
(c) If the source is stationary and the listener is approaching, there is also a Doppler shift, an increase in received frequency. In one period T' of the received signal, the listener travels a distance vT' and the wave travels a distance c_sT'. The sum of these is one wavelength:

$$vT' + c_sT' = \lambda$$

Calculate the received frequency $\nu' = 1/T'$ in terms of the emitted frequency $\nu = c_s/\lambda$.
(d) If the listener approaches the source at 20 m/sec, what is the percentage increase in the frequency he hears? How does this result differ from that of part (b)?

★★ **39. Ultrasonography** Taking pictures with ultrasound is called ultrasonography. It is the acoustic analogue of radiography, taking pictures with x-rays. Used on the brain, it is called echoencephalography. X-rays do not distinguish fat from muscle at all well. Sound waves do. Different kinds of soft tissue have different **acoustic impedance** $z = \rho c_s$. When a sound wave meets a boundary between two media of different acoustic impedance, the fraction of the sound energy reflected is

$$\left(\frac{z_1 - z_2}{z_1 + z_2}\right)^2$$

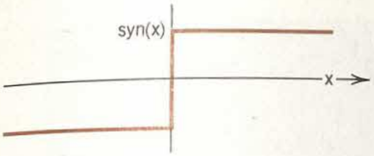
That fraction is called the reflection coefficient of the boundary.
(a) The acoustic impedance of muscle is 20% greater than that of fat. Calculate the reflection coefficient of the boundary. Is it the same going from muscle to fat as going from fat to muscle?
(b) The acoustic impedance of bone is 5 times that of fat. Calculate the reflection coefficient of the boundary.

★★★ **40. Ultrasonography and Doppler Effect** A beam of ultrasound is reflected from the heart of a fetus to "look at" its heart beat. The maximum speed of the chest wall during each pulse is 4 cm/sec.
(a) Find the percentage change in frequency due to the Doppler effect. Consider the medium to have the same sound speed as water, 1500 m/sec.
(b) Why use ultrasound rather than audible frequencies for this?

★★★ **41. Fourier Series (Suggest Using a Calculator)** A certain tone has only odd harmonics. The amplitude of the 3rd harmonic is $\frac{1}{3}$ that of the fundamental. The 5th harmonic is $\frac{1}{5}$ of the fundamental, and so on. Your task is to sketch a graph of the waveform.
HINT: $\sin \omega t + \frac{1}{3}\sin 3\,\omega t + \frac{1}{5}\sin 5\,\omega t + \frac{1}{7}\sin 7\,\omega t + \cdots$
You will not need to do many harmonics before you intuit the shape.

★★★ **42. Waves on the Highway** (Difficult) A row of cars is stopped at a traffic light. The light turns green, and the front car starts to move. The "wave of starting" travels down the row of cars. When the last car starts, the wave has passed down the whole row.
(a) Estimate the speed of this wave down a 20-car queue. Note that the wave goes backward while the cars are starting to go forward. Does the speed of the cars enter your calculation?
(b) After the line of cars has attained highway speed, the front car stops suddenly. Estimate the speed of the "wave of stopping." Is it faster or slower than the speed of the cars? If it is slower, what is the consequence?

★★ **43. Waves in One Dimension** The graph of a mathematical function $f(x)$ may be considered as repre-



PROBLEM 43. Step-function wave. Example: temperature profile of a heat wave.

senting a pattern in space. The variable x is position measured to the right along some "x-axis." That is, some direction in space has been designated the x-direction.

(a) Let t represent time and v represent a speed. Convince yourself that the graph of the function $f(x\text{-}vt)$ is the same pattern just moved to the right a distance vt.

(b) The step function sgn(x) is defined as being +1 for positive x and −1 for negative x. Describe how the graph of the function sgn(x-vt) behaves as t increases. How fast does the wavefront travel?

(c) Does it make sense to ask about the wavelength of this wave? The frequency? Explain.

✶✶ **44. Scalar Wave, Vector Wave** A wave that can be represented as a single function of space and time variables is called a scalar wave. For example, temperature is a function of position and time.

(a) Argue that a heat wave traveling from Chicago to Cleveland is a scalar wave.

(b) Is a water wave a scalar wave? Defend your answer. (HINT: The vertical displacement of the water surface from glassy equilibrium at each point specifies the wave pattern completely.)

(c) A violin string can oscillate both up-and-down and right-and-left. Evidently, it requires two functions to describe the most general wave on a string. We say that these waves have two *planes of polarization.* They are vector waves. How about the waves on a jump rope with a little girl on each end? Scalar or vector? Explain.

(d) When the little girls move the rope so that every point executes a circle, the wave is called *circularly polarized.* In the simplest such wave, all points are in phase and a third little girl might be jumping over the rope in the middle. Argue that this is a standing wave.

✶✶✶ **45. Ultrasonic Cleaner** An ultrasonic cleaner has a vibrating diaphragm in contact with the cleaning solution that transmits the ultrasound into the liquid. Let the diaphragm be in simple-harmonic motion at frequency $\omega/2\pi = 3 \times 10^4$ Hz and amplitude $A = 5 \times 10^{-6}$ meters = 5 microns. The liquid has the density of water, $\rho = 10^3$ kg/m^3.

(a) Calculate the maximum speed of the cleaning liquid due to the passage of the wave: (Eq. 4)

$$v_{\text{max}} = A\omega = \_\_\_\_\_\_\_\_ \frac{\text{m}}{\text{sec}}$$

Estimate how this compares with the speed of the water jet from a teeth-cleaning device like the Water-Pic.

(b) Calculate the intensity of the ultrasound, the rate of energy flow per unit area per unit time. Here is one way: A layer of area a and thickness Δx (small) is moving at the maximum speed $A\omega$. Its kinetic energy is

$$\tfrac{1}{2}\, a\, \Delta x\, \rho v_{\text{max}}^2$$

A time Δt later, the wave will have progressed a distance $c_s\, \Delta t$. If we take that equal to our layer thickness Δx, then we get the energy transported in time Δt. To get energy per unit time, divide by Δt. To get energy per unit time per unit area, divide by the area a:

$$\frac{\text{power}}{\text{area}} = \tfrac{1}{2}\,\frac{\Delta x}{\Delta t}\,\rho v_{\text{max}}^2 = \tfrac{1}{2}\, c_s\, \rho A^2\omega^2 = \_\_\_\_\_\_\_\_$$

Use $c_s = 1500$ m/sec. Give answer in watts per square meter.

(c) The quantity ρc_s is called the *acoustic impedance* of the medium. The maximum velocity v_{max} multiplied by the acoustic impedance gives the pressure amplitude of the wave, the maximum amount by which the pressure in the wave differs from the ambient pressure:

$$p_{\text{max}} = v_{\text{max}}\, \rho c_s = \_\_\_\_\_\_\_\_ \text{ newtons/m}^2$$

Note that this comes out greater than atmospheric pressure. The liquid will therefore *cavitate:* The rarefactions will be hollow, that is, devoid of liquid.

Rays of sunlight made visible by fog among the trees.

11

LIGHT AND VISION– RAY OPTICS

SOME QUESTIONS THIS CHAPTER WILL ANSWER

Why aren't cameras built to focus the way the eye focuses?

Describe how to find the focal length of a positive lens. How about a negative lens?

How is a ray diagram used to find the position of the image of an object formed by a lens? How is the lens equation used for this purpose?

A certain farsighted patient can't focus on anything closer than 1 meter. How strong a lens (how many diopters) would you prescribe to allow that patient to focus at 25 cm?

When you look at the coin in the fountain, why does it look shallower than it is?

11.1 LIGHT RAYS

When you open your eyes to look at something, is there a signal that travels from your eye to the object? Does "casting a glance" have an effect on the object being glanced at? We are so used to thinking about light traveling *from* the object *to* the eye that the question itself evokes thoughts of magic and witchcraft. How about the "evil eye"? Or is such a question out of place in a physics book?

The reason the question is not usually discussed in physics books is that no verifiable effect of the eye on the object has been discovered. Also, the mechanism of seeing can be explained in terms of a one-way trip of the light, emitted or reflected by the object and absorbed in the eye.

EXERCISE 1 How would you look for an effect of the eye on the object being looked at? Design an experiment that could, in principle, detect such an effect. Would it involve more than one observer?

Surveyer using a transit.

The light seems to go in a straight line too. The art of surveying the land, the making of star maps, in fact, seeing anything sharply, all depend on that straight-line propagation. We talk about **light rays.** Indeed we see "rays" when a bright beam of light shines through smoke or fog. (See photo on previous page.)

If you think about it, the fact of seeing those rays tells you that not all the light in the beam travels in straight lines. If it did, how could it get to your eye? You are watching from off to the side of the beam! Quite correctly, you argue that some of the light is scattered by the tiny smoke particles or water droplets.

In this chapter we shall be concerned with various phenomena that show light turning corners. **Reflection** and **scattering** of light are the most familiar.

Light rays breaking through the clouds (see Exercise 2).

Reflections in a window.

We are also familiar with the way light rays are bent by **lenses.** In the next section we turn our attention to the lens of the eye. That will lead into a discussion of **refraction** in general.

EXERCISE 2 Explain the "rays" you sometimes see below storm clouds when the sun is shining through them. Does your explanation involve holes in the clouds? (See photo.)

EXERCISE 3 We distinguish between "specular" reflection and "diffuse" scattering. Mirrors reflect; dust particles scatter. Smooth surfaces reflect; rough surfaces scatter.

Look around you and list at least 10 surfaces in which you see mirror reflections. A smooth floor reflecting a light bulb? A pond? A paper clip?

11.2 LENSES

When you look at a distant object, nearby objects in your field of view appear blurred. Try looking at a distant scene and holding a pencil point in the way. The pencil point is fuzzy as long as you keep your eye focused "at infinity." If you focus on the nearby pencil point, the distant scene will blur. Consciously or not, the lens of your eye **accommodates,** changes its focus. You know how you have to strain to keep the pencil point in focus if you bring it closer and closer. For most young people, about 25 centimeters is the distance of clearest vision. Anything much closer is out of focus. Older people tend to lose the power of accommodation, as the eye·lens loses its flexibility. (See Problem 12, "Bifocals.")

Play with a couple of magnifying lenses to get an intuition for their "focusing'' property. An ordinary magnifying glass is fine. So is a strong eyeglass lens worn by a far-sighted person without astigmatism. (See Exercise 9.) If the sun is shining, try to focus its image on a piece of white paper. With a big enough lens you can set the paper on fire! The stronger the lens, the closer the focal point. We call the distance between the lens and the focal point the **focal length** of the lens.

Experience will convince you that there is no relation between the size of a lens and its strength. The strength of a lens depends on its shape. The stronger lenses bulge more in the middle. Now we can describe how the eye accommodates: To bring a closer object into focus, the muscles of the eye pull the lens into a shape that is thicker in the middle and thinner at the edge.

Try focusing the image of a nearby object on a piece of paper using your magnifying lenses. It helps to have your object brightly illuminated. A small light bulb is ideal.[1] A paper clip in the window will do. You soon discover that the closer the object is to the lens, the further away the image: Object and image move in the same direction if the lens stays in one place (Fig. 11.1). If the object is too close, you can't focus an image at all. ''Too close'' means closer than the focal length. To project an image of a nearby object, you need a strong lens. That means a lens with a short focal length.

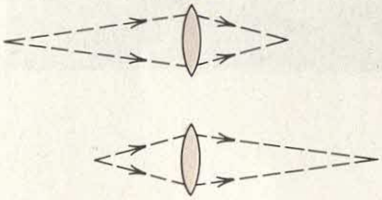


FIGURE 11.1 If the object comes closer, the image moves further from the lens: Object and image move in the same direction.

The Camera and the Eye

With your lens-and-white-paper you have a simple analogue of the lens-and-film of the camera, or of the lens-and-retina of the eye. In the eye, if the image of an object is not sharp, you change the power of the lens: If the image falls behind the **retina,** you accommodate and increase the lens power to put the image *on* the retina. If the image falls in front of the retina, your eye lens is too strong and you accommodate by relaxing those muscles to make it weaker. In the healthy eye, with those muscles completely relaxed, the image of an object at infinity should be in focus on the retina. **Near-sighted** people have too strong an eye lens. For them, the image of a distant object falls in front of the retina even when those muscles are relaxed. **Far-sighted** people can't bring a nearby object into focus. Its image falls behind the retina even when those muscles are contracted. (See Fig. 11.2.)

In the camera, focusing is accomplished not by changing the power of the lens, but by changing the distance between the lens and the film. When the camera is at its shortest, the film is at the focal point of the lens. More precisely, it is in the **focal plane** of the lens. In this setting, an object at infinity is in focus. To bring a closer object into focus, lengthen the camera: Move the film further back, or the lens further forward.

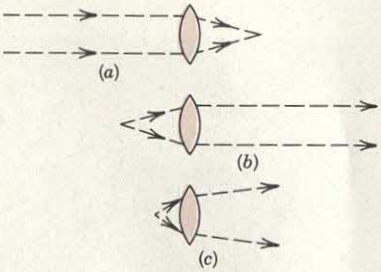


FIGURE 11.2 (*a*) Rays from a distant object are converged to the focal point. (*b*) This is the closest an object can be and still have a real image. The image of an object at the focal point is off at ''infinity'' on the right. (*c*) If the object is brought in closer than the focal length, no real image is formed.

☐ QUESTION FOR FURTHER THOUGHT

Why did Nature not build the eye to focus the way the camera focuses, that is, by changing the distance between the lens and the retina?

---

[1] If you use a flashlight, you have to remove the reflector.

If you are observant, you will have noticed that the image is upside down. The next paragraph shows why. You might ask, if the image of the world we look at is upside down on the retina, why don't we see it as upside down? The answer has to be that we are used to it that way!

Ray Diagrams

In Section 11.4 we talk about how a lens works, based on the law of refraction. Here we just describe which way it bends light rays. Look at Fig. 11.3. Rays that hit the center of the lens go straight through. Rays that hit off center are bent toward the axis of the lens. In other words, a bundle of parallel rays is *converged* in passing through the lens. A magnifying lens is also called a *converging* lens or a *positive* lens. The bundle (or *pencil*) of rays emerging from the lens in Fig. 11.3 is a *convergent* pencil.

A high-quality lens makes a sharp image. This means that all rays coming from one point converge to one point. To each point of the object corresponds one point of the image, not a fuzzy region. Later in this chapter we talk about image quality. Here we just point out that, in general you get a sharper image the narrower the angular width of the pencil of rays you use.

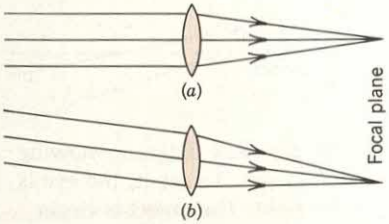


FIGURE 11.3 (*a*) A parallel bundle of rays incident parallel to the axis is converged and comes to a focus at the *focal point* of the lens. (*b*) A parallel bundle of rays coming in off-axis comes to a focus in the focal plane of the lens, a plane through the focal point perpendicular to the axis.

EXERCISE 4 Explain why the pictures made by cheaper cameras usually cover a smaller angle. Sketch!

A ray coming in parallel to the axis will be bent in such a way as to pass through the focal point. We know this, because a really distant object on the axis forms its image at the focal point. This is how we defined the focal point. We are now ready to do "ray tracing," to locate the image of any off-axis object by drawing two rays: One is the ray through the center of the lens, which will emerge at the same angle it enters. The other is the ray parallel to the axis, which will emerge aimed toward the focal point. The two rays meet at a point. That is the "image" point. It is the point where the image is in sharpest focus (see Fig. 11.4).

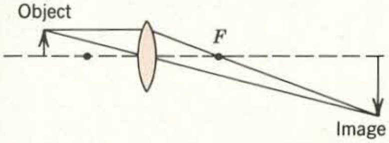


FIGURE 11.4 Ray diagram. The ray through the center of the lens is not bent. The ray parallel to the axis is bent by the lens so as to pass through the focal point *F*.

EXERCISE 5 Convince yourself that the closer the object is placed to the focal plane of the lens, the larger its image. Two or three ray diagrams should be convincing.

EXERCISE 6 Draw the ray diagram for the object placed two focal lengths from the lens. (Assume that the focal length is the same on both sides of the lens.) Show that the image is the same size as the object.

EXERCISE 7 Draw a ray diagram for an object placed more than two focal lengths from the lens. Is the image larger or smaller than the object?

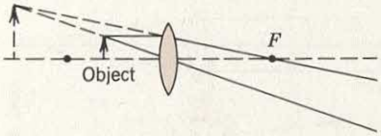


FIGURE 11.5 Ray diagram showing virtual image. To see it, the eye is on the right. The object is closer to the lens than the focal plane. Although the ray parallel to the axis is bent down so as to pass through the focal point, the two rays still diverge after passing through the lens. They both appear to come from a point further to the left of the lens than the object. This is the *virtual image.*

Virtual Image

What happens if the object is close to the lens, closer than the focal length? Experience tells us that we don't get a real image. We can trace the two rays and see why (Fig. 11.5): They emerge from the lens still diverging. The lens is not strong enough to converge the rays when their source is so close. It is useful to ask what point they appear to diverge *from.* If we extrapolate those rays backward (dotted lines in Fig. 11.5), they meet at a point in back of the object, that is, further away from the lens. This is the position of the *virtual image.* Of course, none of the light rays from the object actually get to the virtual image. All of them *appear* to come from there.

With most optical instruments, a virtual image is exactly what we want. When we use a magnifying glass, we don't want it to project a real image of an object. We want to hold the lens close to an object and see it *as if* it were further away. We see the light appearing to come from the virtual image, so we can focus our eye further away than the object. The magnifying lens permits us to get our eye closer to the object than the accommodation of our naked eye lens would permit.

EXERCISE 8 Sketch a ray diagram for an object placed half the focal length away from a converging lens.

Lens Equation

If I know the object position and the power of the lens, there must be a formula for telling me the image position. We are going to derive such a formula using the ray-tracing procedure. But before we do, we shall need to define the concept of a **wavefront.**

In the **pencil** (bundle) of rays leaving the object, the speed of the light is the same along every ray. In technical language, the light is traveling in a homogeneous, isotropic medium. If we think of following a light signal out along each ray, then points of equal travel time lie on a spherical surface called the wavefront. Repeat: The wavefront is defined to be the locus of points of equal travel time.

What the lens does is to change the **curvature** of the wavefront that passes through it. In forming a real image, the lens converts a diverging pencil of rays into a converging pencil, a spreading wavefront into a shrinking wavefront (Fig. 11.3). In forming a virtual image, it converts a diverging pencil into a less diverging pencil (Fig. 11.5). The lens equation says simply,

$$\text{curvature}_{\text{out}} = \text{curvature}_{\text{in}} + \text{power of lens} \qquad \text{(Eq. 1)}$$

The lens is a device for adding a given constant curvature to the curvature of the incident wavefront. Here is what that means:

The curvature of a spherical surface is defined to be the reciprocal of the radius. In geometrical optics, it is called positive if the pencil is converging, negative if diverging. Since the rays diverge from the object, the wavefront incident on the lens has negative curvature. The **power** of the lens is defined as the reciprocal of its focal length. The equation says that a converging lens (a lens with positive power) can convert a negative-curvature wavefront into a positive-curvature wavefront. That is what it does in forming a real image.

From the definitions, *curvature* and *power* are both measured in reciprocal meters. For lens power, we call the $(\text{meter})^{-1}$ the diopter:

$$1 \text{ diopter} = 1 \text{ m}^{-1}$$

Two meters away from an object, the wavefront has a curvature of $-1/2\text{m} = -0.5 \text{ m}^{-1}$. If at this point it meets a lens of $+2.0$ diopters, the curvature of the wavefront coming out of the lens is

$$\text{curvature}_{\text{out}} = -0.5 \text{ m}^{-1} + 2.0 \text{ m}^{-1} = +1.5 \text{ m}^{-1}$$

That is a positive curvature: The lens has made the rays converge to a point 0.67 meters beyond the lens. Repeat: The image is $1/(1.5 \text{ m}^{-1}) = +0.67$ meters from the lens (Fig. 11.6). The "+" sign says "real image," or converging rays.

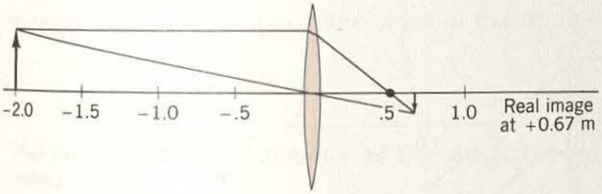


FIGURE 11.6 Object 2 m from +2.0 diopter lens.

Now suppose that the object is brought closer to the lens, say 1.25 m away (Fig. 11.7). When the wavefront gets to the lens, its curvature is $-1/(1.25 \text{ m}) = -0.8 \text{ m}^{-1}$. On emerging from the $+2.0$ diopter lens, the curvature is

$$\text{curvature}_{\text{out}} = -0.8 \text{ m}^{-1} + 2.0 \text{ m}^{-1} = +1.2 \text{ m}^{-1}$$

The lens has made the divergent rays from the object converge to a real image $1/(1.2 \text{ m}^{-1}) = 0.83$ m past the lens. Remember our generalization from experience: Object and image move in the same direction.

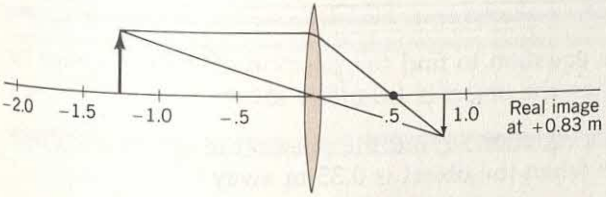


FIGURE 11.7 Object 1.25 m from +2.0 diopter lens.

In Fig. 11.8, the object is still closer, only 0.25 m from the +2.0 diopter lens. This time the lens is not strong enough to change the divergent rays

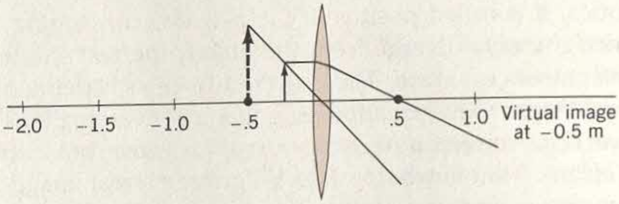


FIGURE 11.8 Object 0.25 m from +2.0 diopter lens.

into convergent rays. It just makes them diverge less. The incident curvature is $-1/(0.25\text{ m}) = -4\text{ m}^{-1}$.

$$\text{curvature}_{\text{out}} = -4\text{ m}^{-1} + 2.0\text{ m}^{-1} = -2.0\text{ m}^{-1}$$

The wavefront diverges as if the rays came from a point $1/(2\text{ m}^{-1}) = 0.5$ m to the left of the lens. The "−" sign says "left"; therefore, "virtual image."

What if the object is in the focal plane (Fig. 11.9)? Then the (negative) curvature of the incoming wavefront is equal to the (positive) power of the lens. A +2.0 diopter lens has a focal length of $1/(2\text{ m}^{-1}) = 0.5$ m. If the object is 0.5 m from the lens, we have

$$\text{curvature}_{\text{out}} = -2.0\text{ m}^{-1} + 2.0\text{ m}^{-1} = 0$$

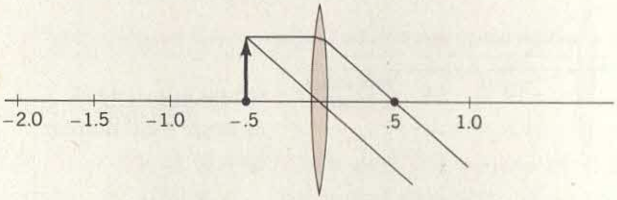


FIGURE 11.9 Object 0.5 m from a +2.0 diopter lens.

The outgoing wavefront is neither convergent nor divergent. The rays are all parallel. We can say that the image is at infinity. Or should we say it is at minus infinity? It doesn't really matter. As you know, when you bring the object in closer to the lens, as you approach the focal distance the real image gets huge. When you are right there, it blurs out. Closer than the focal distance, the image is virtual. As the object is brought toward the focal plane, the virtual image moves out toward infinity on the left, that is, minus infinity.

EXERCISE 9 Use the lens equation to find the position of the real image for the +2.0 diopter lens when the object is 1.0 meter away.

EXERCISE 10 Use the lens equation to find the position of the virtual image for the +2.0 diopter lens when the object is 0.33 m away.

Magnification. Derivation of the Lens Equation[2]

Your experience in following the real image as you moved an object in closer to the lens taught you that the image gets bigger and therefore fainter. The same amount of light is spread over an ever-increasing area. It grows until the object distance is equal to the focal length. At this point the image is infinitely large and disappears in a dark blur. We shall now show how the image size depends on the object position.

We define the **magnification** of the lens as the ratio of image size to object size. Simple geometry shows [Fig 11.10 *a*] that the magnification is equal to the ratio of image distance to object distance. The ray going straight through the center of the lens forms the hypotenuse of two similar triangles. The heights of the triangles are the image size I and the object size O, respectively. Their bases are the image distance q and the object distance p, respectively. The proportionality of corresponding sides of similar triangles yields

$$\text{magnification} = \frac{I}{O} = \frac{q}{p} = \frac{\text{image distance}}{\text{object distance}}$$

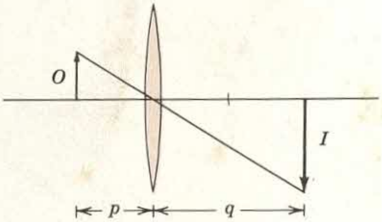


FIGURE 11.10*a* Magnification formula: $I/O = q/p$.

Now trace our second ray, the one leaving the object parallel to the axis (Fig. 11.10 *b*). Again you get two similar triangles. Again the ray forms the hypotenuse of both. Again the sides of the triangles are proportional:

$$\text{magnification} = \frac{I}{O} = \frac{x_I}{f}$$

The distance x_I is the distance of the image behind the focal plane:

$$x_I = q - f$$

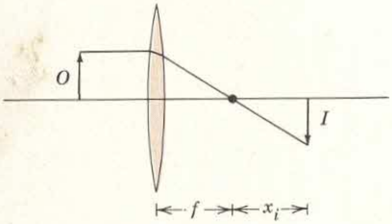


FIGURE 11.10*b* Magnification formula: $I/O = x_I/f$.

Each of these two ray constructions has given us an independent expression for the magnification. Setting them equal, we have a relation between the variables p, q, and f:

$$\text{magnification} = \frac{q}{p} = \frac{q - f}{f}$$

Solve for the curvature of the outgoing wavefront:

$$\frac{1}{q} = -\frac{1}{p} + \frac{1}{f}$$

EXERCISE 11 Do the algebra!

---

[2] You may want to skip this section on first reading.

That is the lens equation! The curvature of the outgoing wavefront is $1/q$. The curvature of the incoming wavefront is $-1/p$. The minus sign indicates that the rays are diverging. Remember that $1/f$ is defined to be the power of the lens. The equation is the familiar

$$\text{curvature}_{\text{out}} = \text{curvature}_{\text{in}} + \text{power of lens}$$

EXERCISE 12 To have the image 10 times the height of the object, the image distance has to be 10 times the object distance.

(a) How far behind the focal plane is the image plane? Answer in the form, "_______ times the focal length."

(b) How far in front of the other focal plane must the object be placed?

Some writers call the focal point in front of the lens the "first principal focus," the one in back the "second principal focus."

EXERCISE 13 For a thick lens, it is hard to know from where to measure p and q. But the principal foci can always be located, so that x_I, the distance of the image plane behind the second principal focus, is easy to measure. So is x_O, the distance of the object plane in front of the first principal focus. We wrote the magnification as

$$\frac{I}{O} = \frac{x_I}{f}$$

Prove also

$$\frac{I}{O} = \frac{f}{x_O}, \quad \text{hence} \quad x_I x_O = f^2$$

This last is sometimes called Newton's form of the lens equation.

EXERCISE 14 Use Newton's form of the lens equation to solve Exercise 12, where the magnification = 10.

EXERCISE 15 Use Newton's form of the lens equation, $x_I x_O = f^2$, to prove that if you move the object, the image moves in the same direction.

EXERCISE 16 Where must the object be placed for the image to be the same size as the object, that is, magnification = 1? Show algebraically and geometrically.

Negative Lens

Examine an eye-glass lens worn by a very near-sighted person. Instead of magnifying, it makes objects appear smaller. Notice how small the glasses make her eyes! A divergent pencil of rays incident on such a lens emerges more divergent. The object appears to have been brought closer. The power of such a lens is defined to be negative.

A positive lens is thicker in the middle than near the edge. You can tell by looking closely at a magnifying glass. A negative lens is thinner in the middle. With an ophthalmic lens this may be hard to tell, especially since it is fashionable to make lenses of either sign convex toward the front (Fig. 11.11).

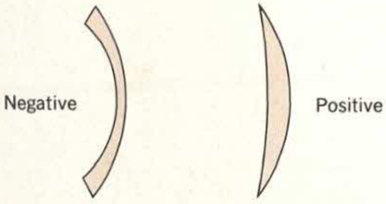


FIGURE 11.11

You won't succeed in forming a real image with a negative lens. When you look through it, you are seeing a virtual image. You remember that a virtual image formed by a positive lens is further away than the object. With the negative lens, the virtual image is closer than the object.

EXERCISE 17 If you have access to a mixed collection of lenses including some negative ones, train yourself to distinguish the positive from the negative. For weak lenses, it may be hard to tell whether the image is larger or smaller than the object. An easy way to tell is to wiggle the lens (or your head) from side to side. Does the image move *with* or *against* the motion of the lens. Sketch.

EXERCISE 18 Hold a pencil vertically; behind it, hold a finger of your other hand. If you move your head to the right, the pencil appears to move to the left. This is called parallax. If you were looking through a negative lens and moved your head to the right, which way would the image appear to move? Relative to what?

EXERCISE 19 The parallax test can be used to locate the image plane with either positive or negative lenses. Look at your pencil through a lens. Support it; you'll need the other hand. Look at a finger by peering over the top of the lens. Try to position the finger so that the image of the pencil does not move relative to the finger when you move your head (or the lens). You have found the image plane!

Even though a parallel pencil of rays is not brought to a focus by a negative lens, the lens still has a focal plane and hence a focal length. A pencil of rays incident parallel to the axis emerges diverging as if it all came from one point. That point is the focal point of the lens (Fig. 11.12). The image (virtual!) of any distant object is in a plane perpendicular to the axis through the focal point. The distance of this *focal plane* in front of the lens is the *focal*

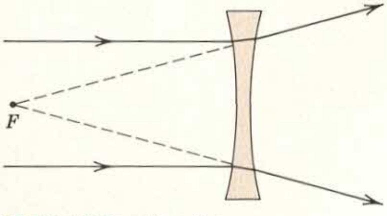


FIGURE 11.12 Parallel rays are diverged by a negative lens so that they appear to come from the focal point F.

length of the lens. Since it is measured backward, in the direction the light comes from, it is designated $-f$, that is, f is negative. We can still use our lens equation (Eq. 1) for negative lenses. Just notice that the power $1/f$ is a negative number. For a negative lens, the equation

$$\text{curvature}_{\text{out}} = \text{curvature}_{\text{in}} + \text{power}$$

says that the curvature of the emerging wavefront is *less* than that of the incident wavefront. If the incident pencil is diverging (negative curvature), the emerging wavefront will diverge more, that is, it will have a larger negative curvature: The pencil of rays will appear to be diverging from a point closer to the lens than the object. That is what we just got through saying.

EXERCISE 20 An object is placed 0.75 m from a -2.0-diopter lens. Where is the image? Solution: Curvature of the incident wavefront is $-1/(.75\ \text{m}) = -1.33\ \text{m}^{-1}$. Curvature of the emerging wavefront $= -1.33\ \text{m}^{-1} - 2.0\ \text{m}^{-1} = -3.33\ \text{m}^{-1}$. The image is ______ m from the lens. The minus sign says that it is in front of the lens, that is, virtual image.

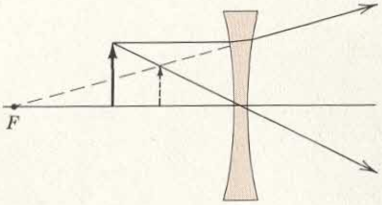


FIGURE 11.13 Ray diagram for negative lens.

EXERCISE 21 Trace the usual two rays through a negative lens (Fig. 11.13). One ray goes straight through the center of the lens. The other ray leaves the object parallel to the axis and is bent away from the axis so that it appears to come from the focal point. The only way the two rays meet is if this second ray is extrapolated backward. The intersection is in front of the lens. The image is virtual. Evidently, it is smaller than the object and erect, right-side up.

Repeat the ray tracing for an object distance greater than the focal length. Is the image smaller than the object? Is it erect or inverted?

11.3 HOW EYEGLASSES WORK

An ordinary spectacle lens helps the wearer to see an object more clearly by forming a virtual image of it at a more comfortable distance. The farsighted wearer wants the image farther away than the object. The near-sighted wearer want it closer.

A farsighted person's eye lens is too weak. Without glasses, the image of a close object falls behind his retina. He sees it blurred. With glasses, the positive power of the lens he wears is added to that of his eye lens. A pencil of rays diverging from the object is converged by the lens so that it appears to diverge from a point further away.

The nearsighted eye is unable to *flatten* the lens sufficiently to focus the image of a distant object on the retina. The eye lens is too strong, so the image falls in front of the retina. When the rays from it get to the retina, they have diverged again. What falls on the retina is a blur. Adding the negative power of an eyeglass lens to the power of her too-strong eye lens allows the wearer to see the distant object clearly. What she sees is a virtual image closer in: The negative lens has diverged the pencil of rays from the distant object so that it appears to come from a nearer point.

The virtual image formed by the eyeglass lens is, of course, a **virtual object** from the point of view of the eye. This terminology is useful if we are to trace rays. The virtual object can be the source of the two rays we conventionally use in our construction of the image. For example, a −2.0-diopter lens worn in front of a nearsighted eye would diverge the parallel pencil of rays coming from a distant object so that it would appear to be coming from a virtual image 0.5 m ($= 1/(2.0\ \text{m}^{-1})$) away. This would be the virtual object for the nearsighted eye, which cannot focus at infinity but may be able to focus comfortably at 50 cm.

EXERCISE 22 A not-very-nearsighted child cannot focus on objects farther than 2.0 meters away.

(a) How strong a negative lens would you prescribe to enable the child to see distant objects clearly?

(b) How far away does this lens form the virtual image of an object 2.0 meters away?

(c) An object 50 cm away?

EXERCISE 23 Suppose that a farsighted eye could not accommodate at all but had a "fixed focus" at infinity, like some cheap cameras. How strong a lens would you prescribe for clear reading at 25 cm?

EXERCISE 24 A farsighted patient wears a +2.0 diopter lens for reading, holding the book 25 cm from his eye.

(a) How far away is the virtual image of the book?

(b) How much stronger must an eye lens be for reading at this distance than for looking out of the window?

Now you know how eyeglasses help farsighted and nearsighted eyes. We have not talked about what goes on at the surface of the glass, about how a lens forms an image. The next section describes the refraction of light in general. Then it shows how the law of refraction can be used to design a lens.

11.4 REFRACTION: SNELL'S LAW

When a light ray strikes a polished glass surface, some of the light is reflected and some is transmitted into the glass. Most of the transmitted rays are bent as they cross the surface. We say they are *refracted*. Only at normal incidence does a ray go straight in, unbent. If you are asked which way a

Reflection in a cylindrical mirror.

ray is bent, a qualitative answer is that it is bent *toward the normal* and remains in the plane of incidence. This means that the incident ray, the refracted ray, and the normal to the surface all lie in the same plane. Asked how much the ray is bent, you might answer that the greater the angle the incident ray makes with the normal, the more it is bent. You might even risk the guess that the angles the incident and refracted rays make with the normal are proportional. That turns out to be a good guess for small angles. But you know it can't be true as you get near grazing incidence, angles of incidence near 90°. There, rather large changes in the angle of incidence give quite small changes in the angle of refraction. The correct law of refraction, discovered by Snell in the early seventeenth century, is a bit subtle and requires some geometry to appreciate.

Draw a circle of unit radius centered at the point of incidence. "Unit" can mean 1 inch, 1 centimeter, or any unit; it doesn't matter. This circle defines a unit vector in the direction of the incident ray, as well as a unit vector in the direction of the refracted ray. Now sketch in the components of those two unit vectors parallel to the refracting surface, the horizontal components in our picture (Fig. 11.14). Snell's Law of Refraction states that for all angles of incidence, these two components are proportional.

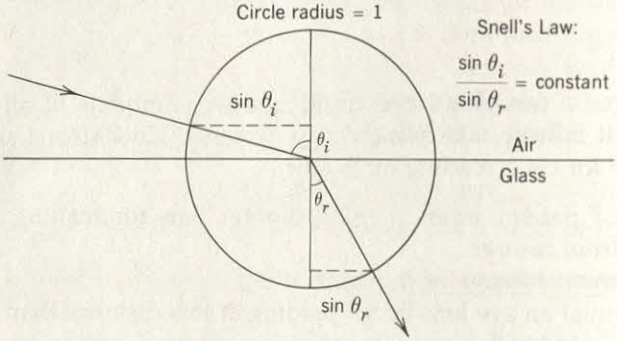


FIGURE 11.14 Snell's Law: $\sin \theta_i / \sin \theta_r$ = constant.

The two rays and the surface normal define two right triangles. Both hypotenuses have unit length. Remembering the definition of the sine of an angle,

$$\sin \theta = \frac{\text{opposite}}{\text{hypotenuse}}$$

you can convince yourself that the parallel component of the unit "incident" vector is $\sin \theta_i$. The parallel component of the unit "refracted" vector is $\sin \theta_r$. Snell's Law is

$$\frac{\sin \theta_i}{\sin \theta_r} = n = \text{constant} \qquad \text{(Eq.1)}$$

The constant is a quantity characteristic of the refracting material, called the *index of refraction*. For light going from free space into a solid or liquid, it is always greater than unity. For most kinds of glass it is around 1.5.

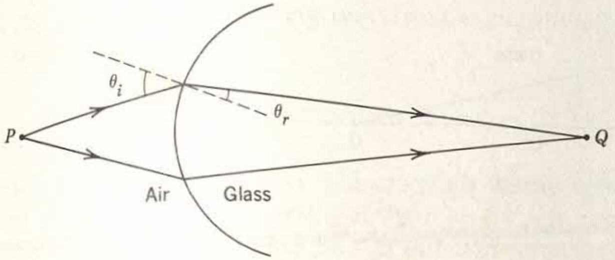


FIGURE 11.15 Formation of a real image by a curved glass surface.

Equipped with Snell's Law, you can see intuitively how light rays can be converged by a curved glass surface. Figure 11.15 shows two rays from a point P outside the glass being refracted so they intersect again at a point Q inside the glass. Both rays have been bent toward the surface normal. The surface has to be convex to be converging. To be sure, two rays diverging from P will meet again only if P is not too close to the surface. If θ_i is too large, the steeply upward-bound ray is bent, to be sure, but is refracted into the glass less steeply upward bound (Fig. 11.16).

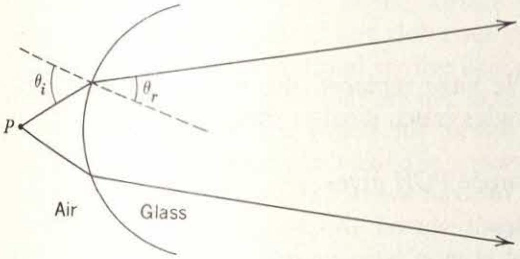


FIGURE 11.16 The object point P is too close to have a real image.

You now have a feeling for the process of image formation. If other rays from P converge to the same point Q, then Q is the image of P. Without saying anything about the shape of the glass surface, we don't know whether rays leaving P at other angles will converge on the same point Q or not. If the surface is rough or ripply, rays will go in various directions, depending on where they strike the surface. But if the surface is reasonably smooth, rays from a reasonably narrow pencil will form a reasonably good image. That is not a strong statement, but it is useful. We now show how this works for a *spherical* surface with rays making small angles with the axis.

Image Formation by a Spherical Surface

Most polished glass surfaces are spherical, just because it is easier to make them that way. They are ground on rotating machinery. In Fig 11.17 the horizontal axis is supposed to be an axis of rotational symmetry. To see how the spherical surface of the glass forms an image, consider rays making small angles with the axis —"paraxial rays." The ray leaving an object point P along the axis strikes the glass surface at right angles ($\theta_i = 0$), so it goes in unbent. Now trace a ray from P that strikes the glass surface at the point

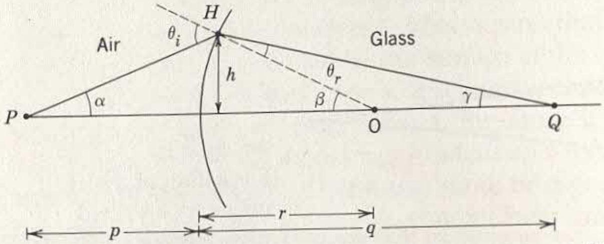


FIGURE 11.17 How a spherical surface forms an image. The incident ray PH makes an angle α with the axis. The outgoing ray HQ makes an angle γ. The radius OH makes an angle β. For small angles, the approximations $h = p \tan \alpha \cong p\alpha$, $h = r \tan \beta \cong r\beta$, and $h = q \tan \gamma = q\,\gamma$ are adequate, giving $\alpha = h/p$, $\beta = h/r$, and $\gamma = h/q$.

H, a height h above the axis. The angle α the ray PH makes with the axis is assumed small, so the angle of incidence θ_i is also small. For small angles the approximate form of Snell's Law,

$$\frac{\theta_i}{\theta_r} \approx n = \text{constant}$$

will be a good approximation: We have replaced the sines by the angles. (See Problem 16.) Think of the angles much smaller than they are drawn in Fig. 11.17.

The geometry of the obtuse triangle POH gives

$$\alpha + \beta = \theta_i = \text{angle of incidence}$$

Similarly, for the triangle QOH, we get

$$\gamma + \theta_r = \beta$$

Substituting these two equations into the approximate Snell's Law $\theta_i = n\theta_r$ gives

$$\alpha + \beta = n\,(\beta - \gamma)$$

or

$$\frac{h}{p} + \frac{h}{r} = n\left(\frac{h}{r} - \frac{h}{q}\right)$$

Now you can see that all the rays striking at different heights h converge to the same point, because the h cancels out. The relation between the object distance p and the image distance q is independent of the direction of the ray. Canceling the h and rearranging gives

$$\frac{n}{q} = -\frac{1}{p} + \frac{(n-1)}{r} \qquad \text{(Eq. 2)}$$

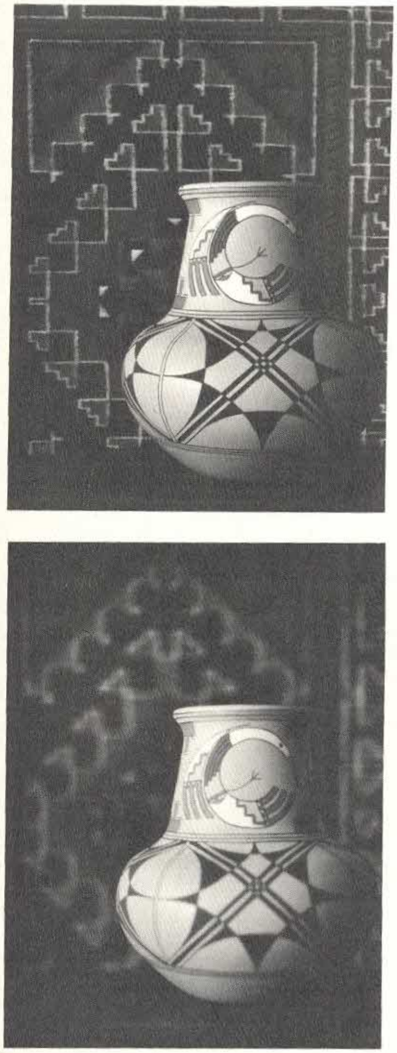
That reminds you of the lens equation (Eq. 11.2-1). Bear in mind that the curvature of the incoming wavefront is negative ($= -1/p$) and you can see the analogy:

$$n \times \text{curvature}_{\text{out}} = \text{curvature}_{\text{in}} + \frac{n-1}{r}$$

Read as a lens equation, this ascribes to the spherical surface the power $(n-1)/r$. If you wonder about the n on the left, remember that the outgoing ray is in the glass, which has an index of refraction n.

That was a lot of algebra. Now let's criticize what we have done. The derivation involved several small-angle approximations. So rays from P making *large* angles with the axis do *not* converge to the same point Q. This is called **spherical aberration**. A large "opening," allowing rays for which those small-angle approximations are bad, will give a fuzzy image. We pointed that out early in this chapter. The smaller the angular aperture, the sharper the image. Of course, there is nothing in principle that restricts us to spherical surfaces for making optical components. They are just easier to grind. Modern computer-controlled machinery can grind **aspherical** glass surfaces for optical systems requiring large apertures. Notice that for small apertures, the exact shape of the surface is not crucial. Any smooth curve looks like part of a circle if you don't see much of it.

Consider the plane defined by the axis and a point on the object. Points off the axis send out rays that are not in this plane. We call them skew. The more skew they are, the worse our formulas for describing where they converge. Our diagrams are not good at representing skew rays. But we look at a three-dimensional world, so we have to be concerned with the images of three-dimensional objects. All lenses distort. Figure 11.18 shows two kinds of distortion of two-dimensional objects lying in a plane perpendicular to the axis. As for the third dimension, the depth dimension, this is what our discussion of focusing has been about. We can state a general rule: If you want greater *depth of field*, that is, if you want objects at different distances to be in good focus, confine your rays to a narrower aperture. Repeat: If you want a lot of depth, you have to sacrifice width. (See Problem 32.)



At f/16 the rug and the jug are all in focus. At f/1.4, which is 128 times "faster," the rug in the background and the design in the foreground are both blurred; only the neck of the jug is in good focus. (Problem 32)

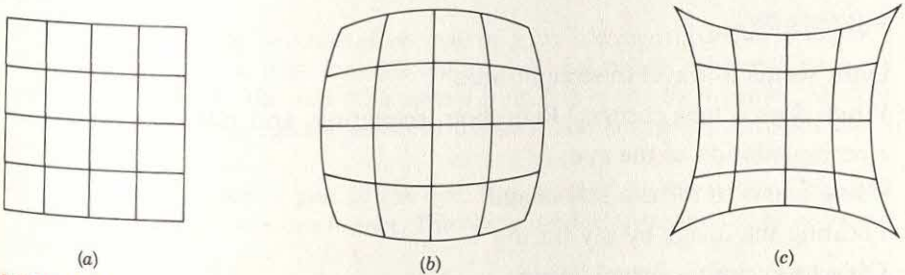


FIGURE 11.18 Distortion: (*a*) With a tiny aperture, the graph paper looks square. (*b*) Barrel distortion arises when the magnification decreases toward the edge of the field. (*c*) Pincushion distortion arises when the magnification increases toward the edge of the field.

Lens Maker's Formula

We have shown how a curved glass surface forms an image. In Eq.2 we obtained an expression for the *power* of the surface:

$$\text{power} = \frac{n - 1}{r}$$

The more curved the surface, the greater its power. The greater the index of refraction of the glass, the greater the power. But a lens has two surfaces, and we are interested in rays that come in the front and go out the back surface. So we have to consider not only light going from air to glass, but also light going from glass to air. In Problem 34 the reader is invited to complete the proof of the **Lens Maker's Formula,** which gives the power of a lens in terms of the radii of curvature of the front and back surfaces:

$$\text{power} = (n - 1)\left(\frac{1}{r_1} + \frac{1}{r_2}\right) \qquad \text{(Lens Maker's Formula)} \qquad \text{(Eq.3)}$$

The powers of the two surfaces just add. Our proof is for convex lenses. Problem 35 extends it to concave lenses by interpreting the radius of curvature as negative if the surface is concave.

The Lens Maker's Formula is presented here so that you know that such a formula exists. It tells you that convex surfaces have positive power and concave surfaces have negative power. We are not going to do anything with the formula. But now you know why people whose eyes need strong corrections have to wear thick glasses.

EXERCISE 25 Explain. Notice that the size of the glasses does not appear in the formula. Why do lenses of larger diameter have to be thicker to have the same power?

SUMMARY

Light seems to travel in straight lines
When does it turn corners? Reflection, scattering, and refraction
Accommodation of the eye
Using a lens to focus a real image
Locating the image by ray tracing
Object too close—virtual image
Locating the image using the lens equation
How to tell negative from positive lenses
Parallax

Glasses for farsighted and nearsighted eyes

Snell's Law

Spherical lens surface—Lens Maker's Formula

KEY TERMS

Rectilinear propagation Light goes in straight lines.

Scattering Reflection by small particles so the light goes out in many directions.

Ray A line or curve that describes the path of the light. The "ray approximation" of geometrical optics neglects the wave nature of light, that is, neglects diffraction. In a homogeneous medium, a ray is a straight line.

Pencil A bundle of rays from the same source.

Reflection The law of reflection states that the angle of reflection is equal to the angle of incidence. *Specular reflection* means *mirror* reflection, that is, smooth-surface reflection.

Refraction The bending of light when it passes from one medium to another. If the optical properties (refractive index) of a medium vary gradually in space, light is bent gradually, that is, the rays are curved.

Snell's Law of Refraction $\sin\theta_i/\sin\theta_r = n =$ index of refraction.

Refractive index Index of refraction, a constant characterizing a medium in which light can travel, usually designated n. The refractive index changes with frequency, a property called *dispersion*. See Snell's Law; also speed of light.

Lens A device for changing the curvature of a wavefront. A bundle of rays is converged by a positive lens, diverged by a negative lens.

Accommodation The ability of the eye to focus at various distances. More generally, the ability of a lens system to change its power.

Focus The place where a bundle of rays comes together to a point. Also used as a verb: To focus a beam means to bring it to a focus.

Focal point In a lens or mirror system: the place where a beam parallel to the axis comes to a focus.

Focal length The distance from the lens to the focal point.

Focal plane A beam of light coming from infinity is brought to a focus in the focal plane.

Ray diagram Requires two rays. One may be a ray coming in parallel to the axis. This is bent so as to pass through one of the focal points. One may be a ray emerging parallel to the axis. This comes from one of the focal points. Another special ray is one through the center of the lens. This emerges in the same direction it came in.

Retina The light-sensitive part of the eye. It is attached to the sclera in back. In front, it bounds the vitreous humor. The retina is often considered to be an extension of the brain.

Cornea The transparent curved cover through which light enters the eye. The cornea holds in the aqueous humor. Together they are responsible for most of the power of the eye's optical system. This is why patients who have had their cornea removed because it became cloudy (cataract) need to wear very strong glasses.

Aqueous humor Watery liquid contained between the cornea and the crystalline lens of the eye. Its refractive index is 1.33 (the same as water), while that of the cornea is 1.37. Since its front surface is more curved than its rear surface, it also contributes to the power of the optical system.

Crystalline lens Layered structure responsible for the accommodation of the eye. Muscles causing it to bulge more in the middle increase its power. The refractive index is greater near the middle (1.40) than at the edges (1.38).

Vitreous humor The transparent jelly filling most of the eyeball.

Cones The photoreceptors of the retina that are responsible for the sensation of color.

Rods The photoreceptors of the retina that sense low light levels. The rods are involved in night vision.

Blind spot The place on the retina where the optic nerve attaches.

Fovea Also called *macula*. The region of the retina in which visual acuity is greatest. It has maximum density of cones.

Iris Diaphragm covering the periphery of the crystalline lens. The ciliary muscle closes the iris down in strong light to give a smaller aperture.

Virtual image The position the light appears to come from in a diverging pencil. Note that no light ray need ever pass through the virtual image. A negative lens forms only a virtual image of a real object, never a real image. A positive lens forms a virtual image of a real object when the object is closer than a focal length.

Lens equation Curvature$_{\text{out}}$ = curvature$_{\text{in}}$ + power, or $\frac{1}{q} = -\frac{1}{p} + \frac{1}{f}$

where q = image distance, p = object distance, and f = focal length.

Sign convention f is positive for a converging lens, negative for a diverging lens. p is distance in front of the lens, so it is positive for a real object. q is positive for a real image, which is in back of the lens; it is negative for a virtual image, which is in front of the lens.

Virtual object When light goes through two or more lenses, a virtual image formed by the first lens may be considered to be the source of rays for the second lens. Since the light rays do not really come from this point, it is called a *virtual* object.

Wavefront Locus of point of equal travel time from a point source. Surface which is everywhere perpendicular to the ray through the point.

Power of a lens $1/f$, the reciprocal of the focal length. Measured in diopters = reciprocal meters.

Diopter = (meter)$^{-1}$ Measure of lens power. Positive for converging lenses, negative for diverging.

Curvature Reciprocal of the radius of curvature. The curvature of a wavefront is considered positive if it is converging, negative if diverging. A spherical surface has a unique curvature; a nonspherical surface does not.

Plane wave All the rays are parallel. All the wavefronts are planes. The source of a plane wave is at infinity.

Magnification Ratio of image size to object size. If the object (or the image) is at infinity, this definition loses its meaning and is replaced by the *angular* magnification, the ratio of the angle subtended by the image to the angle subtended by the object.

Newton's form of the lens equation $x_I x_O = f^2$, where

x_O = object distance in front of the focal plane
x_I = image distance in back of the other focal plane
f = focal length

Lens Maker's Formula

$$\frac{1}{f} = (n - 1)\left(\frac{1}{r_1} + \frac{1}{r_2}\right)$$

Speed of light In vacuo = 2.9979×10^8 m/sec. This is a universal speed limit for the propagation of any signal. The *refractive index* of a medium is the ratio of this speed to the speed of light in the medium.

Ophthalmic lens Spectacle lens. A lens intended to be worn close to the eye, either in eyeglasses or as contact lens.

Cylindrical lens One or both of its surfaces are cylindrical. Used for correcting astigmatism of the eye. If an eyeglass prescription calls for a cylindrical correction, this is usually ground on the inside as a negative correction, allowing a spherical correction to be ground on the outside.

Farsighted Having hyperopia. The eye lens is too weak for the shape of the eyeball, so the image of a nearby object falls behind the retina. Corrected with a positive (converging) eyeglass lens.

Nearsighted Having myopia. The lens is too strong for the shape of the eyeball, so the image of a distant object falls in front of the retina. Corrected with a negative (diverging) eyeglass lens.

Astigmatism If for an off-axis object, rays that stay in the plane of the axis come to a focus different from that for skew rays, the lens is said to be astigmatic. Symptom: When a vertical wire is in focus, a horizontal wire is not. Corrected with a cylindrical lens.

Plane of incidence Defined by the incident ray and the normal to the surface. For refraction at the surface of an isotropic medium, the plane of refraction is the same as the plane of incidence.

Critical angle The largest angle at which a ray can strike the surface and still get out of the refracting medium:

$$\sin \theta_{\text{critical}} = 1/n.$$

For larger angles, get *total internal reflection*.

Coefficient of reflection The fraction of the incident light intensity that is reflected. At large angles of incidence—close to grazing incidence—this is close to 100%. The larger the change in index of refraction at the surface, the larger the coefficient of reflection.

Aspherical lens At least one surface is not spherical. To be sure, a cylindrical lens is not spherical, but the term is usually applied to the fancy surfaces intended to correct spherical aberration.

Spherical aberration Nonparaxial rays, that is, rays making appreciable angles with the axis of a spherical lens surface, do not converge to the same focus as paraxial rays.

Simple microscope Magnifying glass.

Compound microscope A high-power object lens is placed close to the object—just a bit more than the focal length away. The lower-power eyepiece, placed just beyond the real image thus formed, forms a virtual image.

Astronomical telescope A low-power objective lens forms a real image of a distant object. A high-power converging eyepiece placed just behind the focal plane of the objective forms a virtual image—upside down.

Field glass Or binoculars. Same as an astronomical telescope, with a prism used to make the virtual image erect.

Galilean telescope Or opera glass. A converging objective lens forms a real image of a distant object. A diverging eyepiece forms an erect virtual image. Magnifications from twofold to sixfold are useful but not greater.

Slide projector The transparent slide is placed just outside the focal length of the *condensing lens* and illuminated with diffuse light so that the distant real image on the screen is a magnified inverted image.

***f*-number** The numerical aperature of a lens:

diameter of lens = focal length ÷ *f*-number

Thus the diameter of an *f*/2 lens is half its focal length.

REVIEW QUESTIONS

- How do we see objects that are not self-luminous, that is, are not themselves sources of light?
- When you see your own clothes, is it by specularly reflected or by diffusely reflected light?
- While driving at night, long before you see the headlights, you often see a headlight beam coming over the crest of a hill. What is scattering the light out of the beam toward your eyes?
- When you look up from the book and glance across the room, does your eye lens get thicker or thinner?
- Define focal point, focal plane, and focal length.
- The real image in the film plane of the camera is erect / inverted.
- The real image on the retina of the eye is: erect / inverted.
- What two rays can be used in ray tracing?
- When $p = 2f$, is the image real or virtual? Erect or inverted? Calculate the magnification.
- Explain the sign convention for the lens equation.
- When $p = f$, where is the image? Explain both answers.
- As p is increased, the magnification increases / decreases.

- If a stronger lens is used, the magnification: increases / decreases.
- If a stronger diverging lens is used, the magnification: increases / decreases.
- When does a converging lens give a virtual image?
- Why is a wavefront spherical?
- In the lens equation, "curvature$_{\text{in}}$" is usually negative. Why? When is it positive?
- What is meant by a virtual object?
- Derive the formula magnification = x_I/f by tracing rays.
- Why can a diverging lens not form a real image?
- What is meant by the focal plane of a negative lens?
- The glasses worn by farsighted people are converging / diverging lenses. They form a real / virtual image of an object, apparently bringing it nearer to / farther from the eye so that it looks larger / smaller.
- The glasses worn by nearsighted people are converging / diverging lenses. They form a real / virtual image of an object, apparently bringing it nearer to / farther from the eye so that it looks larger / smaller.
- To increase the angle of reflection θ_r, you increase / decease the angle of incidence θ_i (Snell's Law).
- Doubling θ_i can double θ_r. True / False. Explain.
- A narrower pencil of rays makes a sharper image. True or False. What is spherical aberration?
- A narrower pencil of rays gives greater depth of field. True or False. Explain.
- According to the Lens Maker's Formula, if two lenses have the same shape, the one having the larger index of refraction will be stronger. True or False. Explain.

PROBLEMS

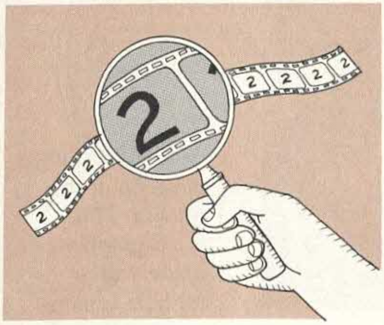
★★ **1. Magnifying Glass** A pocket magnifier might have a focal length of 20 cm.

(a) If the lens is held 1 cm from the book, hardly any magnification is achieved (see figure). Explain. Approximately where is the virtual image of the print?

(b) If the lens is held more than 20 cm from the book, the print looks upside down. Explain.

(c) When the lens is held 10 cm from the book, the magnification is 2. That means the print looks twice as big as it is. As the lens is moved away from the book, does the magnification keep increasing?

(d) Is it possible to achieve a magnification of 4 with this hand lens?



PROBLEM 1 Magnifying glass (hand lens).

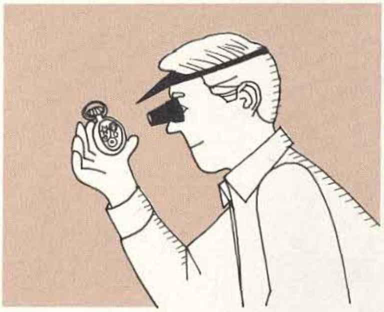
✶✶ **2. Eye Loupe** Jewelers sometimes wear a strong magnifying lens close to one eye in order to get close to their work (see figure). The lens might be as strong as +10 diopters.

(a) A young jeweler can bring the work to 24 cm from the naked eye. Argue that a loupe that permits bringing the work to 8 cm achieves a three-fold angular magnification.

(b) If the work is held more than 10 cm from the lens, it looks blurred. Why? Convince yourself that you could not see the real image formed by the lens. Where is it?

(c) If the work is brought very close to the lens, it also looks blurred. Why? Where is the virtual image?

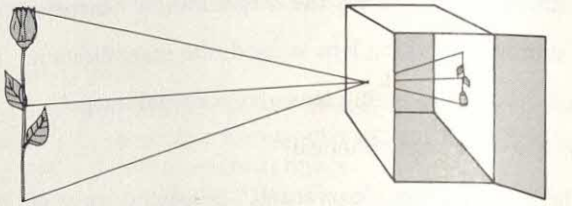
(d) If the work is held 8 cm from the lens, the image is nice and sharp. Approximately where is it?



PROBLEM 2 Jeweler's eye loupe.

✶✶ **3. Pinhole Camera** If you exposed a sheet of photographic film briefly to the scene in front of your eyes, no image would be recorded. The film would be uniformly exposed all over. This is because each point of the scene (object) scatters light in all directions, so each point of the film receives light from all points of the object, the bright ones and the dark ones. But if all the light has to go through a pinhole, an image is formed. Each spot of the film receives light from only one spot of the object (see figure).

Problem 3 Finger pointing at the camera. This photograph was made with a pinhole camera.



PROBLEM 3 Pinhole camera.

(a) Suppose that the object to be photographed is 20 cm long and the film is 10 cm long. Show in a sketch how to place the pinhole and the film in order to get the whole image on the film.

(b) If the pinhole is to be 1 meter from the object, how far in back of the pinhole should the film be? (HINT: Similar triangles.)

(c) The smaller the pinhole, the sharper the image. What is the disadvantage of too small a pinhole?

(d) What are the advantages of a lens versus a pinhole?

✶✶ **4. Speed of Light** You always see the lightning before you hear the thunder. So you know that light travels faster than sound. How much faster? Almost a million times. Speed-of-sound measurements are done by timing echoes over measured paths.

(a) With good electronics (fast oscilloscopes), times can be measured to nanosecond accuracy. How long a round-trip path is needed in order to measure the speed of sound in air (330 m/sec) to ±1%?

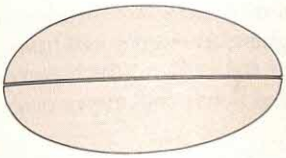
SOLUTION: 1 nsec should be 1% of the round-trip time, which should therefore be 100 nsec. Round-trip path = 330 m/sec × 100 nsec = ________________.

(b) The speed of light is 3×10^8 m/sec. Can we do a ±10% measurement in the class room? Argue.

(c) Why can't we do even an order-of-magnitude measurement with a stopwatch?

(d) In 1926 Michelson timed light over a 44-mile round-trip path and measured the speed to one part in 10^6. How accurate was his timing device? His surveying of the distance was good to one part in 10^7.

✶ **5. Two Lenses** A lens with one plane face and one convex face is called planoconvex. Two identical planoconvex lenses are pressed together, flat faces joined (see



PROBLEM 5

figure). Compare the power of the combination with the power of each.

✶✶ **6. Compound Microscope** The objective is a very strong lens. The object is placed just outside its focal length so that it forms a real image well beyond the other focal plane. The eyepiece, a weaker magnifier, forms a virtual image at the distance of most comfortable viewing (see figure).

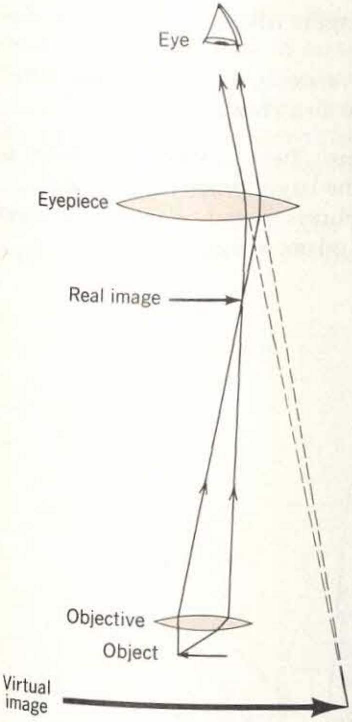
(a) Explain why the virtual image is upside down.

(b) If the object is placed closer than the focal length of the objective, all you see is a blur. What is happening?

(c) If the object is too far away from the objective, the microscope has little magnification. Show that the real image formed by the objective is actually smaller than the object if the object distance is more than twice the focal length of the objective.

(d) If the object is just barely outside the focal plane of the objective, the real image will fall behind the eyepiece. Show that the eyepiece can not then form a virtual image.

(d) The stronger the eyepiece, the greater the magnification of the microscope. Yet most microscopes are made with low-power eyepieces. Why is it easier to use a microscope with a low-power eyepiece?



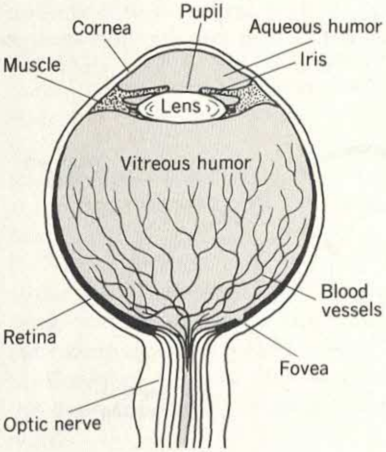
PROBLEM 6 Compound microscope.

PROBLEM 6 Using a binocular microscope.

✶✶ **7. Oil-Immersion Microscope** To reduce spherical aberration of the objective, some microscopes are designed so that the object is immersed in an oil having roughly the same index as the glass of the lens, with which it is in contact. The front surface of the lens is usually planar.

Show that the curvature of the front surface is arbitrary, that is, does not affect the lens power.

✶ **8. Eye** The human eye is about 24 mm in diameter (see figure). Show that when focused at infinity, its



PROBLEM 8 Human eye.

optical system (cornea, humors, and crystalline lens) has a power of more than +40 diopters.

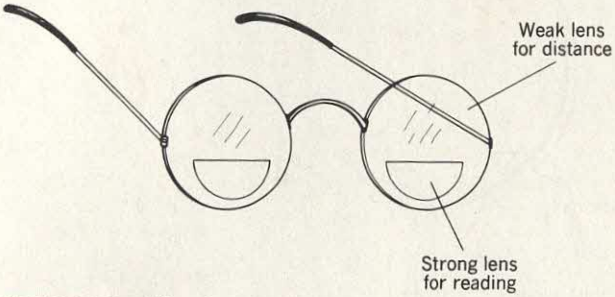
NOTE: The optician won't sell you an ophthalmic lens much stronger than +12 diopters.

✶ **9. Graph the Lens Equation** Take $1/f = +2.0$ diopters. Plot q versus p for positive values of the object distance p.

✶✶ **10. Laser Surgery (Photocoagulation)** A noninvasive technique for the treatment of a detaching retina is to make tiny burns with a narrow laser beam. The scar tissue tends to adhere well to the sclera which backs the retina. Good aim is crucial to avoid hitting the fovea, the part of the retina where vision is most acute. Suppose that the surgeon permits himself an error no greater than 1 mm on the retina. How closely does he have to control the aiming angle of the laser? Express the permissible aiming error in degrees. Assume a +40-diopter optical system. Compare with the accuracy with which a gun must be aimed in order to hit a 30-mm bull's eye at 30 meters.

✶✶ **11. Reciprocity** There is a theorem that for any optical system, the object can be interchanged with the image. That is another way of saying that light can go backward along any ray path.
(a) Recount an experience you have had with lenses or mirrors to support this theorem.
(b) If you turn a Galilean telescope (opera glass) around, it makes the scene appear smaller. Why does this not violate the reciprocity theorem?

✶ **12. Bifocals** People who have lost some of their power of accommodation often own two pairs of glasses, one for "distance," one for reading or close work (see figure).
(a) For a farsighted patient, which pair has the stronger lenses?

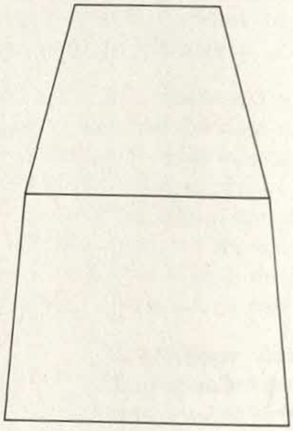


PROBLEM 12 Bifocals.

(b) Instead of having two pairs of glasses, the two lenses can be ground on one piece of glass, the upper half having one prescription, the lower half the other. Which part of this *bifocal* would be the reading lens, the upper or the lower? Explain.
(c) Suppose a patient needs a +1.0-diopter lens to see distant objects clearly. How strong a lens does he or she need to focus on an object 25 cm away?

✶✶ **13. *f*-Stops** The aperture of a lens is conventionally given in terms of *f*-numbers. The larger the lens, the smaller its *f*-number. An *f*/4 lens has a diameter equal to its focal length divided by 4.
(a) If that lens is now stopped down to *f*/8 (that means diameter = focal length divided by 8), only half the diameter lets in light. Show that for photography the exposure time should be 4 times as long.
(b) If you wanted an *f*-stop giving exactly double the exposure time of the *f*/4, show that you should stop down to *f*/5.6. (HINT: $\sqrt{2} = 1.41$.)
(c) If you wanted only half the exposure time, you would need a larger lens. What should be its *f*-number?
(d) On most cameras, the available *f*-stops are arranged so the exposure time doubles from one *f*-stop to the next. After *f*/5.6 the next smaller aperature is *f*/8. Then comes *f*/11. What is the next one?
(e) For a 50-mm focal length, what is the diameter of an *f*/1.4 lens?
(f) Conjecture why camera stores don't sell lenses with *f*-numbers less than 1. (See also Problem 32.)

✶✶ **14. Prism → Lens** Two prisms are placed as shown, the one having the larger prism angle on top.
(a) Show that two parallel rays, one incident on the upper prism, one on the lower prism, are converged.



PROBLEM 14

(b) A **cylindrical lens** can be thought of as made up of many tiny prisms joined so the prism angle changes smoothly. Show that the power of the lens is greater the more rapidly the angle changes across the lens.

✶✶ 15. Ray Through a Window

(a) Show that a ray incident on a glass plate with parallel sides comes out the other side going in the same direction.
✶✶✶(b) Show that the emerging ray is offset for oblique incidence and that the amount of offset increases with the angle of incidence. A graphical "proof" is sufficient.

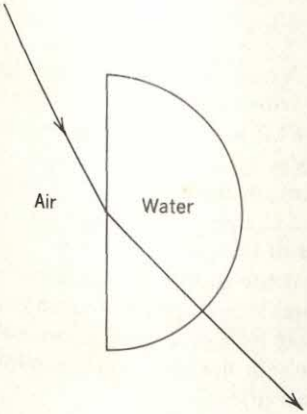
✶ 16. Snell's Law

(a) A ray strikes the horizontal water ($n = 1.33$) surface at a 1° angle with the vertical. What angle does the refracted ray make with the vertical?
(b) If the angle of incidence is doubled to 2°, what is the angle of refraction?
(c) If the angle of incidence is 45°, what is the angle of refraction?

✶✶ 17. Snell's Law

(a) Is there any angle of incidence on water that would be twice the angle of refraction? Yes/no? If No, prove. If Yes, give an example.
(b) Same question for crown glass, $n = 1.52$.

✶ 18. Snell's Law A semicircular plastic cheese box makes a good apparatus for measuring angles of refraction (see figure). Fill it with water or some other liquid. (Avoid carbon disulfide; it dissolves the plastic.) Shine a light ray so that it enters at the center as shown. It will leave the box at normal incidence, permitting easy measurement of the angles.
✶(a) At what angle should the incident ray strike so the emerging ray leaves at 45°? Solve for water, $n = 1.33$.



PROBLEM 18 Snell's Law. Fill a transparent cheese box with water. All you need is a protractor to measure the angles.

✶(b) Argue that the parallel sides of the plastic box have no effect on the direction of the rays.
✶✶(c) Design an experiment using the box to measure the critical angle of water.

✶✶ 19. Atmosphere as a Lens The index of refraction of air is just a bit greater than unity—about 1.0003 at atmospheric pressure. Sketch a ray to show how refraction by the atmosphere actually makes the sun appear higher than it is. Would this make sunset earlier or later than the astronomers predict?

✶✶ 20. Two Refractive Media: A Glass Aquarium Snell's Law tells us how a light ray is bent in going from air to water:

$$\frac{\sin \theta_{\text{air}}}{\sin \theta_{\text{water}}} = n_{\text{water}}$$

Similarly, for a ray going from air to glass, we have

$$\frac{\sin \theta_{\text{air}}}{\sin \theta_{\text{glass}}} = n_{\text{glass}}$$

How about continuing that ray from the glass into the water? If you write the Snell's Law equation

$$\frac{\sin \theta_{\text{glass}}}{\sin \theta_{\text{water}}} = n$$

What is the n? Answer: $n = n_{\text{water}}/n_{\text{glass}}$
(a) Prove.
(b) Why are you not surprised that this index is less than unity?
(c) Show that the index of refraction of the glass does not affect how far from the side of the aquarium the fish appears to be when viewed through the thin glass. Only the index of water matters. Sketch!

✶✶ 21. Coin in the Fountain [Some Trigonometry Required] The drain at the bottom of the swimming pool is always at the deepest place.
(a) Why?
(b) If the pool is 4 meters deep there and you are looking at the drain from directly overhead, how deep does it appear to be?
(c) Suppose that you are standing 2 meters from the edge of the pool. With your eyes 2 meters higher than the water level, you see the drain appearing at the water's edge. How deep does it appear to be?
(d) Convince yourself that the further away you are from the drain the higher its image. Then put the argument in words.

★★ **22. Coin in the Fountain** Looking down from above, the onion at the bottom of the martini does not appear to be as far below the surface as you know it really is. Let's trace two rays to see why (see figure). One ray can be the one going straight up; this is not bent by the water-to-air surface. The other ray makes an angle θ_{water} with the surface normal and emerges into the air making an angle θ_{air}. The size of the angles is exaggerated in the sketch. Both rays have to enter the eye! According to Snell's Law, we have

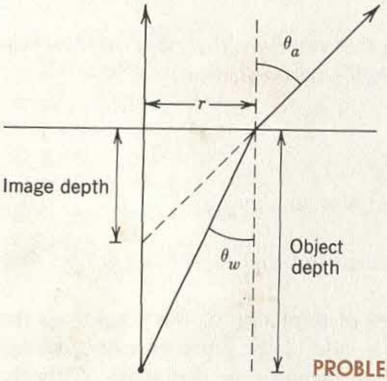
$$\sin\theta_{\text{air}}/\sin\theta_{\text{water}} = n_{\text{water}}$$

Geometry:

$$\text{object depth} = \frac{r}{\tan\theta_w}, \qquad \text{image depth} = \frac{r}{\tan\theta_a}$$

$$\frac{\text{image depth}}{\text{object depth}} = \frac{\tan\theta_w}{\tan\theta_a} \approx \frac{1}{n} \qquad \text{for small angles}$$

Take $n = 1.33$ (water?) and calculate how deep the onion appears in a 3-inch martini.



PROBLEM 22

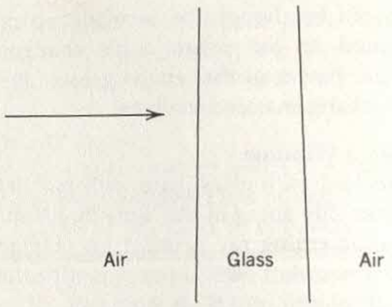
★★★ **23. Law of Refraction** In certain crystals, the refracted ray is not in the plane of incidence for all angles of incidence. Use a symmetry argument to prove that it has to be in the plane of incidence for a glass, which is an isotropic material.

★★ **24. Infrared and Ultraviolet** Ordinary glass is rather opaque in the ultraviolet but quite transparent in the infrared. Infrared photography requires special emulsions; film sensitive deep into the IR spectrum can be bought at most photo stores. Why don't photo stores sell special film for ultraviolet photography? (HINT: UV photography is possible with lenses made of quartz.)

★ **25. Prism** Unlike a lens, the faces of a prism are plane. Unlike a window, the faces are not parallel. In an optical system a prism can be used for bending a beam of light through an angle. An incident parallel beam comes out as a parallel beam.

(a) Show that a horizontal ray incident on the prism of the figure emerges bent downward.

(b) If the incident ray were aimed slightly upward instead, would it emerge more or less inclined than before? Show by drawing rays.



PROBLEM 25

★★ **26. Small-Angle Prism** A crown-glass prism ($n = 1.52$) has a 5.7° angle between the opposite faces. A ray normally incident on one face meets the other face at an angle $\theta_{\text{glass}} = 5.7°$.

(a) Show.

(b) At what angle θ_{air} does it emerge?

(c) Show that it is turned through an angle of 2.96°.

(d) Show that for small angles of incidence and small prism angles α, the angular deviation of an incident ray is $(n - 1)\alpha$.

★★ **27. Chromatic Aberration** The index of refraction of crown glass varies about 1.4% across the visible spectrum:

Wavelength	Index of Refraction
4047 Å (violet)	1.532
7682 Å (red)	1.512

(a) A certain lens has a focal length of 50 mm with violet light. What is its focal length with red?

(b) A certain camera has an $f/1.2$ lens, which means that its diameter is equal to its focal length divided by 1.2. A ray parallel to the axis incident on the edge of the lens is bent through an angle of ________ degrees. Remember, it comes to a focus at the center of the picture. Sketch.

(c) In part (a) you noticed that the index of refraction enters the lensmaker's equation as $n - 1$. In Problem 26 you found that the deviation of a ray by a prism is proportional to $n - 1$ also. By what angle will the violet be separated from the red for the ray of part (b)?

✶ **28. Using Prisms as Mirrors** This is an application of total internal reflection useful in fine optical instruments.

(a) The index of refraction of crown glass is 1.52. Calculate the critical angle. Note that it is less than 45°.

(b) Show by a sketch that a ray entering a 45° right-angle prism normal to one of the small faces emerges bent through 90°.

(c) Show that a ray entering such an isosceles right-angle prism normal to the large face emerges bent through 180°.

(d) Show how to use the prism as in part (b) to see an inverted image.

(e) (Harder) Show that a ray incident on the large face at a small angle emerges from that face at the same angle, that is, turned through 180°.

✶✶ **29. Opera Glass or Galilean Telescope** The objective lens, a low-power magnifier, forms a real inverted image of the distant object. The eyepiece is a negative lens.

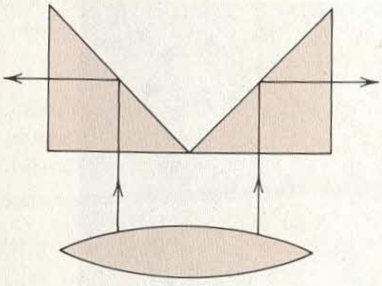
(a) Show that this allows the eye to see an erect virtual image.

(b) Show that a positive eyepiece can only give an inverted virtual image. Such a device is called an **astronomical telescope** and gives greater magnifications.

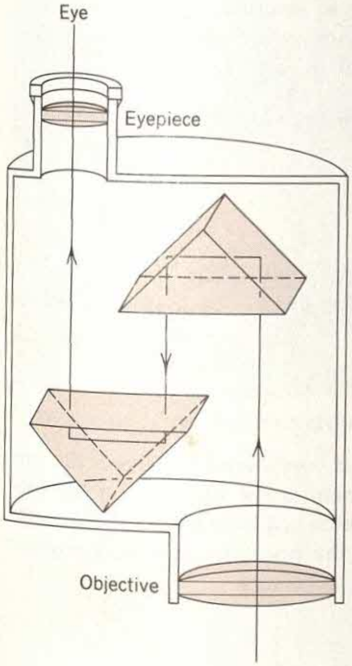
✶✶ **30. Prism Binoculars** Each ocular is an astronomical telescope with two inverting prisms in the optical train (see Fig. *a*) One turns the image right-side up; the other gives a right-left inversion, to make it "true."

(a) Show that four reflections, each a right-angle turn, can do exactly that.

(b) Two 45° prisms placed behind a microscope objective can be used to split the beam for making a binocular microscope (see Fig. *b*). What possible advantages might a binocular microscope offer over a monocular instrument?



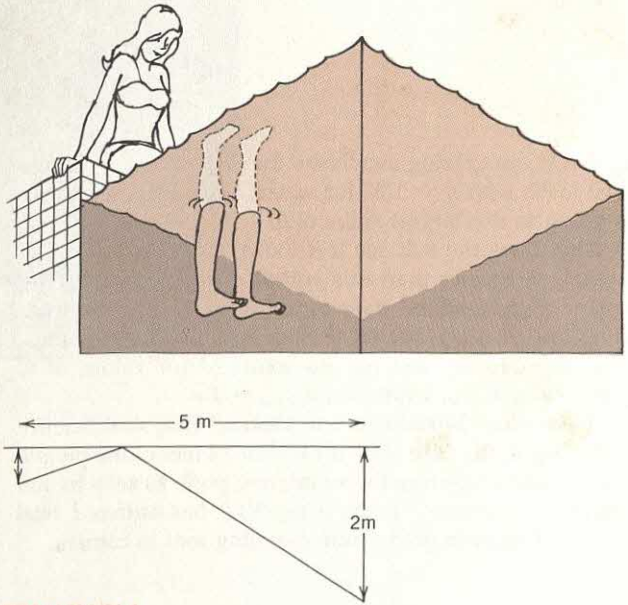
PROBLEM 30*b* Binocular microscope. Light goes through a single objective, is split by the 45° prisms.



PROBLEM 30*a* Prism binoculars—really two astronomical telescopes, one for each eye. The objectives lens gives an inverted image. The first prism reinverts it; the second prism interchanges left and right so that the distant object looks natural.

✶✶ **31. Critical Angle** In Snell's Law, $\sin \theta_i/\sin \theta_r = n$, the angle θ_i is for the ray outside the refracting medium, the angle θ_r for the ray inside, regardless of which way the light travels.

(a) A fish sees the world outside the water compressed into a rather narrow cone. Substitute $\theta_i = 90°$, correspond-



PROBLEM 31*d*

Total internal reflection. Several distinct pencils of light enter the glass prism from below at slightly different angles. The first four from the left are refracted at the long surface of the glass. The others reach that surface at angles greater than the critical angle and hence are totally reflected there. If you look carefully, you can see that the first four are also partially reflected, increasingly so as the critical angle is approached.

ing to a ray at grazing incidence, the largest that θ_i can be. Solve for θ_r with $n = 1.33$ for water. Sketch the ray corresponding to this largest value of θ_r.
(b) What does the fish see if it looks up at the surface at an angle θ_r greater than this critical angle? It sees the reflection of the bottom! Rays incident on the surface from below are totally reflected if they strike at large angles. Show that Snell's Law has no solutions for values of θ_r greater than θ_{critical}, where $\sin \theta_{\text{critical}} = 1/n$.
(c) In the film "Introduction to Optics" (Physical Science Study Committee) are shown two sets of feet of the people dangling their legs into the swimming pool, as seen by the underwater camera. Sketch a ray that has suffered *total internal reflection* in going from dangling foot to camera.
(d) (Harder) If the foot is 0.5 m below the surface and the camera is 2 m below the surface and 5 m from the pool edge, at what angle does it have to point to see the "flying" feet? (See figure) (HINT: If you have forgotten your trigonometry, use a sheet of graph paper.)

✻✻✻ **32. Depth of Field and *f*-Number** (For definition, see Problem 13.) A camera with an *f*/4 lens (a lens whose diameter is $\frac{1}{4}$ its focal length), when focused at an object 3 meters away, also has in good focus objects between 2.6 m and 3.6 m. When the lens is "stopped down" to *f*/11, the depth of field becomes greater: All objects between 2.1 m and 5.4 m are in focus. When the lens is further stopped down to (a tiny aperture) *f*/22, everything between 1.6 m and infinity is in good focus.

The same camera is now focused at 1.5 meters, by increasing/decreasing the distance between lens and film plane. Wide open (i.e., *f*/4), only objects between 1.4 m and 1.7 m are in good focus. At *f*/22, everything between 1.1 m and 2.4 m is in good focus. Estimate the depth of field at *f*/11. (HINT: The data are rough, so your estimate is going to be a rough estimate. You have not been taught how to solve this problem. What you have to go on is your physical intuition and your recollection that, in geometrical optics, it is *reciprocal* distances that enter the calculations. Try making a table of the data. If you make a graph, plot the reciprocal distances.)

✻ **33. Swim Goggles** In swimming, when you open your eyes under water, things look blurred. With goggles, you can focus sharply. Explain carefully.

✻✻ **34. Proof of Lens Maker's Formula** Start from Eq. 11.4–2,

$$\frac{n}{q_1} = -\frac{1}{p_1} + \frac{n-1}{r_1}$$

for the front surface of a lens with radius of curvature r_1.
(a) A similar equation holds for the back surface, where light goes from glass to air:

$$\frac{1}{q_2} = \frac{n}{p_2} \oplus \frac{n-1}{r_2}$$

The radius of curvature r_2 is considered positive if the surface is convex. The argument is the same as for the previous equation, but the sketch is a little different, since the rays are bent away from the normal. Draw it. Convince yourself that the circled + is correct.

(b) If the lens is thin enough, the image distance q_1 for the front surface is the same as the object distance p_2 for the back surface. Make this substitution and solve for $1/q_2$ in terms of $1/p_2$. Your result will have the form of the lens equation 11.2–1. Show that the power is given by Eq. 11.4–3.

✶✶ **35. Negative Lens** Repeat the above derivation for concave surfaces. The algebra is all the same; only the sketches are different.

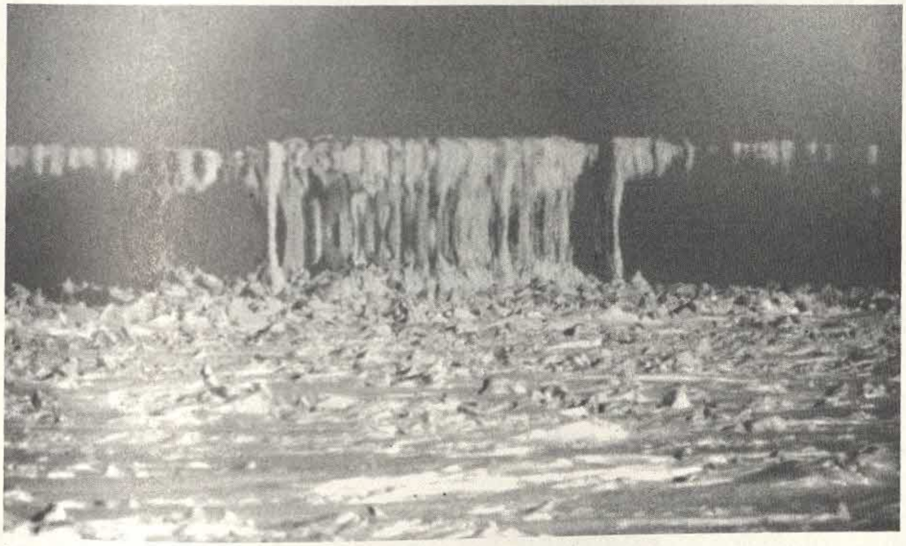
✶ **36. Lens Design** Use the Lens Maker's Formula to calculate the power (in diopters) of a plano-convex lens made of glass of index $n = 1.50$ with front surface curvature $4.00\ \text{m}^{-1}$, that is, radius of curvature 0.25 m.

☐ QUESTIONS FOR FURTHER THOUGHT

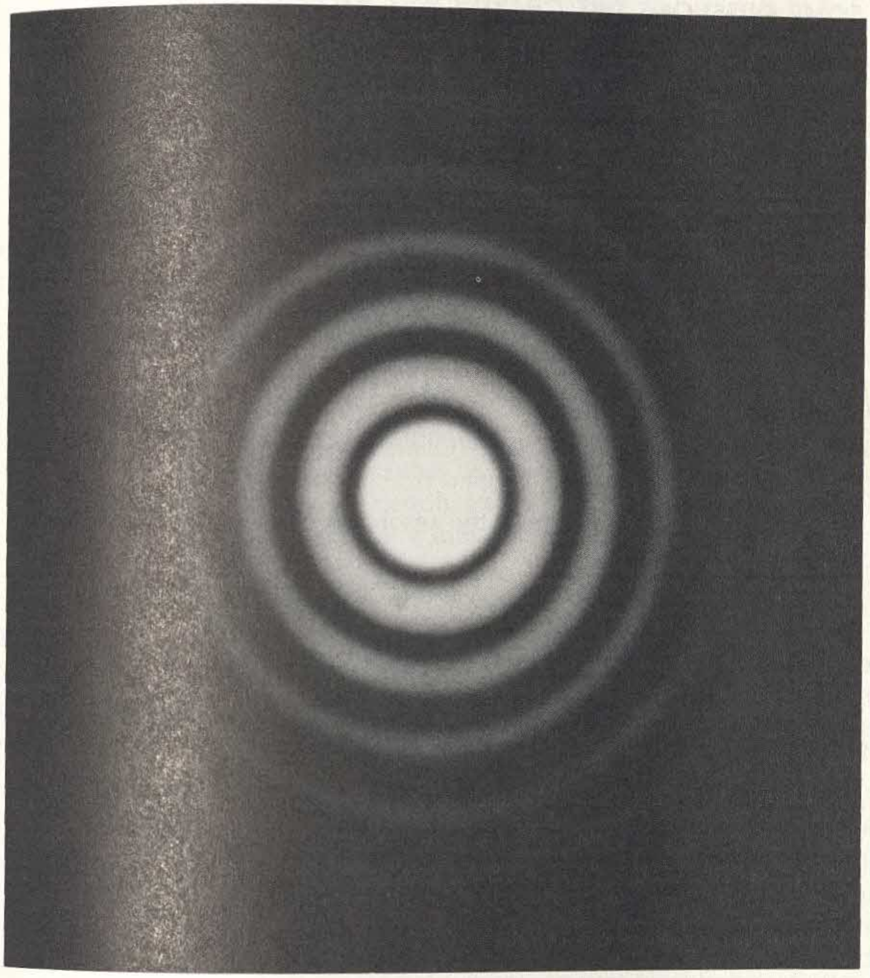
α. Whan an astronaut looks at the moon, it has very sharp edges. When he looks back at the earth, its edges are fuzzy. Explain.

β. **Eye Surgery** The aqueous humor ($n = 1.33$) is contained between the cornea ($n = 1.37$) and the crystalline lens ($n = 1.4$). If the cornea is removed surgically, the power of the eye's optical system is cut by more than half. Explain.

γ. **Mirages** Hot air has a lower index of refraction than cold, simply because it is less dense. In the desert, air close to the ground tends to be much warmer than air higher up. Show how such an air structure can give inverted images of distant objects. Sketch!



Mirage: Fata Morgana over arctic ice. (See also Problem 11.19.)



Diffraction pattern made by monochromatic light from a point source going through a circular hole.

12

LIGHT AND VISION– WAVE OPTICS

SOME QUESTIONS THIS CHAPTER WILL ANSWER

How do you know that red light has longer wavelengths than blue light?

Why is the sky blue?

Why is the sunset red?

In a double-slit interference pattern, how do you calculate at what angle from the central maximum (bright line) the first-order maximum (bright line) can be seen?

What determines how many orders appear on each side of the central maximum?

Why are the bright lines narrower when there are more than two slits?

Why is there no "single-slit pattern" for a slit much narrower than one wavelength?

Explain how diffraction limits the resolving power of an eye or telescope.

What is a hologram?

Give evidence that the glare from a highway is polarized light.

12.1 COLOR AND WAVELENGTH

Long before you studied physics you may have known about "light waves." How did you know? Somebody told you that light travels like a wave or you read it in a book. In this chapter we will discuss some properties of light that require a wave explanation.

You may already know that red light has longer wavelengths than blue light. Remember the order of the colors in a rainbow? The red is on the outside; then comes the orange, the yellow, the green, finally the blue, trailing off into violet on the inside of the **primary rainbow.** That is the one we usually see when the sun is shining during the rain. It is caused by internal reflection of the sunlight in the rain drops. That is why you see the rainbow in a direction away from the sun. Do you know how that works? If light hits a good-sized spherical drop at a small angle, that is, pretty squarely, most of it goes through; the drop functions as a lens. If the angle of incidence is larger, the light is refracted (bent) as it enters, and more of the light suffers *internal reflection.* In leaving the drop when it gets to the surface again, the light is refracted a second time. Green light is turned through an angle of 138° or less. Red light is turned through a slightly smaller angle, blue light through a slightly larger angle. (See Problem 4)

This says that the index of refraction of water is different for different wavelengths. The variation of the index of refraction with wavelength is called **dispersion.** The higher the index of refraction of a substance, the more we tend to notice the dispersion. That is what makes the pretty colors of a cut diamond. Without attempting an explanation of dispersion, we just tell you that it goes the same way in all transparent materials: The short wavelengths have the higher index of refraction. Whether we are talking about water or glass or diamond, the blue light travels more slowly than the red.

How about the color of the sky? On a clear day it is blue. On a cloudy day it is white or gray. Explanation: The droplets in the clouds are much smaller than the raindrops that cause rainbows, but they tend to be bigger than any of the wavelengths of visible light. Consequently, waves having all those wavelengths are scattered in all directions. All colors of light go every which way. When all the colors of light are mixed, what you see is white or gray. What makes the sky blue? That has to do with scattering by air molecules. A single molecule of nitrogen is so much smaller than a wavelength of light that it hardly scatters. Appreciable molecular scattering requires a clump of molecules. There has to be a statistical fluctuation in the local air density in a region many molecular diameters in size that has a refractive index slightly larger than average. To be sure, small clumps occur more often than large clumps. But small clumps scatter short-wavelength light more effectively than long-wavelength light, blue more than red. If there were no scattering from air molecules, the sky would look black. That is how astronauts above the atmosphere see it. Down here in the atmosphere, thanks to density fluctuations, blue is scattered more effectively than red, and the clear sky looks blue.

How do you know that long wavelengths are less effectively scattered than short wavelengths? You learned that about sound waves when you read Chapter 10 on Sound and Hearing. A sound wavelength about the size of your head is roughly in the middle of the range of audible wavelengths. Longer wavelengths (lower tones) **diffract** around your head easily so that you hear them with both ears regardless of what direction the signal may be coming from. Shorter wavelengths (higher tones) tend to be reflected if they hit your head. So if the signal comes from the right, the direct signal is heard mainly by the right ear. Just how sensitive diffraction is to the size of the obstacle is very apparent at a rocky beach or with a branch sticking out of the brook. Only the waves with wavelengths short compared with the dimensions of the obstacle are appreciably reflected. The longer waves go right around the obstacle. In the bathtub those long waves you make by sliding your whole body back and forth barely notice an arm or a leg. The short-wave ripples can reflect even from a finger poking through the surface.

☐ QUESTION FOR FURTHER THOUGHT Sunsets

In our polluted cities, the air close to the ground often contains dust particles whose sizes are comparable to the wavelengths of visible light. These will scatter the short wavelengths more effectively than the long. Explain why, at sunrise and at sunset, when the direct rays of the sun have to traverse a lot of dirty low-altitude air, the sun often looks very red.

The Visible Spectrum

From Chapter 10 on "Sound and Hearing" you may remember that the highest sound frequencies you can hear are about a thousand times the lowest. This means that the shortest audible sound wavelengths are about one-thousandth the length of the longest. With light waves, the longest reds are about twice as long as the shortest violets. There is, as with sound, some variation among different people in the cutoff at both ends of the spectrum, but only a few percent. We like to use the word *light* even for wavelengths we can't see. Wavelengths somewhat longer than red are called infrared (IR). Wavelengths somewhat shorter than violet are called ultraviolet (UV).

☐ QUESTION FOR FURTHER THOUGHT

Infra means below; *ultra* means above. If infrared wavelengths are longer than red and ultraviolet wavelengths shorter than violet, aren't the prefixes backward? Conjecture why we call the red end of the **visible spectrum** the low end, the violet end the high. Compare with the piano keyboard.

The Sensation of Color

The sensation of pitch is well correlated with the frequency of the incident sound, and different frequencies of sound excite different parts of the basilar membrane of the cochlea. Is there a comparable correlation between color and light frequency? Is there a comparable mechanism for color perception in the retina of the eye? No. It has to be very different, because the retina has to perceive a two-dimensional image. Each little spot on the retina receives light of different color and intensity when we look at the scene before our eyes. The angular resolution of which we are capable requires millions of separate photoreceptors. These are the rods and cones of the retina. The brighter the light, the faster the rate at which these receptors fire, sending electrical signals to the optic nerve. The rods are mostly on the peripheral parts of the retina. They are sensitive to low light intensities and are used for night vision. The part of the retina capable of acutest vision, the **fovea centralis,** has mostly cones, about 10^5 per square millimeter. This central part of the retina is also where our color discrimination is keenest.

There are three different kinds of cones, having maximum sensitivity to red, green, and blue light, respectively. According to one theory of color vision, each wavelength of light is characterized by a certain pattern in the rate of firing of the three sets of cones. But with only three coordinates to specify each color, you can see why mixtures of two or three very different wavelengths could look the same as one pure wavelength. Painters use this fact to achieve a continuous range of colors by mixing relatively few pigments. In stage lighting, all colors can be produced with three colors of light bulbs.

☐ QUESTION FOR FURTHER THOUGHT Color Mixing

You can get light of a pure spectral color using a prism to break up a beam of white light into separate colored beams. Different wavelengths come off at different angles because the glass is *dispersive:* Its index of refraction is different for different wavelengths.

If you mix a beam of spectrally pure yellow and a beam of spectrally pure blue in the right proportions, the result is white light. This is called color addition. This addition occurs in the brain. Two colors that add to give white are called complementary colors.

(a) Explain why laundries use bluing to make a yellow wash look white.

In mixing paints, we use color subtraction. The subtraction occurs in the pigment. Blue paint absorbs strongly wavelengths on the red end of the spectrum. Yellow paint absorbs in the red and in the blue. If we mix blue and yellow pigments, the only part of the spectrum that is not strongly absorbed is the green.

blue + yellow → green (paint)

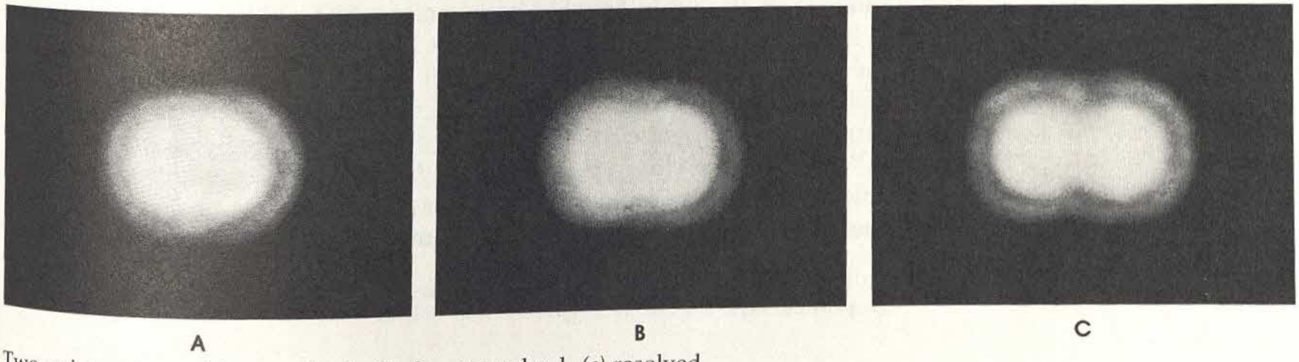
(b) If you mix all the paints on the artist's palette, will you get white or black? Experienced painters will avoid the question and answer gray. But explain what goes on. Addition or subtraction?

(c) Now explain why bluing works in the laundry and does not make the yellowed wash look green. (HINT: The pigment has to be sparsely distributed.)

12.2 DIFFRACTION

A car approaches on a dark road. Does it have two headlights or only one? From several miles away, the headlights blur together, and you can't be sure. As the car gets closer and the angular separation of the headlights increases, you are able to **resolve** the two sources of light.

The blurring together of two distant points of light occurs no matter how sharp the focus, no matter how perfectly your eyes may be focused at infin-



Two point sources: *(a)* unresolved, *(b)* almost resolved, *(c)* resolved.

ity. Also, it is not a peculiarity of headlights. It has to do with light not quite traveling in straight lines. In Chapter 10 on "Sound" we talked about the diffraction of sound waves. Not surprisingly, we find that light waves too are diffracted. Try moving farther and farther away from the printed page while you read. As the angular size of the letters decreases, they become indistinct. The details blur.

How close can two point objects be and still be seen as two objects? At 10 meters, the answer is about 1 millimeter for a person with excellent visual acuity. At 100 meters, the closest separation is about 1 centimeter. So the smallest angular separation we can resolve with the naked eye is one in ten thousand, or 10^{-4} radians. Can you do that well? Can you distinguish a I from a II at 10 meters, in this type face? I can't. (See Fig. 12.1.) But if we had bigger eyes, we could do better. Here is an experiment to convince you that it is the size of the opening the light goes through that limits your resolving power.

I II

FIGURE 12.1

Punch a small hole in a piece of paper, hold it right up to your eye, and look through the hole. A needle or pin makes a nice round hole, but a pencil point will do. If you use aluminum foil instead of paper, you can experiment with tiny holes. You find that you have lost much of your ability to resolve fine detail.[1] Is it just because less light is getting to your eye? Notice that very bright objects also become indistinct when viewed through the small hole. You can do quantitative experiments using your book as an eye chart, and holes of various sizes. When the hole gets as large as the pupil of the eye, it no longer limits your resolving power. The size of your pupil does.

In the previous chapter we discussed various ways in which light can turn corners. We mentioned *reflection* and *scattering*. We discussed *refraction* at length. Now we turn our attention to the diffraction of light. As the light passes through the hole, it is diffracted by the edge of the hole. If you have not read Sec. 10.6 on diffraction of sound, you might do that before going on.

Diffraction by a Circular Hole

With a sewing needle make a small hole in a sheet of aluminum foil and wrap the foil around a strong light bulb so that one bright ray of light emerges. It should be roughly horizontal so that you can look at the point of light from various distances. Look at it through a small hole in a second sheet of foil. If you get your eye right in line with the filament of a 100-watt bulb, you will see the diffraction pattern: a set of concentric bright rings surrounding a bright circular disk of light. (See chapter-opener photo)

When you are close up, the central disk is just the size of the "source" hole next to the bulb. As you move away, the hole gets smaller at first, just because it subtends a smaller angle. But as you move further away, it remains the same angular size. It is as though the hole were actually getting larger as you back off. At a distance of several meters that bright disk is much larger than the size of the source hole. The reason is that the light is diffracted at the edge of the hole close to your eye. As you continue to back away, the diffraction pattern gets less bright, but the angular size of each ring remains the same.

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[1] At 10 meters. At close distances the pin hole may actually make images sharper, by limiting the pencil of rays. See Sec. 11.2

If you repeat the experiment with a smaller hole in the foil next to your eye, the pattern is bigger. If you measure, you will find that the angle subtended by the bright inner disk is inversely proportional to the diameter of the hole you are looking through. Except when very close up, it is independent of the size of the source hole. This is true for any of the concentric rings. Note that it takes a very bright bulb to see more than two rings of the diffraction pattern.

Now you can understand the difficulty of resolving two adjacent points of light. Make a second hole in the foil of the bulb, a millimeter or two from the first. Seen from close up, the two bright "central" disks have a dark region between them. As you back away, they start to overlap, to blur together. Since each disk subtends a constant angle, the dark region between them gets thinner and vanishes when that angle is no longer smaller than the angular separation between the centers of the two source holes. Repeat: The angular separation of the two points of light is equal to their linear separation divided by your distance from them. But the angular radius of each disk of light is independent of distance once you are a meter or two away. *When the angle between the sources is smaller than the angular radius of each disk, you can no longer resolve them.* Skill and judgment play a role, of course, but what we have here is a rule of thumb for the limit of resolution. This rule is called the **Rayleigh criterion.** You might be able to do better but not much better.

We have not answered the question of why we get those rings. Such alternating regions of light and dark are called diffraction patterns or interference patterns. The word *fringes* is also used, especially in situations where the stripes are straight lines rather than circles. In Sec. 12.3 on Interference, we show quantitatively how some of the simpler patterns are produced.

☐ LAB AT YOUR DESK Diffraction by a Round Hole

Use a magnifying glass to get rough measurements of the diameters of the holes in the foil close to your eye. Even with a sharp needle you will probably not succeed in making very round holes, so 30% accuracy is quite good. Then measure the angular radius of the bright central disk. This can be done by comparing its apparent size with that of an object whose size you measure. For example, if the radius of the disk is the same size as a 1-cm object held 2 meters away, its angular radius is 5×10^{-3} radians. That would be roughly the angular radius of the bright central disk with a 0.1-millimeter hole.

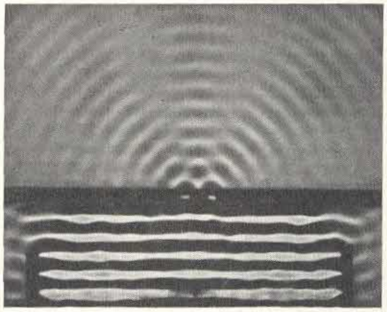
NOTE: A U.S. nickel (5-cent piece) is 10.5 mm in radius.

12.3 INTERFERENCE

In the Sound and Hearing chapter we saw how standing waves arise when progressive waves are reflected. The superposition of a wave traveling to the right and a wave traveling to the left results in a stationary pattern because the two waves retain their phase relationship. In Chapter 10 we were content to limit the discussion to one dimension. When a pattern of nodes and antinodes is two- and three-dimensional, physicists call it an **interference** pattern. You might try to stump your teacher by asking about the difference between "standing waves" and "interference." But you already know what is happening: Where crest meets trough and trough meets crest, there is a node. Where crest meets crest and trough meets trough, there is an antinode. With sound waves, you can move your ear to sample the alternating loud-soft pattern. With light waves, the pattern may look something like zebra stripes; alternating light-dark.

Before we analyze this "single-slit diffraction pattern," it will be instructive to learn about the pattern from two parallel slits. With two very narrow slits very close together this is called **Young's double-slit experiment.**

Double Slit



Double-slit interference pattern in the ripple tank.

Light of a well-defined wavelength falls on the double slit. The light source is equally distant fròm the two slits and far enough away so that we don't have to worry about the curvature of the wavefront: We have normally incident, monochromatic plane waves. Our slits are narrow enough so that the waves spread out after going through them. That is how we get interference between the two wavetrains. This is an important point that is often missed. Diffraction is necessary for interference to occur. Incidentally, interference is necessary for diffraction to be observed.

To observe the interference pattern, put a screen parallel to the slits and far away from them. "Far away" means many wavelengths away. This will allow some simplifying approximations later. On a line equidistant from both slits the waves from both will arrive in phase: crest with crest, trough with trough. That line is a line of antinodes, of *constructive interference*. How far away from this *central maximum* do the two wavetrains arrive with opposite phase? If you go either way from the central bright stripe, you get a dark stripe, a line of nodes. Here one slit is half a wavelength closer than the other. Half a wavelength is the distance between a crest and a trough. So a crest from the one gets there at the same time as a trough from the other. The interference between the two wavetrains is *destructive*. Somewhat farther there is a line a whole wavelength closer to one slit than to the other. This is a line of antinodes again, a bright line. Here a crest from one slit meets the preceding crest from the closer slit, giving constructive interference (see Fig. 12.2). The bright stripes are located where one slit is a whole number of wavelengths closer than the other. In between these interference maxima are interference minima, lines of destructive interference. Here one slit is an odd-half number of wavelengths closer than the other. Repeat:

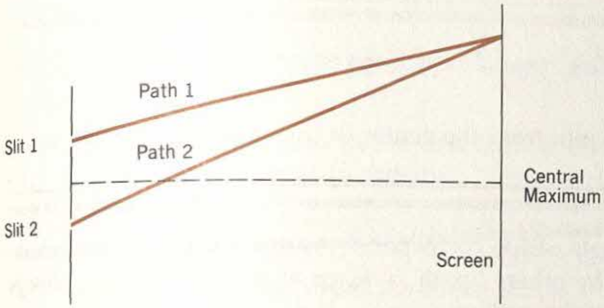
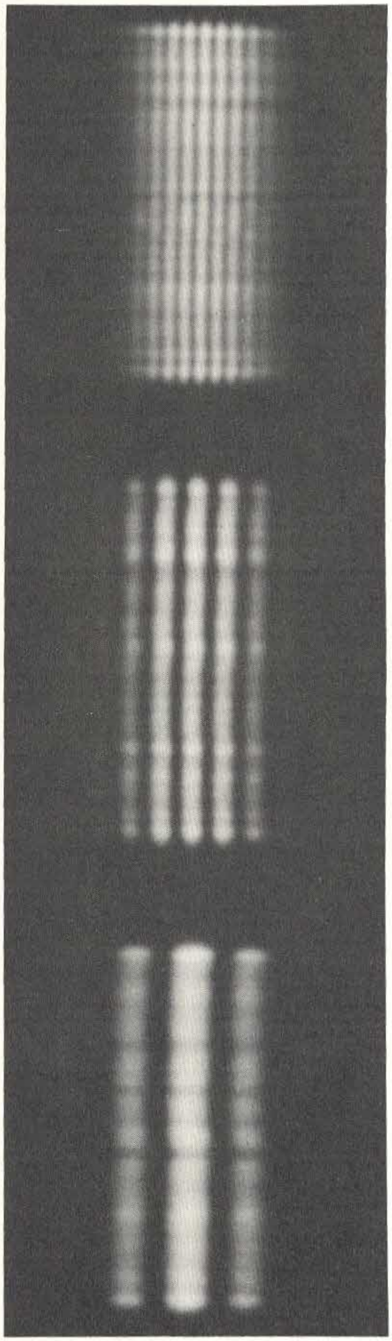


FIGURE 12.2 If the path difference = path 2 − path 1 is a whole number of wavelengths, the interference is constructive.



Double-slit interference patterns. The closer the slits, the farther apart the stripes.

$$\text{maxima} \rightarrow \text{path difference} = m\lambda$$
$$\text{minima} \rightarrow \text{path difference} = (m + \tfrac{1}{2})\lambda$$

The letter m stands for an integer. It is called the *order* number. The central bright stripe, or central maximum, corresponds to order number $m = 0$.

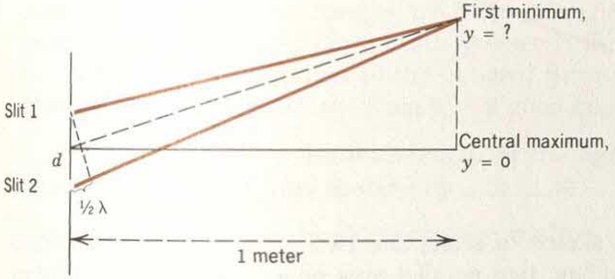


FIGURE 12.3 How far above the central maximum do the two wavetrains arrive half a period out of phase?

Let's ask again, how far away from that central maximum is the first minimum? Where on the screen is $\Delta\text{path} = \frac{1}{2}\lambda$? Take a wavelength in the middle of the visible spectrum, $\lambda = 5 \times 10^{-7}$ meters. We could also say 5000 Å or 500 nm; it's green. Take narrow slits, 0.2 millimeters apart, and a screen 1 meter away. The sketch of Fig. 12.3 is badly out of scale, of course. The angles are grossly exaggerated. Path 2 is supposed to be half a wavelength longer than path 1, or 2.5×10^{-7} meters longer. Both paths are just over 1 meter long. The dotted line that meets path 2 half a wavelength from slit 2 is the base of a long, thin isosceles triangle. It also defines a small triangle with the slits at two vertices. The geometry is made easy by the fact that this small triangle is so close to being a right triangle. That makes it almost similar to the big right triangle whose hypotenuse is the dashed line drawn from midway between the slits to the "first minimum" on the screen. That hypotenuse is just over 1 meter long. From the similar triangles we have the proportions

$$\frac{y_{\text{1st min}}}{\text{dashed hypotenuse}} = \frac{\frac{1}{2}\lambda}{d}$$

giving

$$y_{\text{1st min}} = \frac{2.5 \times 10^{-7}\ \text{m}}{2 \times 10^{-4}\ \text{m}} \times 1\ \text{m} = 1.25 \times 10^{-3}\ \text{m}$$

The first dark line is 1.25 mm from the center of the central bright stripe.

EXERCISE 1 The next bright stripe corresponds to one slit being one whole wavelength closer than the other: $\Delta\text{path} = \lambda$, or $m = 1$. Show that this is 2.5 mm from the center of the pattern.

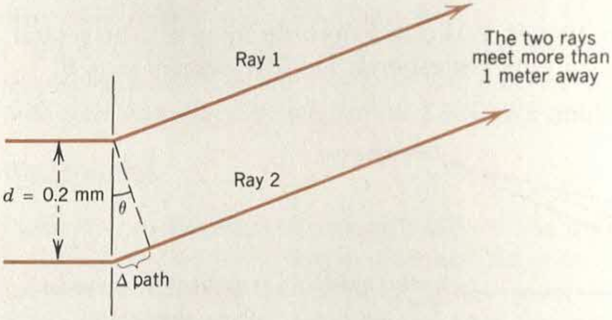


FIGURE 12.4 The two rays shown have been turned through an angle θ. From the right triangle, $\sin\theta = \Delta_{\text{path}}/d$.

If we had drawn the sketch to scale, the two rays would have looked parallel. Well, you might say, two parallel rays never meet. But these start one-fifth of a millimeter apart, and they meet 1 meter away. They are so nearly parallel that we are not confused by the question, through what angle θ have they been diffracted? To be sure, ray 2 has been turned through a somewhat larger angle than ray 1, but you hardly notice (Fig. 12.4). Consequently, you are content to label with the same symbol θ the angle made by our dotted line with the line connecting the slits. To the same approximation, that dotted line defines a right triangle, so

$$\sin\theta = \frac{\Delta\text{path}}{d}$$

Describing the interference pattern in terms of the angle θ of diffraction has the advantage that the description is independent of where we put the screen. If we replace the screen with a camera or an eye, angles are all we care about. We want to know at what angle from the central stripe the other stripes appear. The answer is that the bright stripes, the interference maxima, are at angles θ given by

$$\sin\theta_{\text{max}} = \frac{m\lambda}{d}$$

EXERCISE 2 At what angles are the dark lines? $\sin \theta_{\min} = ?$ Answer in terms of λ and d.

EXERCISE 3 At what angle is the first dark line with $\lambda = 5 \times 10^{-7}$ m and $d = 2 \times 10^{-4}$ m? Answer in radians.

EXERCISE 4 How many maxima are there on each side of the central bright fringe for our example, $\lambda = 5 \times 10^{-7}$ m, $d = 2 \times 10^{-4}$ m? [Answer: 400] The fringes having high order numbers, like $m = 398$ and $m = 399$, will be very dim. The last fringe, $m = 400$, is almost impossible to see. Why? Where do you have to look for it? Note that the pattern is symmetrical: It doesn't matter on which side of the central maximum ($\theta = 0$, $m = 0$) you call angles θ negative with negative order numbers m.

EXERCISE 5 Exercise 4 cont'd. How many fringes are there in the 1° angle above $\theta = 0$? [Answer: 6] Is the approximation $\sin \theta = \theta$, with θ measured in radians, good enough here?

EXERCISE 6 How many fringes are there at angles greater than 30°? [Answer: 200 on each side] How much greater is the spacing of fringes near $\theta = 30°$ than near the center of the pattern? (HINT: Compare $m = 200$ and $m = 201$. Is the approximation $\sin \theta = \theta$ good enough here?)

EXERCISE 7 Increasing the wavelength of the light by 1% increases / decreases the fringe spacing by ______%.

EXERCISE 8 Increasing the slit separation by 1% increases / decreases the fringe spacing by ______%.

Many Slits → Diffraction Grating

Some students like to say the formula

$$m\lambda = d \sin \theta$$

as if it were a poem. What is its magic? Well, it tells you the diffraction angles of the interference maxima whether you have two slits or two thousand.

Try the argument for three slits. The spacing between centers is d. The slits are narrow. Light is normally incident from a distant point source so that it arrives at all three slits in phase. For an angle θ for which wavetrains from slits 1 and 2 interfere constructively, wavetrains from slits 2 and 3 also interfere constructively:

$$\Delta\text{path} = d \sin \theta = n\lambda \qquad \text{for the brightest maxima}$$

At these angles the path difference between slits 1 and 3 is $2d \sin \theta$, which is $2m\lambda$, also an integer number of wavelengths. All the interference is con-

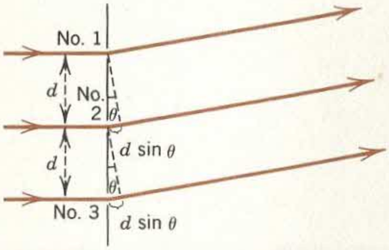


FIGURE 12.5 Three slits. If the path difference between rays 1 and 2 is an integer number of wavelengths, then the path difference between rays 1 and 3 is also an integer number (exactly twice) of wavelengths, and all three waves arrive in phase.

structive (see Fig. 12.5). You can see that the argument remains the same regardless of how many slits you have, as long as they are evenly spaced a distance d apart.

A **diffraction grating** is any evenly spaced set of lines or ridges or scratches placed so that waves reflect off it. Any periodic straight-line pattern of scatterers will do. Nowadays transparent plastic sheets with tens of thousands of lines are produced photographically and can be bought for pennies. Such *replica gratings* can be used for high-precision spectroscopy. The next paragraph shows why. These gratings are also very pretty to look at. They break up white light into all the colors of the rainbow.

The more slits (or scratches, or dark lines) in your grating, the narrower the bright stripes. The photograph shows how narrow they get as the number of lines in the grating is increased. Here is the argument for six slits. (Fig. 12.6.) You already know where the maxima are: $d \sin \theta = m\lambda$. The path difference between rays from adjacent slits is equal to an integer number of wavelengths for **constructive** interference. How about **destructive** interference? Look at small angles, angles near the central maximum. With two slits, we got destructive interference when $d \sin \theta = \frac{1}{2}\lambda$. At this angle, rays 1 and 2 arrive out of phase and cancel. Rays 3 and 4 cancel, and so on. But there is a smaller angle, too. Try the angle at which ray 1 and ray 4 differ by half a wavelength and thus arrive out of phase. At this same angle ray 2 will cancel 5, and ray 3 will cancel 6: complete cancellation. That angle θ is given by

$$3d \sin \theta = \tfrac{1}{2}\lambda$$

since slits 1 and 4 are $3d$ apart. Repeat:

$$\sin \theta_{\text{1st min}} = \frac{1}{6}\frac{\lambda}{d}$$

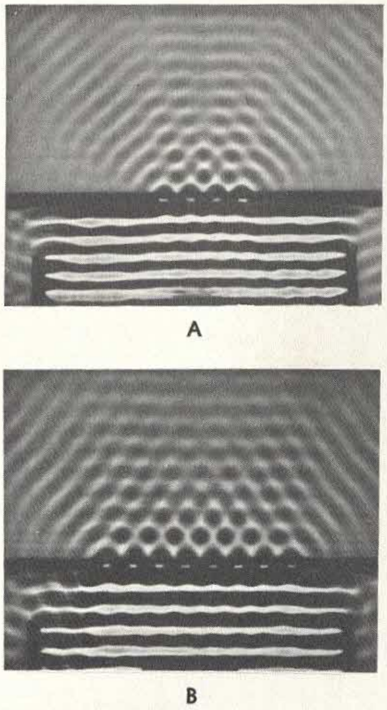
gives the smallest angle at which the destructive interference is total. That tells you how wide the central bright stripe is. With six slits it is one-third as wide as with two. Now generalize. With $N = 100$ slits, there is a smaller angle at which we get totally destructive interference. It is an angle for which the wave from slit 1 cancels the wave from 51, 2 cancels 52, . . ., 50 cancels 100:

$$50d \sin \theta = \tfrac{1}{2}\lambda \qquad \text{or} \qquad \sin \theta_{\text{1st min}} = \frac{1}{100}\frac{\lambda}{d}$$

With N slits the smallest angle at which the waves cancel pairwise to give total darkness is given by

$$\sin \theta_{\text{1st min}} = \frac{1}{N}\frac{\lambda}{d}$$

The generalization may be stated in words: The width of the central stripe



Interference pattern in the ripple tank made by *(a)* four slits, and *(b)* eight slits.

is inversely proportional to the number of slits, N. That statement required substituting θ for $\sin\theta$, surely a good approximation for large N, that is, narrow stripes. We now state without proof that the width of all the other stripes, all the orders m, are also inversely proportional to N. The more lines in the grating, the narrower the stripes. Because the region between these stripes is not completely dark, we refer to the maxima given by $d \sin\theta = m\lambda$ as **principal maxima.** Not only do they get narrow with many slits, they also get very bright.

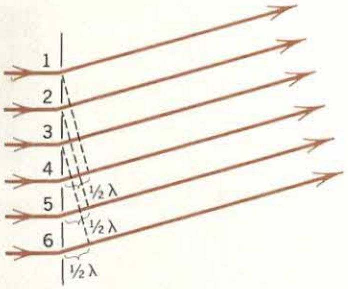


FIGURE 12.6 Diffraction grating with $N = 6$. The first minimum is at the diffraction angle for which rays interfere destructively in pairs; 1 cancels 4, 2 cancels 5, 3 cancels 6. Consequently, $3d \sin\theta_{\text{1st min}} = \frac{1}{2}\lambda$. The more lines in the grating, the narrower the bright stripes.

EXERCISE 9

(a) How wide is the central bright stripe for a grating with $N = 200$ lines?

SOLUTION: At the smallest angle for destructive interference, the wave from slit 1 interferes destructively with the wave from slit 101, and so forth, in pairs. Those pairs of slits are $100d$ apart. Hence, for this angle $\theta_{\text{1st min}}$, we have

$$100\, d \sin\theta_{\text{1st min}} = \tfrac{1}{2}\lambda \qquad \text{or} \qquad \sin\theta_{\text{1st min}} = \tfrac{1}{200}\lambda/d$$

Solve for $\theta_{\text{1st min}}$ if $\lambda = 500$ nm and $d = 0.2$ mm. [Answer: 1.25×10^{-5} radians]

(b) Why must this grating be 4 cm wide for this answer to hold?

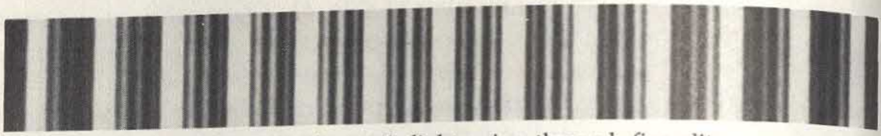
(c) At what angle is the first principal maximum?

EXERCISE 10 Suppose that you double the number of lines in your grating. Or suppose (same thing) you double the width of the grating illuminated by your light source. You might think you would double the intensity of the principal maxima, because twice as much light gets in. In fact, you quadruple it. Explain. (HINT: How much more dark area is there in the pattern?)

EXERCISE 11 Show that the intensity of the principal maxima is proportional to N^2, the square of the number of lines in the grating.

Single Slit

The mathematicians are always talking about "passing to the limit as N approaches infinity." Physicists only talk that way about *thought* experiments, imaginary experiments nobody ever carries out. Try this one: What happens as you crowd the lines of the grating closer and closer together ($d \to 0$) and increase their number ($N \to \infty$) in such a way that $(N - 1)d$ remains constant? The width w of the grating is $(N - 1)d$ and can be approximated as Nd.



Interference pattern of monochromatic light going through five slits.

EXERCISE 12 Prove that the width of the grating is $(N - 1)d$.

EXERCISE 13 Show that the $N = 6$ grating has width 5d. Sketch!

You will probably object quickly that the separations between the "slits" of the grating will get thinner and thinner, and all you will be left with is an open region of width w. But that is precisely the point of the story. We want to ask, how wide is the central bright stripe for a single slit of width w? We made the single slit by taking so many tiny slits close together that there is no space between them. The answer to the question is

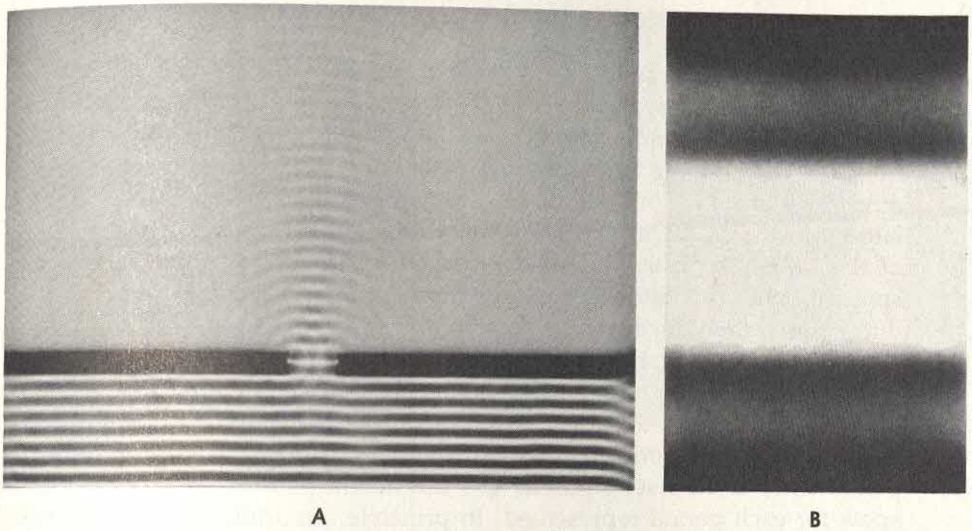
$$\sin \theta_{\text{1st min}} = \frac{1}{N}\frac{\lambda}{d} = \frac{\lambda}{w}$$

The wider the slit, the narrower the stripe. This is a result you observed with the ripple tank in Chapter 10. You saw the analogous effect with light diffracted at a round hole in Sec. 12.2. The size of our pinholes is always many wavelengths, so our diffraction patterns are always small:

$$\sin \theta_{\text{1st min}} << 1$$

That allows us to make the approximation $\sin \theta \approx \theta$: *The width of the pattern is inversely proportional to the width of the opening.*

Now you may object that this goes against intuition. A narrower slit looks wider? If you double the width of a window, it *looks* twice as wide, too, doesn't it? It certainly doesn't look half as wide. But remember, we are talking about the width of the central bright stripe of the diffraction pattern. You probably won't even notice diffraction effects unless you are far from the window. You have to be far enough away so that the angle subtended by the window is less than the width of that central bright stripe. Only then does the outline of the window blur appreciably. When you are much closer than that, the outline of the window is its geometrical shadow. When you are close in, effects other than diffraction are more important in making shadows fuzzy. One is that you don't really have a point source. For example, the sun subtends an angle of about one-hundredth of a radian (about half a degree). Such a large angle swamps most diffraction effects. Remember when you did the pinhole experiments! When you were close to the source hole you saw its geometrical shadow. Only as you walked away from it did the diffraction pattern become noticeable.



Diffraction by a slit *(a)* in the ripple tank, and *(b)* with monochromatic light.

Our quantitative treatment of the single slit made a number of assumptions. To arrive at the result that the halfwidth of the central bright stripe is λ/w, we made some approximations. One of our assumptions was that the screen or the observer is far away from the diffracting edge. That allowed us to consider all the rays approximately parallel. Parallel rays means plane waves. The planewave regime is called **Fraunhofer diffraction.** This book does not discuss what happens when the point of observation is closer in, the regime called **Fresnel diffraction.** There the geometry gets harder, but the physics is much the same.

EXERCISE 14

(a) How many radians wide is the central bright stripe when a parallel beam of 500-nm light is diffracted from a 1-mm slit? [Answer: $2\,\theta_{\text{1st min}} = 10^{-3}$ radians]

(b) Why do you have to be more than 1 m away from it before the slit appears to be wider than 1 mm?

EXERCISE 15 In our double-slit example we had $d = 0.2$ mm and $\lambda = 500$ nm, giving the third-order interference maximum at $\theta = 7.5 \times 10^{-3}$ radians.

(a) Show that this is true.

(b) For what slit-width w would this third-order maximum be *missing* because it coincides with the first minimum of the single-slit diffraction pattern?

(c) Show that there is no slit-width w for which the first-order interference maximum is missing.

12.4 TWO-DIMENSIONAL GRATINGS: HOLOGRAMS

Suppose that you were to pass a beam of monochromatic light through two line gratings placed with their rulings at right angles. Think, say, of a piece of graph paper. What happens is that the interference stripes formed by the horizontal grating are broken up into spots by the vertical one. The result is a rectangular lattice of light spots. The more closely spaced the rectangular lattice of rulings, the more widely spaced the lattice of spots. The more area of the diffracting rulings is illuminated, the smaller and more intense the spots of light. This diffraction in two dimensions is completely analogous to diffraction in one dimension.

Now suppose that lattices of two different periods are superposed. They could be etched or printed on the same piece of glass or film. The resulting diffraction pattern is usually a superposition of the individual patterns. We don't have to limit ourselves to two different periods. A large number of periodicities can be superposed. The diffraction pattern will have a set of spots for each period represented. In principle, an arbitrary design of light spots could be achieved by the proper superposition of periodic graph-paper rulings. The remarkable feature of such a multiperiodic lines-and-squares film is that the whole design—the whole light picture—is formed by any tiny scrap of the film. To be sure, if you use only a small area of film, you get a fuzzy picture. The more area of film the incident light beam illuminates, the sharper the individual spots of light in the picture. The next paragraph describes how to make a practical hologram.

Holography

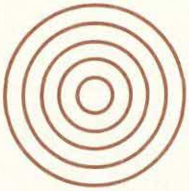
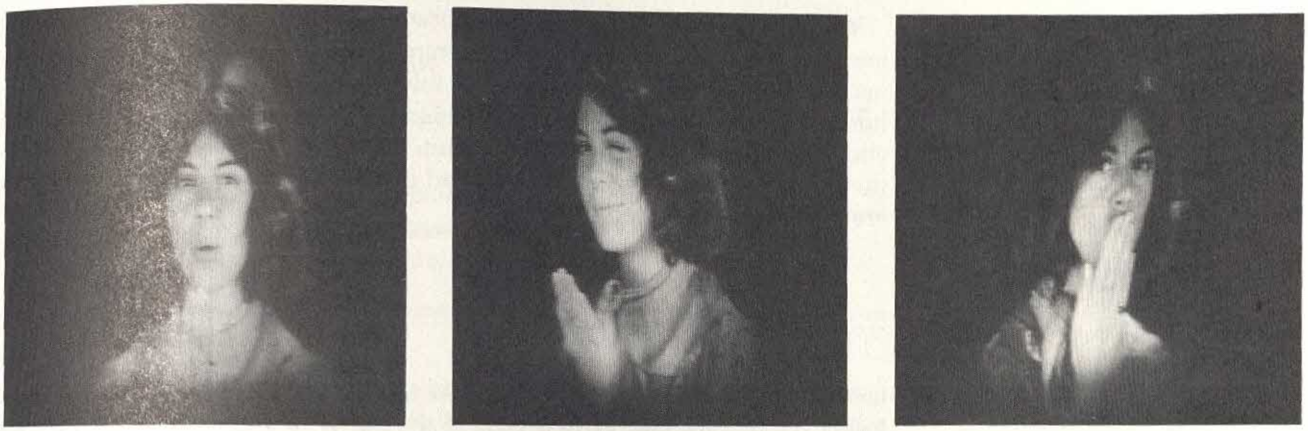


FIGURE 12.7 Hologram (i.e., interference pattern) formed by a plane wave and a spherical wave that remain coherent.

Practical holography was made possible by the invention of light sources giving highly monochromatic parallel light: lasers. A hologram is made by letting two laser beams interfere in the plane of a photographic film. The film will be exposed at the interference maxima. So that is where it will be blackened. Figure 12.7 shows an interference pattern formed by a plane wave and a spherical wave of the same wavelength. To see why you get this sharpshooter's-target effect, suppose that the plane wave has a crest in the plane of the film. Then the places where the spherical wave crests happen to be at that time will be antinodes, hence intensity maxima. Another spherical wave coming from a different point source would give a concentric-rings pattern with a different center. Any number of such patterns can be superposed on the same film, one for each point source of light. Indeed, if a three-dimensional object is illuminated by laser light, each point of the object scatters that light so that it acts as a point source of spherical waves.

The film is developed. It is the hologram. What if you now illuminate it with a monochromatic parallel beam of light? The diffraction pattern will be an image of the original object. What is more, as you turn the hologram, you can see the object from different angles. Its three-dimensional character has been preserved in the two-dimensional hologram.

Problem 26 shows why a hologram forms both a real and a virtual image

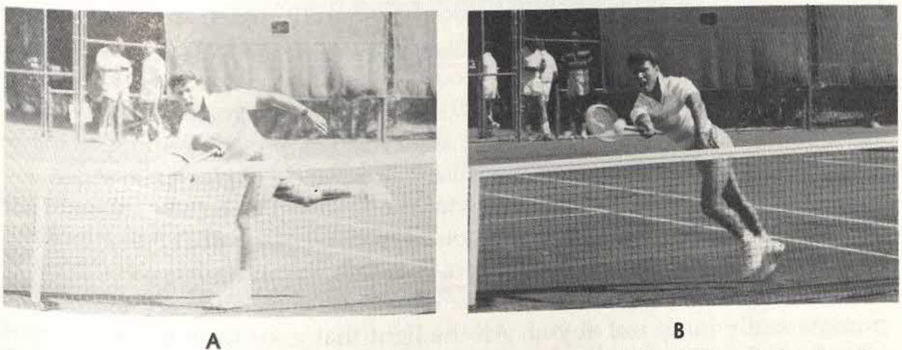


A hologram gives a truly three-dimensional view. Here are three ordinary photographs of the same hologram seen at three different angles.

of the object. For naked-eye viewing, the virtual image is what you want. It is fun to "look around the corner": One object in the field of view may block another at one angle. Change the angle of viewing and you can see behind it!

12.5 POLARIZATION

Polaroid sunglasses help suppress road glare on a sunny day. If you want to test whether your sunglasses are polarizing or not, hold them so that you are looking at a bright light reflected at somewhere near 45° incidence from a floor or table top. Then rotate the glasses in their own plane. If the intensity of the reflected light changes as you rotate, the lenses are polarizing.



Polaroid sunglasses on the tennis court? *(a)* Without. *(b)* With. If the angle of reflection is close to Brewster's angle, the reflected light is close to 100% polarized, and a correctly oriented polaroid filter will attenuate it considerably. Look at the dark-colored court, the wind barriers in the background, and the wire fence on which they are mounted.

You will notice that there are two orientations that give maximum transmission and two that give minimum transmission of the reflected light. The maxima are 180° apart, and so are the minima. The maxima are 90° from the minima. See if your sunglasses are designed so that the orientation in which you wear them corresponds to minimum transmission of the glancing reflection. That glancing reflection is the road glare you want to cut out when you are driving in bright sun.

☐ LAB AT YOUR DESK

Look at a patch of blue sky through a piece of polaroid plastic and rotate it. Does it matter whether the blue patch is close to the sun or in a direction away from the sun?

In Chapter 10 we distinguished between scalar waves and vector waves. A transverse wave is obviously a vector wave. When we say that a wave disturbance is perpendicular to the direction of propagation, that still leaves its direction unspecified. All this sounds very abstract. We didn't say what vector we are talking about.* We just said, if the waves are transverse, there has to be a vector. The direction of that vector is called the direction of polarization of the wave. It is actually more conventional to talk about the plane defined by that direction and the direction of propagation and to call that the *plane of polarization*.

If light waves were longitudinal waves, turning your sunglasses about the direction of propagation as an axis could not have any effect. That argument is just based on symmetry. We conclude that light waves are indeed transverse. But most of the light we see is a mixture of all possible "polarizations." Turning the polaroid sunglasses does nothing to most of the light that meets the eye. What kind of light is polarized? What kind of light has one predominant plane of polarization? Evidently, light that has been reflected at just the proper angle from a sufficiently smooth surface. If you vary the angle of reflection at the floor or table top, you can find an angle at which hardly any of the reflected light goes through in the "minimum" orientation of the polaroid. At that angle, called Brewster's angle, the reflected light is 100% polarized. For glass, Brewster's angle is about 57°.

By now you are getting the idea that the polaroid polarizes the light that passes through it. To test this idea you need two polaroids. Look at ordinary unpolarized light through one polaroid. Rotate. Nothing happens. Now interpose the other polaroid and rotate either one. The intensity maxima and minima really jump out at you. All the light that goes through is now modulated by the rotation. The relative orientation of the two plastic sheets at minimum transmission is called "crossed" polaroids.

To appreciate what is happening, we need one more fact: As one of the polaroids is rotated, the intensity varies according to a cosine-squared law:

---

*It is called the electric vector.

$I(\theta) = I(0) \cos^2 \theta$ **Law of Malus**

EXERCISE 16

(a) Show that this law predicts a pair of maxima and a pair of minima at right angles to each other, as observed.

(b) Is $\theta = 0$ a maximum or a minimum?

Interpretation: The direction of polarization is the direction of an oscillating electric field. The intensity of the light turns out to be proportional to the square of the field amplitude. The polaroid is a filter that projects the electric field vector on the direction of its optic axis, that is, allows only the component of field parallel to the optic axis to pass. Thus

$$E_{\text{after}} = E_{\text{before}} \cos \theta$$

for an ideal polarizer, and

intensity $I \propto E^2$

If the two optic axes are parallel, light that gets through the first polarizer also makes it through the second. If the optic axes are perpendicular, nothing gets through. For unpolarized light incident on the first polaroid, the angle θ is random. Consequently, this first polarizer cuts the intensity in half, if it is "ideal."

EXERCISE 17 Prove that the average value of $\cos^2 \theta$ is $\frac{1}{2}$ if θ is random. (HINT: Look at a plot of $\cos^2 \theta$. Then turn it upside down.)

The above interpretation of the transmission of two polaroids may seem far-fetched at this point. Do come back to it after reading the electricity chapters, especially Chapter 17, Radiation. But even if you don't see the whole picture now, one conclusion is inescapable from the bare facts of polarization: Light waves are transverse.

EXERCISE 18

(a) Argue why you think that sound waves are longitudinal.

(b) Waves on a violin string are transverse. Argue why they might be polarized in the plane containing the bow.

SUMMARY

Rainbow and dispersion

Blue sky, white sky: scattering by small particles

IR and UV are separated by one octave of visible light.

Rods and cones

Color addition and subtraction

Diffraction blurs two headlights into one.

Larger aperature improves resolution.

Closer slits → wider interference fringes: $d \sin \theta = m\lambda$

Wider grating → narrower and brighter maxima

Narrower slit → wider central maximum

How to make a hologram

Polarization is evidence that light waves are transverse.

KEY TERMS

Visible spectrum Approximately 400 nm (violet) to 700 nm (red), just a little less than one octave.

Infrared IR. Electromagnetic waves of wavelength longer than 700 nm, that is, longer than visible light. The long-wavelength end of the IR is undefined, but waves longer than 0.1 mm are usually called microwaves.

Ultraviolet UV. Electromagnetic waves of wavelength shorter than 400 nm, that is, shorter than visible light. The short-wavelength end of the UV is undefined, but waves shorter than 10 nm are usually called x-rays.

Color blindness Probably due to the absence of one (or two) of the three kinds of cones on the retina. This explains why there are several different kinds of color blindness. (Not discussed in the text.)

Dispersion Variation in the index of refraction n with wavelength of light: the shorter the wavelength, the greater the n. Called *dispersion* because a prism breaks up (disperses) white light into spectral colors. The red is the least deviated; the violet is turned through the largest angle. In glass, n changes about 4 parts in 10^5 per nm of wavelength change.

Monochromatic Having only one frequency or wavelength. Light of a single color. Of course, no light is truly monochromatic. The question "how monochromatic" can be answered in terms of the range of frequencies or wavelengths present.

Primary rainbow Light from the sun enters a spherical water droplet off center, is internally reflected from the back surface, and re-emerges. In this refraction-reflection-refraction the light is turned through a maximum angle of 139°. (See Prob. 4) More precisely, red light is turned through a maximum of 138°, violet through a maximum of 140°. In the secondary rainbow (rare!), there are two internal reflections. Rainbows with up to 14 internal reflections have been observed.

Rayleigh scattering Scattering of light by small scatterers, with linear dimensions considerably smaller than the wavelength. Scattered intensity is proportional to the fourth power of the frequency (not discussed in the chapter). This strong wavelength dependence explains the blue color of the sky. The short wavelengths are more strongly scattered by small clusters of air molecules than the long ones.

Red sunsets The blues and greens have been scattered out by dust and water droplets.

Interference Superposition of two or more wavetrains having a constant phase relationship.

Young's double-slit experiment The two slits must be illuminated by the same source. With monochromatic light, the interference pattern is a set of zebra stripes.

Diffraction The bending of a light ray at the edge of an obstacle into the geometrical-shadow region. Diffraction is responsible for the interference pattern in the double-slit experiment.

Constructive interference Two or more wavetrains arrive in phase.

Destructive interference Two wavetrains arrive half a period (180°) out of phase.

Coherence Two sources of waves are *coherent* if they have the same frequency and their phase difference stays the same. If a source starts and stops emitting wavetrains at random times, it will not be coherent with another source. Light from two noncoherent sources does not give an interference pattern.

Diffraction grating Glass or transparent plastic with equally spaced parallel lines (rulings) to simulate "slits." Also called *transmission grating*. A *reflection grating* is usually ruled on a metal surface, which may be curved to achieve focusing without a lens (useful for IR and UV).

Principal maximum Of a grating. Angle at which wavetrains from all slits interfere constructively. The angular position θ is given by $\sin \theta = m\lambda/d$, where m is an integer called the *order* of the maximum.

Laser *L*ight *a*mplification by *s*timulated *e*mission of *r*adiation. A source of highly monochromatic, coherent, parallel light.

Resolving power Measure of the smallest angular separation between two pencils of light that can still be distinguished. For a diffraction grating, the resolving power is the smallest wavelength difference that gives rise to two distinguishable lines. For some authors, resolving power is the ratio of the average wavelength to this smallest wavelength difference.

Rayleigh criterion Two adjacent point sources (or line sources) of light are considered resolvable if the first minimum of the diffraction pattern of one falls nearer than the central maximum of the other.

Fraunhofer diffraction The pattern is observed far enough away so that the interfering rays may be considered parallel.

Fresnel diffraction The pattern is observed close enough to the obstacle so that the interfering rays are noticeably oblique (converging).

Hologram A pattern produced by the interference of two monochromatic beams in the plane of the film, one plane wave and one scattered from the object. When the finished hologram is illuminated with monochromatic plane waves, an image of the original object is seen.

Newton's rings The interference pattern formed by a thin layer of air between two glass surfaces. Typically, one surface is plane, one is convex, for example, a lens surface. In this case, the "rings" surround the point of tangency. The quality of the lens, that is, how evenly it is ground, can be tested by the roundness of the rings.

Michelson interferometer Optical device for measuring the wavelength of light, or alternatively, for calibrating distances in terms of a standard wavelength. (See the figure accompanying Problem 22) The beam of light is split by the half-silvered 45° mirror. One beam is reflected by the stationary mirror. The other beam, perpendicular to the first, is reflected by a movable mirror attached to a calibrated screw. Reflection of the "straight-through" beam from the back surface of the half-silvered mirror brings the two beams back into line, so they can interfere (superpose) at the eye or other detector.

Plane-polarized light When passed through a polarizing filter (polaroid), the intensity of polarized light varies with the angle of the filter axis. Light that is 100% polarized can be completely stopped when its plane of polarization is perpendicular to the axis of the filter.

Polaroid A plastic film whose absorption of polarized light is a function of the direction of polarization. Made by pulling the film when soft (warm), cooling rapidly.

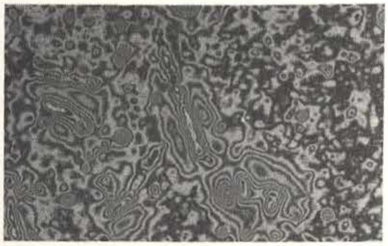
Law of Malus The fraction of incident polarized light transmitted by a polarizer is proportional to the square of the cosine of the angle made by the plane of polarization with the axis of the polarizer: $I = I_O \cos^2 \theta$

Brewster's Law Polarization by reflection from a plane surface having index of refraction n is complete when the angle of reflection ϕ is given by $\tan \phi = n$.

REVIEW QUESTIONS

- What is meant by pure spectral color? What other kinds of color are there? Do they look the same?
- What are the functions of the rods and cones of the retina?
- Why does the size of our cornea limit the resolution of the eye to an angle of about 10^{-4} radian?
- State the Rayleigh criterion.
- Derive the formula $\sin \theta = m\lambda/d$ for a double slit.
- Is it possible for d to be so small that no first-order maximum appears?
- The width of the central bright stripe is proportional to the number N of lines in the grating. True/False Explain.
- When is the width of the image of a single slit inversely proportional to the width of the slit?
- If you cut a hologram in half, you see approximately half the picture. True/False Explain.

☐ LAB AT YOUR DESK Interference in Thin Films (Newton's Rings)



Newton's rings.

Apparatus

Two glass slides. (Microscope slides can be borrowed from the biologists.)

Monochromatic light source, illuminating a white screen or wall in a darkened room.

Make a glass-air-glass sandwich by pressing two slides together. Hold with thumb and one finger by the edge, to prevent finger prints. Use it as a mirror to view the bright screen. You will see the interference fringes as "zebra stripes."

Try slightly deforming the glass by gentle bending. The fringes move dramatically.

Argue that what you are seeing is a contour map of the thickness of the air layer. Trade slides with your neighbors to marvel at how different the patterns are. Manufacturing tolerances are not good to one wavelength!

Confirmatory Experiment: Use only one slide. The fringes vanish. It really *was* reflection at the top and bottom of the air layer that gave the two interfering beams. With only one slide you might still see some very closely spaced straight fringes that don't move much when you deform the slide.

HINT: A sodium lamp is best; it has highly monocromatic yellow light. A mercury lamp gives three or four different wavelengths but still shows fringes. Ordinary fluorescent lamp works with a colored-cellophane filter, but the fringes are harder to see.

Measurement: Make an air wedge by squeezing a thin piece of paper into one edge of the sandwich. Now the fringes will be parallel to that edge, and closely spaced, therefore hard to count. If you can count them, then measuring the thickness of the paper amounts to a measurement of the wavelength of the light. Argue that $N\lambda = 2 \times \text{thickness}$.

PROBLEMS

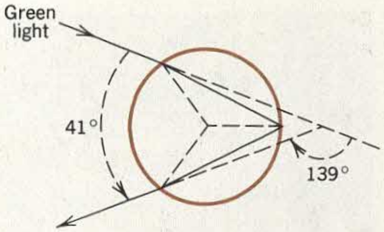
* **1. Spectral Range** The human singing voice has a frequency range of 2 to 3 octaves. The piano keyboard covers 7 octaves. The range of human hearing is over 10 octaves wide. How many octaves is the frequency range of the eye? Remember, when the higher frequency is twice the lower frequency, they are 1 octave apart. (See Chapter 10, Sound and Hearing.)

* **2. Frequency of Visible Light** The shortest violet wavelengths are about 400 nm; the longest reds are about 700 nm. Taking the speed of light to be 3×10^8 m/sec, calculate the highest and lowest visible-light frequencies.

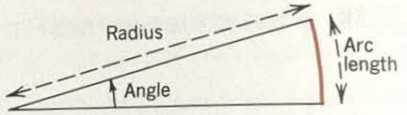
*** **3. Color Blindness** Defective color discrimination is presumed to result from the absence of one (or two) of the three sets of cones. For example, people who lack the green-sensitive cones are called green-blind. They still see green objects and call them green, because of the differential activity of the red-sensitive and blue-sensitive cones.

Design a behavioral test for this visual defect.

NOTE: Some degree of **color blindness** affects about 10% of the male population and about 1% of the female population. Color-blind aviators have been used to penetrate camouflage against aerial reconnaissance.



PROBLEM 4
Sunlight refracted by spherical water droplets to form the primary rainbow.



PROBLEM 5

✶✶ **4. Rainbow—Good Practice on the Calculator** A light ray incident on a spherical water drop at an angle θ_i is bent toward the normal so that it makes an angle θ_r with the radius drawn to the point of incidence, where $\sin\theta_r = (1/n)\sin\theta_i$. (Remember Snell's Law.) (See figure.)

(a) Prove by a geometrical argument (including sketch) that the ray makes the same angle θ_r with the radius drawn to the point where the ray next meets the surface of the drop. Some of that light does not get out but is reflected internally. According to the law of equal-angle reflection, the reflected ray again makes an angle θ_r with the radius at the point of reflection. It therefore meets the surface of the drop a third time, again at an angle θ_r.

(b) Convince yourself that it emerges from the drop making the original angle θ_i with the radius at the point of emergence.

(c) On the first refraction, the ray is turned through an angle $\theta_i - \theta_r$. Upon reflection, it is turned through an angle $180° - 2\theta_r$. Show by means of a clear diagram.

(d) On the second refraction, the ray is again turned through $\theta_i - \theta_r$. Verify that the total angle of deviation is $180° + 2\theta_i - 4\theta_r$. Some people prefer to say that the ray emerges at an angle $\phi = 4\theta_r - 2\theta_i$ *below* where it came in. Calculate the angle ϕ for $n = 1.33$ (water) and $\theta_i = 45°$.

(e) Repeat for $\theta_i = 72°$. Are you surprised?

(f) Find the maximum value of ϕ by trial and error. (HINT: Make a table of ϕ versus θ_i.)

Note that angles θ_i between 50 and 68° all give ϕ within a couple of degrees of maximum. A large range of incidence angles corresponds to outgoing light in a small range of angles of deflection. Not surprisingly, the rainbow is seen about 42° above the "antisolar" direction. Now you know why you can't see a rainbow close to sunset, that is, when the sun is less than 42° above the horizon.

✶✶ **5. Resolved Headlights** The headlights of a car are 1.5 meters apart. One kilometer away they subtend an angle of 1.5×10^{-3} radians.[2]

(a) What angle do they subtend 10 km away?

(b) The human eye cannot resolve two point objects if their angular separation is less than about 10^{-4} radians. From how far away will the headlights start to blur together from the effect of diffraction?

(c) You may have observed headlights blurring together from cars at much smaller distances. Discuss the effect of:

Uneven temperature and moisture content of the air.

The finite size of each headlight. They are not point sources.

A nearsighted eye.

Astigmatism.

Dirty windshield (seriously).

✶ **6. Double Slit** With slits 1 mm apart, what is the angular spacing of the fringes near the center of the pattern? Take $\lambda = 5000$ Å ($= 500$ nm).

✶ **7. Double Slit** With light of wavelength 5×10^{-7} m, how close must the slits be to space the fringes 0.1° apart near the center of the pattern?

✶ **8. Double Slit** A sodium vapor lamp gives light of wavelength 589 nm. What is the angle between the $m = 0$ and the $m = 1$ maxima? Take $d = 0.2$ mm.

✶✶ **9. How Many Maxima?** For the double slit, the angle θ_1 at which the "first-order" maxima appear on either side of the ($\theta = 0$) central maximum is given by $d\sin\theta = \lambda$.

(a) Prove, using a simple sketch.

(b) Let $\lambda = 620$ nm (red light) and $d = 2$ μm. Solve for θ_1. Assume that the slits are much narrower than 2 μm.

(c) The second-order maxima are at angles $\pm\theta_2$, where $d\sin\theta_2 = 2\lambda$. Why is θ_2 greater than twice θ_1?

(d) Find θ_3, the angular position of the third-order maximum.

(e) Show that there is no fourth-order maximum.

(f) Sketch roughly the seven rays that point toward interference maxima. (HINT: Don't forget $\theta = 0$.)

[2] Recall the definition of radian measure. In radians, the angle subtended by the arc of a circle is the arc length divided by the radius. Note that for very small angles we don't bother to distinguish between the arc length and the chord of the arc, that is, between the angle and its sine.

★★ **10. 20 Slits** A small "grating" has 20 narrow slits spaced 2 μm apart. It is illuminated normally by monochromatic light of wavelength 620 nm, orginating at a distant point source.

(a) At what angle does light from the second slit travel just one wavelength further to the eye than light from the first slit?

(b) Is this answer the same as if the other 18 slits were not there?

(c) How many wavelengths further does light from the third slit travel at this same angle than light from the second slit?

(d) Does light from all 20 slits arrive at the eye (or screen or film) in phase at this angle?

(e) At what angle does light from the eleventh slit travel one-half wavelength farther than light from the first slit? Remember, half a wavelength means destructive interference.

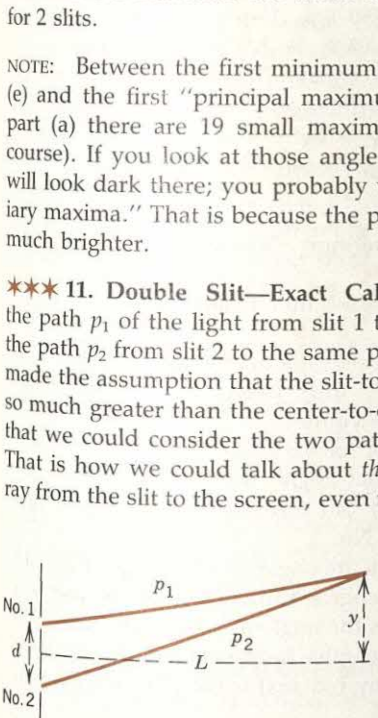
(f) Argue that there will be a interference minimum at that angle.

(g) Argue that there is no smaller angle at which pairwise cancellation occurs for light from all 20 slits.

(h) Show that the width of the central maximum for 20 slits is about one-tenth the width of the central maximum for 2 slits.

NOTE: Between the first minimum you calculated in part (e) and the first "principal maximum" you calculated in part (a) there are 19 small maxima (and 19 minima, of course). If you look at those angles, though, the pattern will look dark there; you probably won't see the "subsidiary maxima." That is because the principal maxima are so much brighter.

★★★ **11. Double Slit—Exact Calculation** In drawing the path p_1 of the light from slit 1 to the screen and then the path p_2 from slit 2 to the same point on the screen, we made the assumption that the slit-to-screen distance L was so much greater than the center-to-center slit separation d that we could consider the two paths essentially parallel. That is how we could talk about *the* angle θ made by *the* ray from the slit to the screen, even though there are really two different angles. In this problem you are going to see how good (how bad?) this approximation is in a practical situation. Take $L = 1$ meter, $d = 0.2$ mm and wavelength $\lambda = 0.5$ μm (green light). We are going to calculate the position y of the first-order interference maximum above the central maximum.



PROBLEM 11

The following sketch is not to scale. According to Pythagoras, the hypotenuse p_1 is given by

$$p_1^2 = L^2 + (y - \tfrac{1}{2}d)^2$$

Similarly,

$$p_2^2 = L^2 + (y + \tfrac{1}{2}d)^2.$$

(a) Why is the first-order maximum characterized by

$$p_2 - p_1 = \lambda?$$

(b) Put the values into this equation and solve it for y. This may be unfamiliar! You can get rid of those square roots by squaring through twice. After a little algebra, only terms in y^2 remain. Or you might prefer to do the entire solution by trial and error on your calculator. Note that you need to keep nine significant figures in order to handle the small-differences-of-large-numbers situation.

(c) To compare with our usual approximation,

$\sin\theta = \lambda/d$, calculate y using $y = L\tan\theta$.

Express the error in parts per billion. (HINT: y comes out very close to 2.5 mm.)

(d) Argue that in calculating the position of the second-order maximum, the error in the usual approximation would be: greater/less

12. Third-Order Maximum Calculate the position y of the third-order maximum for the data of Problem 11.

★★ **13. Radio Antennas** Chapter 17, Electromagnetic Radiation, discusses the electromagnetic nature of light waves. There it is pointed out that other electromagnetic waves differ from light waves only in having different frequencies. Radio waves and microwaves, for example, have much lower frequencies and therefore longer wavelengths than waves in the visible spectrum. The mechanism of radiation of light may seem obscure. We say light "comes from excited atoms." But we all know that radio and radar waves are radiated by antennas.

Four vertical radio antennas positioned on a north-south line 2.5 meters apart radiate radio waves (short wave) of wavelength 3 meters. They are driven by the same transmitter, so they radiate in phase. A receiver is moved in a

large circle around them, several miles in radius. To label directions, we call east $\theta = 0$ and west $\theta = 180°$.
(a) Argue that there are two "central maxima," $\theta = 0$ and $\theta = 180°$.
(b) Argue that there are no other directions in which waves from all four antennas arrive in phase.
(c) Argue that the smallest angle θ at which there is complete destructive interference is given by $\sin\theta = \frac{1}{4}\lambda/d = 0.3$ so that the "beam" is 35° wide.

★★ **14. 100 Slits** Same parameters as "20 slits" above: $d = 2\ \mu\text{m}$, $\lambda = 0.62\ \mu\text{m}$.
(a) Argue that the grating is 0.2 mm wide. (HINT: You multiplied 99 times 2 μm. Why 99?)
(b) At what angle is the first-order principal maximum?
(c) The first minimum is at such an angle that light from the first slit travels half a wavelength less than light from the 51st slit. Calculate that angle.
(d) Verify that the width of the central maximum is one-fifth what it was for 20 slits.
(e) Since 5 times as much light goes through 100 slits as through 20, argue that the intensity of the central maximum is 25 times the intensity with only 20 slits.

★★ **15. Single Slit—Problem 14 cont'd.** Now try 200 slits in a 0.2-mm-wide grating. They would be 1 μm apart.
(a) Argue that the width of the central maximum is the same as in Problem 14 (100 slits).
(b) Now try 400 slits, 0.5 μm apart; then 800 slits, 0.25 μm apart. If you keep going this way, what happens to the width of the central maximum? What happens to the first-order principal maximum?
(c) As the number of slits N increases and the slit separation d decreases in such a way that the product Nd remains constant (call $Nd = w$, for *width*), you see that the angular position of the first minimum (i.e., the angular half-width of the central maximum) remains the same. Argue that making the width of each slit equal to d, the center-to-center distance between slits, does not change the argument as far as the position of the first minimum is concerned. What it means is that both N and d have lost their meaning. Only one big slit remains, of width w.
(d) Write an expression for $\theta_{1\text{st min}}$ in terms of λ and w.

★★ **16. Single Slit, at Various Distances** The single slit of Problem 15 (width $w = 0.2$ mm), when looked at from far away, looks wider the farther away the observer stands.
(a) Argue that with the observer's eye (or screen) at $L = 1$ meter, the slit looks 6.2 mm wide.
(b) How wide does it look when the observer is $L = 10$ meters away?
(c) Argue, on the other hand, that with the observer only 1 cm away, the slit looks 0.2 mm wide. With the observer 2 cm away, it also looks 0.2 mm wide. Remember, it *is* 0.2 mm wide. Adding 0.062 mm would be barely noticeable.

NOTE: The arguments made possible by the Fraunhofer approximation break down at such close distances. The character of the pattern changes completely. At certain (close) distances the central maximum even becomes a central minimum. These phenomena, called *Fresnel diffraction*, will not be discussed in this book.

★★★ **17. 6 Slits—Width of Various Maxima** In Exercise 4 through 9 we had $d = 0.2$ mm and $\lambda = 500$ nm. That gave the angular position of the first-order maximum as $\theta_1 = 2.5 \times 10^{-3}$ radians. (See Fig. 12.6)
(a) Argue that, at that angle, the path difference between light coming from slit 1 and slit 4 is three wavelengths, or 1500 nm. Since all the light from all the slits is arriving in phase, this is indeed a principal maximum, that is, it is very bright.
(b) Argue that the next-largest angle at which there is complete destructive interference is such that the path difference between light from slit 1 and slit 4 is three-and-a-half wavelengths, or 1750 nm. Verify that that angle is 0.4067×10^{-3} radians past θ_1, or 2.9067×10^{-3} radians.
(c) What is the first angle smaller than θ_1 at which there is complete destructive interference?
(d) Verify that the width of the first-order bright stripe is about the same as that of the central bright stripe.

★★ **18. 1000 Slits (Grating)** Take a grating 2 mm wide with 5000 lines per cm.
(a) Argue that the center-to-center distance between "slits" is 2 μm.
(b) How many principal maxima does it give on each side with green light ($\lambda = 0.5\ \mu\text{m}$)? (HINT: Do you count the one at 90°? The choice is yours.)
(c) Calculate the angle θ_1 for the first-order principal maximum. Argue that at that angle, the path difference between light from slit No. 501 travels 500 wavelengths further than light from slit No. 1.
(d) How much greater is the angle at which light from slit 501 travels 500.5 wavelengths further than light from slit 1? How much smaller is the angle at which light from slit 501 travels 499.5 wavelengths further than light from slit 1? How wide (how many radians) is the first-order bright stripe?
(e) Compare this last answer with the width of the central bright stripe.

✱✱ **19. Resolving Power of a Grating** A very fine grating may have as many as 10,000 lines per centimeter, making $d = 10^{-6}$ m, or 1 micron ($= 1\ \mu$m). The first-order maximum for green light of wavelength 500 nm will fall at $\sin\theta = \lambda/d = 5 \times 10^{-7}/10^{-6} = 0.5$, or $\theta = 30°$.

(a) Show that the second-order maximum (i.e., $m = 2$) is not visible.

(b) The angular half-width of the first-order maximum is inversely proportional to the number N of lines of the grating that are illuminated. If $N = 10^4$, we have $\Delta\theta = 0.5 \times 10^{-4}$ so that $\sin\theta$ can be measured to one part in 10^4. That means the wavelength can be measured to that accuracy. At 30°, what fraction of a degree is that uncertainty? Note that using a wide source slit in order to get a lot of light through wastes some of the precision.

✱✱✱ **20. Dispersion of a Grating**

(a) How far beyond 30° is the first-order maximum for that 10,000-line-per-cm grating with a wavelength of 505 nm?

(b) Calculate the slope of the angle-versus-wavelength curve ($d\theta/d\lambda$), which is called the *dispersion* of the grating.

✱✱ **21. Interference in Thin Films** You may have seen the colored patterns formed by a thin film of oil or gasoline on a puddle of water. They are interference patterns. What you are seeing is the variation in thickness of the film. Light reflected at the air-oil interface is interfering with light reflected at the oil–water interface. If the path of the light in the oil (= twice the thickness of the oil layer, for normal incidence) is exactly one wavelength of light, the interference should be constructive. Indeed, the same holds if the path difference between the two interfering beams is any integer multiple of one wavelength. The index of refraction of water is 1.33. Suppose that the index of refraction of the oil is 1.2.

(a) Argue that the condition for constructive interference is $2 \times \text{thickness} = m\lambda/1.2$, where m is an integer.

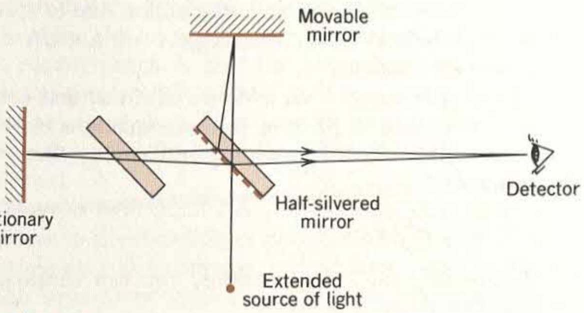
(b) What is the minimum thickness of the film for constructive interference with green light, $\lambda = 500$ nm?

(c) Argue that for violet light ($\lambda = 400$ nm), the film can be 20% thinner. Why are you not surprised that a naturally occurring oil film shows both green and violet regions, usually both quite narrow? (HINT: Wavelength in the refractive medium is shorter than wavelength λ_v in vacuo: $\lambda = \lambda_v/n$.)

✱ **22. Michelson Interferometer**

(a) Make your own sketch of the instrument and the two interfering light rays shown in the figure.

(b) Argue that if the movable mirror is moved backward (by turning the screw) through a distance x, the light path for that ray is increased by $2x$ so that N interference fringes move past the observer, where $N\lambda = 2x$.



PROBLEM 22 Michelson interferometer.

(c) Sodium light has wavelength $\lambda = 589$ nm. How far does the mirror have to be moved to count 100 fringes?

✱✱(d) Roughly how many fringes would you have to count to measure the wavelength to one part in a thousand?

✱✱ **23. Michelson Interferometer with Compensating Plate** In the figure in Problem 22 you see an extra piece of glass, mounted parallel to the half-silvered mirror. In solving Problem 22, you did not need to ask what that extra plate was for. It is for seeing interference fringes with white light.

(a) Argue that the straight-through ray traverses the glass plate with the half-silvered surface 3 times; the other ray traverses it only once.

(b) Glass is dispersive; that is, the long-wavelength light is slower than the short-wavelength light. Argue that without the compensating plate, there would be no position of the movable mirror for which the number of wavelengths in the two rays would be the same for both red light and violet light.

(c) Argue that with the compensating plate, you should be able to see perhaps half a dozen dark interference fringes with white light. (HINT: The last ones are highly colored!)

✱✱✱ **24. Sinusoidal Signals from Two Sources** The waves from two slits may be represented by the sum of two sinusoids differing in phase by

$$\phi = \frac{2\pi(\text{path difference})}{\lambda}$$

The 2π is just to express the phase difference in radians; that is, one wavelength represents 2π radians. If each source gives rise to a disturbance $E(t) = E_o \sin\omega t$, then the received signal may be written

$$E(t) = E_o \sin\omega t + E_o \sin(\omega t + \phi)$$

To be sure, we have neglected attenuation due to spreading of the spherical wave. This neglect will not affect our qualitative conclusions.

(a) Expand the $\sin(\omega t + \phi)$ in terms of $\sin \omega t$ and $\cos \omega t$. Set $E(t) = A \sin(\omega + \beta)$. You get two equations in A and β. You can eliminate β by squaring and adding. Show that the result is

$$A^2 = 4E_o^2 \cos^2 \tfrac{1}{2}\phi$$

(b) By dividing the two equations, you can eliminate A instead. Prove that $\beta = \frac{1}{2}\phi$.

(c) Now set $\phi = 2\pi d \sin \phi/\lambda$ to investigate the double slit interference pattern as a function of θ. Take on faith the "rule" that the intensity of the received signal is proportional to the square of the wave disturbance, hence to A^2. Prove that the maxima of A^2 occur at angles θ such that $d \sin \theta = m\lambda$, where m is an integer. Note that, as you expected, those are the angles that make the phase difference ϕ an integer multiple of 2π. Prove that the minima occur when the phase difference ϕ is an odd integer times π, that is, halfway between the maxima.

✶✶✶ **25. Three Sinusoidal Sources** We use the same algebra as Problem 24. Let

$$E(t) = E_o \sin(\omega t - \phi) + E_o \sin \omega t + E_o \sin(\omega t + \phi)$$

Set equal to $A \sin(\omega t + \beta)$.

(a) Show that for the phase ϕ at integer multiples of 2π, we get $A^2 = 9E_o^2$. These are the principle maxima.

(b) For what values of ϕ does A vanish?

(c) Show that there are two such minima between each pair of principal maxima. The smaller maxima in between are called subsidiary maxima.

(d) In the text it is stated that the width of the central maximum is proportional to $1/N$, where N is the number of coherent sources (slits). Compare the results of this problem with those of the previous one to confirm this result.

(e) In the text it is stated that the intensity of the principal maxima is proportional to N^2. Are the maximum values of A^2 in this problem and the previous one in the ratio 9:4?

✶✶✶ **26. Zone Plate, Hologram of a Point Source** Transparent and opaque concentric rings alternate according to the following scheme: When the plate is placed a distance f from the *focal point*, the nth transparent ring is a distance $f + n\lambda$ from the focal point. A plane wave falling normally on the plate would be diffracted in such a way that the first-order maximum is at the focal point. Evidently, the waves from any two adjacent rings would arrive exactly one period out of phase, since the path difference is exactly one wavelength.

(a) With light of wavelength $\lambda = 5 \times 10^{-7}$m and a zone plate with a focal length $f = 1$ m, the first transparent ring has to be $f + \lambda = 1.0000005$ from the focal point. So its radius r_1 is given by $\sqrt{f^2 + r_1^2} = f + \lambda$. Solve for r_1.

(b) The second transparent ring has to be $f + 2\lambda$ from the focal point, that is, a ray from the ring to the focal point travels two wavelengths farther than one from the center of the plate. The radius r_2 of this ring is given by $\sqrt{f^2 + r_2^2} = f + 2\lambda$. Calculate r_2.

(c) Calculate the radius of the 100th ring.

(d) There is also a "focal point" a distance f in front of the plate. Argue that this is the position of a virtual image, that plane waves coming in will have a first-order diffraction maximum if they are bent as if they came from this point.

NOTE: In holography, either the real or the virtual image can be used. If the one is called an $n = 1$ maximum, the other would be called $n = -1$.

(e) Show that the radius of the 400th ring is approximately double that of the 100th.

(f) Show that the ring with $n = 4 \times 10^5$ is appreciably more than twice as big as the one with $n = 10^5$. Some people say that the radii of the rings of a zone plate are proportional to the square root of n. Why are they almost right?

(g) Show that the hologram (the zone plate) could be produced by the interference (in the plane of the film) of a plane wave and a spherical wave coming from the focal point.

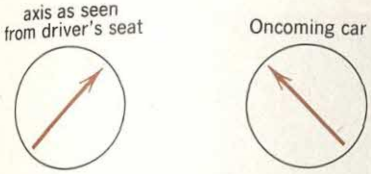
✶✶ **27. Law of Malus**

(a) Two polaroids have their axes at 90°. Show that, according to the Law of Malus, no light can get through. (HINT: $\cos 90° = ?$)

(b) One polaroid is now rotated through another 90°. Prove that this is the position that maximizes the transmitted intensity.

(c) This polaroid is now rotated another 45°. By what fraction is the intensity reduced?

✶✶ **28. Polaroid Windshields** If automobile headlights had polaroid filters with axis at 45°, then a polaroid windshield with the same axis would be "crossed" with the headlights of an oncoming car. The driver could still see the road by the light of his own headlights.



Problem 28

(a) If all cars were thus equipped, explain why it would no longer be necessary to dim your headlights at the approach of another car.
(b) Explain why the intensity of unpolarized light would be cut at least 50% by a polarizing windshield.
(c) Explain why the windshield of an oncoming car would look black.
(d) What other advantages and disadvantages can you think of?

✱✱ **29. Polarization of Sky Light—The Bees** If you look at blue sky through a polaroid, you easily convince yourself that the light is somewhat polarized. In fact, some parts of the sky have the light more polarized than others. You readily conclude that the angle the direction makes with the sun is what counts. But light from a cloudy sky is not polarized at all. Light coming from a cloud has been scattered several times by the water droplets making up the cloud.
(a) There is behavioral evidence that the eyes of bees are sensitive to the direction of polarization of light. They can use their eyes as a compass, to find directions. Would you expect bees to be less able to find their way on cloudy days?
(b) Design an experiment to test your answer to part (a).

NOTE: A Nobel prize was awarded for the research establishing this direction-finding ability of bees.

✱ **30. Brewster's Law** The formula $\tan \phi = n$ relates Brewster's angle ϕ to the index of refraction of the reflecting surface. Calculate ϕ for a glass with index of refraction $n = 1.51$.

Lightning over New York City, c. 1950.

13

ELECTRICITY: ELECTROSTATICS

SOME QUESTIONS THIS CHAPTER WILL ANSWER

What is triboelectricity?

A proton plus an electron bind to make a hydrogen atom. What is the net charge of a hydrogen atom?

What does it mean to "ground" one side of a battery?

What is electric field?

The equipotential surfaces between oppositely charged parallel plates are uniformly spaced. What about the spacing of equipotentials around a point charge?

What is the inverse-square law of electrostatic interaction?

How does the sodium-potassium pump keep the inside of a living cell electrically negative relative to the outside?

What is an excitable cell? Describe how it depolarizes and repolarizes.

Someone says that a large grapefruit might have an electrical capacitance of 10^{-11} farads. Is that about right or way off?

The closer two parallel plates are brought together, the larger their capacitance. True/False?

How does the energy stored in a charged capacitor depend on the voltage?

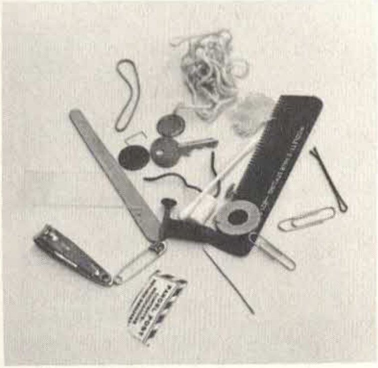
Two capacitances C connected in series have a net capacitance of $2C$ / $\frac{1}{2}C$.

If you comb your hair right after is has been dried, it may get unruly and literally stand on end. On a dry day you might even get sparks. If you walk on certain nylon carpets, you may become so highly charged electrically that you draw a spark just before touching a metal door knob, especially if you shuffle a bit as you walk.

These are examples of **triboelectricity,** charging by rubbing: Two different materials, both electrically insulating, are rubbed against each other. The plastic comb rubs against the dry hair. The shoe sole rubs against the nylon carpet. The door knob, on the other hand, is highly conducting and offers the charge a line-of-least-resistance path to escape, to reach a lower-energy distribution.

The voltages needed for a half-inch spark at a door knob amount to thousands of volts. The spark is an indication that the air, usually a good insulator, has been made conducting by being **ionized.** That is also what happens in a thunderstorm. Highly charged clouds discharge when the voltages are high enough to ionize the air in a path extending from the cloud to the ground: lightning. That requires millions of volts. The light given off in the spark and in the lightning results from the recombination of the electrons (−) and the ions (+) into neutral air (nitrogen and oxygen) molecules.

The words *charge* and *voltage* have been used in these paragraphs without being defined. But you are familiar with the notions of positive and negative charge and you know about voltage from the household. You know that like charges repel; charges of opposite sign attract. You know that the charge of the H^+ ion (the proton) and of the electron are equal and opposite. They are, respectively, the smallest positive and negative charges so far found in nature. You know that all other charges are integral multiples of these "elementary charges." Similarly, you probably know that the house current in most homes in the United States is labeled "115 volts ac",[1] and that flashlight batteries (dry cells) of all sizes are labeled "$1\frac{1}{2}$ volts." You know that it is not advisable to put your finger in the light socket. If you do, that painful vibrating sensation is the 60-Hz ac—*a*lternating *c*urrent changing direction 120 times per second. You also know that you feel no shock from the flashlight battery. The most you get from 1.5 volts is a slightly salty taste if you put both electrodes in your mouth—or wires connected to both electrodes. You have a pretty good intuition about which materials are **insulators** and which are **conductors.** In the class of conductors you are likely to be able to distinguish electronic conductors (metals and semiconductors) and ionic conductors (electrolytes). The practical devices that you "plug in," such as lights, electric motors, and TV's, utilize primarily electronic conduction. Our nerves and muscles, on the other hand, rely almost exclusively on ionic conduction. The electrical signals associated with the muscles of the heart are the basis of the *electrocardiogram*, one of the more familiar medical diagnostic tools. The voltage across the cell membrane of a cardiac muscle fiber is rarely larger than 90 millivolts (less than one-tenth of a volt), and most



Conductors and insulators. All of the metal objects are conductors; the wood, plastic, paper, string, and cloth are insulators.

Patient hooked up to an EKG (electrocardiogram) machine.

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[1] AC means *a*lternating *c*urrent.

other biovoltages tend to be smaller than that. You will meet these terms again in this chapter and in Chapter 14 on Electric Current.

13.1 THE VOLT

But it is time for a quantitative definition of the volt. The volt is a unit of potential. Potential is energy per unit charge. Usually, we are talking about *potential* energy per unit charge. When two points have a **potential difference** of 1 volt, the work needed to bring one elementary charge from the one point to the other is 1 **electron volt** (1 eV). Translated into more familiar energy units, 1 eV is 1.6×10^{-19} joule. That this unit of energy (i.e., of work) is so small is just a reflection of the small size of the elementary charge. To give an idea of comparative sizes, it takes 6.02×10^{23} protons (H^+ ions) to amount to 1 gram. That huge number, called *Avogadro's number*, is the number of atoms in 1 gram atom, or the number of molecules in 1 mole. You may know its historical origin but wonder where the 1.6×10^{-19} came from. Why is 1 eV $= 1.6 \times 10^{-19}$ joule? Of what large charge unit is the elementary charge such a small fraction? [Answer: 1 elementary charge $= 1.6 \times 10^{-19}$ coulomb] The coulomb is the amount of charge that flows past any point when an electric current of 1 ampere flows for 1 second:

$$1 \text{ coulomb} = 1 \text{ amp sec} \qquad \text{or} \qquad 1 \text{ amp} = \frac{1 \text{ coulomb}}{\text{sec}}$$

Historically, the **amp** came first, making the "ampere second" a logical unit of charge. It turns out to be a large one.

Go back over these definitions of the electrical units: To move 1.6×10^{-19} coulomb against an electrical repulsive force through a potential difference of 1 volt requires an amount of work equal to 1.6×10^{-19} joule. Consequently,

$$1 \text{ coulomb} \times 1 \text{ volt} = 1 \text{ joule} \qquad \text{or} \qquad 1 \text{ volt} = \frac{1 \text{ joule}}{\text{coulomb}}$$

If you like to circle important formulas in red, this is a good one to remember.

13.2 ELECTRIC FIELD AND POTENTIAL

Suppose that you take a number of cells and connect them together. For example, take some $1\frac{1}{2}$-volt flashlight batteries (dry cells) and connect them + to −, + to −, and so on. The stack you get is what is properly called a battery. One hundred of them would have a potential difference of 150 volts between their terminals. Remember, this just means that it requires 150 eV of work to bring one electron from one end of the stack to the other. If you now connect each terminal to a large, flat conducting plate and place the two identical plates parallel to each other, there will be equal and opposite electric charges on the two plates. The battery has not *created* any charge.

What the battery does is to separate positive charge from negative charge. It puts positive charge on the plate connected to the + terminal, and equal negative charge on the plate connected to the − terminal. Each plate is an *equipotential surface*, meaning a surface of constant electric potential. For example, the negative plate could be labeled "0 volts" and the positive plate "+150 volts." The labels are quite arbitrary; only their difference has to be 150 volts (see Fig. 13.1).

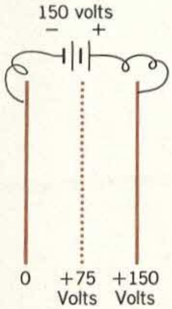

FIGURE 13.1 A 150-volt battery connected between two parallel plates. The 75-volt equipotential surface is halfway between the plates.

As you may remember, that is the way with potential energy. The zero has to be agreed on. It is simply a convention. One simple convention is to choose the largest conductor in the vicinity and call its potential zero. Usually, that is the earth itself; hence the word *ground*. But the zero-potential conductor may be the body of a car, the metal hull of a boat, or the chassis of the TV set.

When no electric current is flowing, any conducting surface is an equipotential surface. If it were not, a charge in one place on the conductor could lower its potential energy by moving to a place having a lower potential. In a conductor, charge is free to move, so current would flow until the charge was redistributed in such a way as to make the potential the same all over the conductor. Repeat: In *electrostatics*—that means when no current is flowing—a conductor is an equipotential.

But how does the potential behave in the space between our two conducting plates? This is empty, and you might wonder how you can talk about the potential at a point in empty space. The idea is, if you did have a charge at that point, what would be its potential energy? The potential at the point is the *work per unit charge* to bring a negligibly small test charge there from the zero-volt equipotential. The locus of points at the same potential is some sort of surface. So the equipotential surfaces are imaginary surfaces connecting all the points of equal potential. In our case, they are planes parallel to the plates. For example, the 75-volt equipotential surface is a plane halfway between the 0-volt plate and the 150-volt plate. If you think about 149 imaginary planes, one at each integer voltage between 0 and 150, they would all be equally spaced. This is stated here as an experimental fact, as a good approximation. The regular spacing of equipotentials actually requires that the plate separation be small compared to the linear plate dimensions. Near the edge of the sandwich the spacing becomes less regular. (See Fig. 13.2.) The exact law that governs the spacing of equipotentials, Coulomb's Law, is discussed in Sec. 13.5.

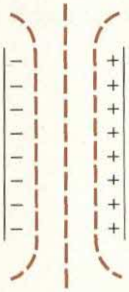

FIGURE 13.2 Three equipotential surfaces are shown in dotted lines. If the negative plate is at 0 volts and the positive plate is at 150 volts, estimate the potentials of the dotted surfaces.

EXERCISE 1 The 75-volt equipotential is halfway between the two plates. Where is the 50-volt equipotential surface? (See Fig. 13.1.)

What characterizes each equipotential surface is that a small test charge placed on it has the same potential energy anywhere on the surface. Imagine taking one elementary charge (+) from the 0-volt equipotential (the negative plate) to the 150-volt equipotential (the positive plate) through the intervening space. That requires 150 electron volts of work, since the potential energy of that test charge has to be raised by that much (= 1 elementary charge × 150 volts). That means there was a force on the charge opposing its displacement from one plate to the other. This is what you expect when you move a positive test charge from the negative plate, which attracts it, to the positive plate, which repels it. The next paragraph calculates that force for a particular plate separation. But before putting in numbers, it is useful to show that the force is the same everywhere in the space between the plates, except near the edges, where it gets weaker. It is just as strong far from the plates as it is close to them. To see that, remember the spacing of the equipotentials: Just as much work is done in going from 0 volt to 1 volt as in going from 99 volts to 100 volts. The corresponding displacements—the spacings of those equipotential planes—are equal. Since the work is the force times the displacement, the forces have to be equal. Now to calculate that force:

Suppose that the plates are 0.15 m (= 15 centimeters) apart. Imagining those 149 equally spaced planes in there, they would be

$$\frac{0.15\ \text{m}}{150} = 10^{-3}\ \text{m} = 1\ \text{millimeter}$$

apart—10 planes per centimeter. The work to move one elementary charge "uphill" by 1 volt is what we have called 1 electron volt (1 eV). Repeat: To move the test charge the 1 millimeter between an adjacent pair of those equipotentials requires 1 eV of work:

$$\begin{aligned} \text{work} &= \text{force} \times \text{displacement} \\ 1\ \text{eV} &= \text{force} \times 10^{-3}\ \text{meters} \end{aligned}$$

Now solve for the force:

$$\text{force} = \frac{1\ \text{eV}}{10^{-3}\ \text{m}} = 10^{3}\ \frac{\text{eV}}{\text{m}}$$

That unit, electron volts per meter, looks like an unfamiliar force unit. But if we remember that 1 eV = 1.6×10^{-19} joule, it becomes

$$\begin{aligned} \text{force} = 10^{3}\ \frac{\text{eV}}{\text{m}} &= 10^{3}\ \frac{\text{eV}}{\text{m}} \times \left(1.6 \times 10^{-19}\ \frac{\text{joule}}{\text{eV}}\right), \\ &= 1.6 \times 10^{-16}\ \text{newton} \end{aligned}$$

(HINT: 1 joule = 1 newton × 1 meter.)

A larger test charge would experience a proportionately larger force, and a statement valid for any size test charge gives the force per unit test charge:

$$\frac{\text{force}}{\text{charge}} = \frac{10^{3}\ \text{eV/m}}{1\ \text{elementary charge}} = 10^{3}\ \frac{\text{volts}}{\text{meter}}$$

Again the units may look unfamiliar until you substitute

$$1 \text{ volt} = 1 \frac{\text{joule}}{\text{coulomb}}$$

so that the unit of "force per unit charge" is

$$1 \frac{\text{volt}}{\text{meter}} = 1 \frac{\text{joule}}{\text{coulomb} \times \text{meter}} = 1 \frac{\text{newton}}{\text{coulomb}}$$

as you might have expected!

The quantity *force per unit charge* is called **electric field.** For the electric field of our example, in which the potential changes by 1 volt for each millimeter displacement, we can write

$$\text{electric field } E = 10^3 \frac{\text{newtons}}{\text{coulomb}}$$

Since force is a vector, electric field is also a vector, a vector defined at each point in space. That is what the mathematicians call a *vector field*. In our example, it is a particularly simple vector field, since it has the same magnitude and direction everywhere inside the sandwich, only falling off and pointing a little skew near the edges of the plates.

One way to describe a vector field is by means of a field map (see Fig. 13.3). That just means putting little arrows in the field direction at representative points of the map. Suppose that lines are drawn following the arrow directions. Those lines are called **field lines,** or lines of force. Where field lines converge, the field gets stronger. Where field lines diverge, the field gets weaker. Where field lines are parallel—as in our example—the field is uniform.

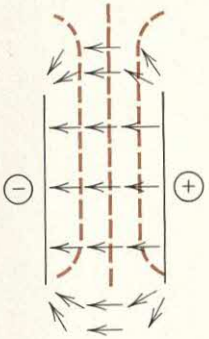

FIGURE 13.3 Field map for oppositely charged parallel plates (Fig. 13.2). The field lines (arrows) are parallel except near the edges, where they diverge from the positive and converge toward the negative plate.

You will see the term *electric field* a lot in this chapter. Why the jargon? Why don't we just talk about electric force? The reason is that the field concept is more powerful. In our example, the force on one electron was 1.6×10^{-16} newtons; the force on a doubly charged ion would have been 3.2×10^{-16} newtons. Saying that the field is 10^3 newtons per coulomb includes both of these statements.

In Sec. 13.4 we look at the electric field outside a spherical conductor. There the field vectors diverge so that the field is not uniform but falls off according to an inverse-square law. That is mathematically more interesting. But many electric fields of biological interest are quite uniform. The transverse electric field across a neural membrane or a muscle sheath is an example: A potential difference of 90 millivolts across a cell membrane 10 nm ($= 10^{-8}$ meters) thick corresponds to an electric field in the membrane of 9 million volts per meter ($= 90 \times 10^{-3}$ volts/10^{-8} meters). In the next section we shall try to get an intuitive idea of the origin of such potential differences by relating them to some notions familiar from chemistry.

13.3 WHAT MAKES CHARGE SEPARATE?

If you have had the experience of biting on a piece of aluminum foil with a tooth that has a deep filling, then you may have felt that most unpleasant sensation caused by electric current flowing through the tooth. Some kinds of candy still come wrapped in foil, and there are restaurants that serve baked potatoes surrounded by a layer of aluminum. Keep the foil away from your fillings!

If the aluminum touches the silver, you have short-circuited what amounts to a battery in your mouth. Aluminum is chemically much more reactive than the silver of the filling and takes on electrons from the saliva—that is the electrolyte here. Indeed, some aluminum dissolves in the saliva in the form of Al^{+++} ions. At the same time, silver, which is a more "noble" metal, converts some H^+ ions from the saliva into hydrogen gas adsorbed at the surface, leaving the silver filling more positive. That is the open-circuited battery, with the silver "electrode" positive, the aluminum "electrode" negative. When the silver filling bites into the aluminum foil we say "the circuit is closed."

This battery can give rise to a steady electric current if the saliva has enough ions to make it a reasonably good conductor of electricity, and it makes electrical contact with the piece of foil. Especially during and after eating, the saliva is likely to be quite a good electrolyte (slightly acid). Making contact also to the gum, the saliva electrolyte conducts so that our silver-aluminum battery can drive some current from the foil through the saliva down through the root of the tooth to the silver filling. By saying "current flows," we mean that charge is transferred all around the circuit. What actually carries the charge is different in different parts of the circuit. Inside the metals, the charge carrier is likely to be an electron, moving in the direction opposite to the electric field, because it is negative. In the saliva and in the soft tissue of the gum it is likely to be a sodium ion (Na^+) or a chloride ion (Cl^-). The Na^+ will move in the direction of the electric field, the Cl^- against it. The source of energy for all this transport, the net chemical reaction that "powers" the battery, is the oxidation of aluminum.

We have used the word *battery* because it is familiar. Good technical jargon is *chemical* **e.m.f.** You may wonder what e.m.f. stands for. It stands for "electromotive force"; but those words are misleading. An e.m.f. is not a force. No one uses those forbidden words, anyway. Some authors like the term *electromotance,* but it has not caught on. People say e.m.f. to refer to a voltage source. They also use the term e.m.f. to mean the voltage of that voltage source when no current is flowing. In Chapter 16 you will read about e.m.f.'s whose energy is magnetic in origin (Faraday induction) rather than chemical. The chemists just use the word *cell* for a chemical e.m.f. That usage gets ambiguous in biological contexts, where we talk about living cells.

Many students ask, when is a voltage an e.m.f. and when isn't it? A simple answer is that an e.m.f. has a source of energy other than electrostatic potential energy to effect the separation of charge. Thus an e.m.f. converts some other form of energy—chemical or mechanical or magnetic—into electrical energy.

In the simple battery, the separation of charge arises from the difference in the binding energy of the two metals for electrons. Electrons are bound more tightly to silver than to aluminum. You may think of this as connected with the chemistry of the metal. Silver is shiny. It does not react easily. Aluminum is dull. Aluminum reacts so fast that what you see is an aluminum oxide layer on the surface. Or you may think of aluminum metal as having a denser electron gas inside it than silver so that the potential difference is associated with a difference in electron concentration. This later interpretation parallels our understanding of bioelectricity, the electrical potentials across cell membranes. These are associated with differences in the concentrations of certain ions on opposite sides of the membrane.

A living cell—a muscle cell, for example—normally has a smaller concentration of chloride ions inside it than in the fluid that surrounds it. Because the membrane is permeable to Cl^- ions, the concentration difference implies a **diffusion potential:** In the absence of electric field, Cl^- ions would diffuse in faster than they diffuse out, just because there are more of them outside than in: The net diffusion is down the concentration gradient. But that means the inside becomes more negative, that is, a potential difference arises as a result of the diffusion. The electric field in the membrane opposes further inward diffusion of negative ions. In equilibrium, the net inward flow associated with diffusion down the concentration gradient is balanced by the net outward flow associated with electrical conduction. Note that the side with the higher concentration of negative ions is the one that becomes more positive. This is quite analogous to our silver-aluminum example. With two metals, the one with the denser electron gas became the positive electrode.

It is important to realize that a diffusion potential—the electric potential difference associated with a concentration difference—requires the possibility of diffusion. In fact, it may be useful to think of the electric field as the cause of the concentration gradient, rather than vice versa. What about the sodium ions, which also have higher concentration outside the cell than inside? Their diffusion potential would be in the opposite sense, positive inside and negative out. Which concentration gradient determines the actual potentials, the Na^+ or the Cl^-? If the membrane were impermeable to sodium, its concentration would not affect the electric potential at all. Indeed, the **permeability** of cell membrane to sodium is much less than the permeability to chloride so that the potential depends more on the chloride than on the sodium concentration. The Cl^- ions move through the membrane much faster than the Na^+ ions so that they dominate the resulting diffusion potential.

The word *equilibrium* was mentioned previously, equilibrium between the inward diffusion and the outward migration driven by the electric field. Actually, the concept of equilibrium in the presence of a concentration gradient is perhaps a bit fuzzy. *Electrochemical equilibrium* is established within times of the order of microseconds: It only takes a few excess negative charges inside to lower the potential dramatically, as we shall see in the next section. But chemical equilibrium, the eradication of the concentration gradient as a result of the diffusion, has a much longer time scale. It is never achieved as long as the cell is alive.

■ SPECIAL TOPIC: The Sodium–Potassium Pump

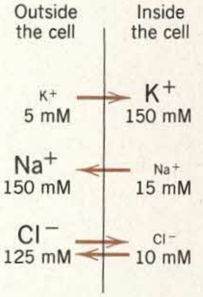

FIGURE 13.4 The sodium-potassium pump keeps the potassium-ion concentration high inside and the sodium-ion concentration low inside the cell. Because the membrane is more permeable to potassium, this makes the inside electrically negative.

What keeps the cell alive is a cycle of chemical reactions that maintains large concentration differences between inside and outside for two ion species: The concentration of sodium ion is kept lower inside the cell than out. The concentration of potassium ion is kept higher inside the cell than out. What is required here is a mechanism that transports potassium in and sodium out, both from a region of low concentration to a region of high concentration. Such a pumping mechanism is exactly the opposite of diffusion, since the movement of ions is *up* the concentration gradient. It is called *active transport*. (See Fig. 13.4.) The mechanism is not well understood yet, but seems to involve a large tubular molecule extending all the way through the membrane. It takes on a potassium ion outside, pushes it through the membrane, releases it inside, takes on a sodium ion inside, pushes it through the membrane, and finally releases it outside, perhaps in such a way as to start the cycle over again. How energy is supplied for this cycle is known. It involves the splitting of the molecule ATP (adenosine triphosphate) to form ADP (adenosine diphosphate) and an inorganic phosphate. So the breaking of the "high-energy phosphate bond" is the energy source for this sodium-potassium pump, which maintains the concentration gradients, consequently the electric potential difference, and thus the life of the cell.

Free potassium ions can also diffuse through the membrane, not as easily as chloride, but more easily than sodium. The potential of the inside of the cell relative to the outside can be as much as −90 millivolts. This is almost exactly what is required to maintain the potassium concentration difference. At body temperature, this means the K^+ concentration inside is 30 times the concentration outside the cell.[2] If the situation were reversed, and the sodium ions traveled through the membrane faster than potassium, the 10:1 concentration ratio of sodium (outside-to-inside) would make the inside 60 millivolts more positive than the outside, rather than 90 millivolts negative, as is the case. In *excitable* cells (nerve and muscle) such a reversal does indeed occur. The membrane's permeability to sodium ions becomes briefly enhanced as much as 600-fold. During the millisecond or so of this increased sodium influx, the potential inside does go positive, although not quite to 60 millivolts. This brief electric pulse is known as an **action potential.** It is the basis of the electrocardiogram (EKG), electromyogram (EMG), and electroencephalogram (EEG). The full voltage excursion can be as much as 120 millivolts. A substantial fraction of this can be developed across different parts of the skin and thus measured noninvasively. ■

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[2] The formula $\Delta V = -(61.5 \text{ millivolts}) \log [K^+]_{in}/[K^+]_{out}$ known as the Nernst equation, will not be derived here.

EXERCISE 2 As a cell dies, what do you think happens to the potential difference between inside and outside?
[Answer: You guessed it. It goes to zero. The pump stops pumping and diffusion wipes out the concentration gradients.]

13.4 CAPACITANCE

If you have just shuffled across one of those nylon carpets and charged yourself up electrically, the man who shakes your hand is likely to draw much the same sort of spark as a door knob. Even if there is no spark, you share your charge, which flows easily through the salty perspiration of your hands. Each of you ends up with half the charge you had, assuming none flows off into the floor, and also assuming you are roughly the same size. If he is much bigger than you are, he gets a larger share of the charge.

This happens because your skin surface is a reasonably good conductor. On the time scale of a handshake, you and he come to the same potential. Actually, it takes only a few microseconds. Your skin becomes approximately an equipotential surface. The voltages in static electrification phenomena can be hundreds of thousands of volts, so the word *approximately* here means + or − a few volts. If the two of you are the same size, your potential *after* is half your potential *before* the handshake. Half as much charge, half the potential. This says that the potential to which a conductor is raised by charging it is proportional to the charge. Restated: The amount of charge needed to raise the potential of a conductor is proportional to the potential rise. The remark about "if he is much bigger than you are" means that the proportionality constant has something to do with size. The bigger conductor can contain more charge than the smaller one for the same rise in potential. We say it has a larger *electrical capacitance*.

Why introduce a new term? The capacitance of an object tells you how much charge flows when its potential is increased by a certain amount. The ratio of the amount of added charge to the potential change when it is added is what is defined as capacitance:

$$\text{capacitance} \equiv \frac{\text{charge}}{\text{potential}}$$

$$C \equiv \frac{Q}{V}$$

If the charge is measured in coulombs and the potential in volts, capacitance is measured in *farads*. Thus

$$1\ \text{farad} = \frac{1\ \text{coulomb}}{1\ \text{volt}}$$

As for orders of magnitude, it requires about 10^{-10} coulombs to raise the potential of an adult human body by 1 volt; so its capacitance is roughly 10^{-10} farads or a hundred picofarads.[3] If you remember that one electron has a charge of 1.6×10^{-19} coulomb, a charge of 10^{-10} coulombs no longer seems so small; it represents losing close to a billion electrons. Raising the potential of the body by 10,000 volts requires one microcoulomb—ten thousand times as much charge.

That figure of 10^{-10} farad is just an estimate for an average person. For simpler geometrical shapes there are exact formulas for how capacitance depends on the dimensions. For a spherical conductor the capacitance is simply proportional to the radius. The formula, which you need not memorize, is

$$C_{\text{sphere}} = \frac{\text{radius of sphere}}{9 \times 10^{9}\ \text{meters/farad}} = \frac{r}{k_e} \qquad \text{(Eq. 1)}$$

Thus a sphere 1 centimeter in radius has a capacitance of about 1 picofarad. The whole earth has a capacitance of 0.007 farads. You will meet k_e again in Sec. 13.5. The enormous size of the farad is, of course, just a reflection of the enormous size of the coulomb.

EXERCISE 3 Show that a charge of 1 coulomb on that 1-centimeter sphere would raise its potential by almost 10^{12} volts.

One other geometry is particularly simple: oppositely charged parallel conducting plates. As you recall, the field in between the plates is uniform. This is essentially the situation in a cell membrane, where the fluid inside is at one potential, the fluid outside at a different potential. The capacitance of the membrane is a measure of how much charge has to be transferred through the membrane per unit change of the potential difference. We shall use the formula for the spherical conductor to derive the formula for parallel plates.

Parallel Plates

Just as in Sec. 13.2, consider a sandwich whose length and width dimensions are much greater than its thickness. That is because we want a uniform field and would like to neglect the nonuniformity, the tapering off at the edges. If we double the area of such a sandwich while keeping the potential difference between the plates constant, the field inside stays the same; but it will require twice the charge on each plate. This is the same as laying two identical sandwiches side by side to make a sandwich twice the area of each: The capacitance is simply proportional to the area.

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[3] pico = 10^{-12}; micro = 10^{-6}.

How does it depend on the thickness of the sandwich, the plate separation d? [Answer: The thinner the sandwich, the larger the capacitance. The formula

$$C_{\text{sandwich}} = \frac{1}{4\pi k_e} \frac{\text{area}}{d} \qquad \text{(Eq. 2)}$$

is proved in the box. But it is suggested you read Sec. 13.5 first, and then come back and read this proof.]

EXERCISE 4 Having shuffled across one of those luxurious nylon carpets, you are charged up to 10,000 volts. That means your skin is an equipotential surface at a potential 10,000 volts higher than your surroundings ("ground").

(a) If you share your charge by touching another person of your size (i.e., same capacitance), each of you will be at ______ volts.

(b) After this you touch a third person, also the same size. Both of you are now at ______ volts.

(c) If you now touch the second person again, the charge on each of you will distribute uniformly on both of you. At what potential will you both be after this?

(d) Argue that the total charge on the three of you is equal to the charge you started out with.

EXERCISE 5 A capacitance of 1.0×10^{-10} farads is charged up to a potential of 1.0×10^4 volts.

(a) Argue that its charge is one microcoulomb.

(b) An uncharged capacitance of 100 picofarads is briefly connected to the charged one. After being disconnected, each capacitance has a charge of half a microcoulomb. What is the potential of each?

(c) The first one is again charged up to 10,000 volts. Argue that, when the two are reconnected, they share a charge of 1.5 microcoulombs.

(d) What is now the potential?

(e) They are disconnected, and the process is repeated. Argue that the potential is now 8750 volts.

PROOF OF PARALLEL-PLATE CAPACITANCE FORMULA[4]

The argument goes back to the sphere. Imagine two concentric conducting spherical shells whose radii are almost the same—differing by that small distance d. The inner one carries positive charges the outer one an equal negative charge. The space between them looks, at any one place, very much like the space between parallel plates, just as our round earth looks flat to us as long as we stay close to the ground. For example, take an inner sphere of radius 1 meter, charged up to a potential of 1000 volts. Equation 1 tells how much charge that requires.

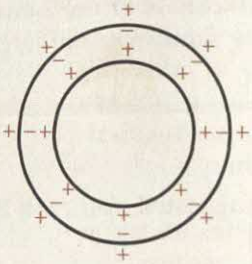

FIGURE 13.5 The inner metal shell is charged; the outer one is uncharged. The positive charge on the inner sphere induces an equal negative charge on the inner surface of the outer sphere. This leaves an equal positive charge on the outer surface.

EXERCISE 6 Carry out the calculation. $Q = ?$

Any spherical surface concentric with our 1-meter sphere is an equipotential surface; that we can conclude from symmetry considerations. But Eq. 1 also tells you the potential for any given radius. For example, a sphere 1.01 m in radius is at 990 volts: The radius, and therefore the capacitance, is 1% greater; the potential has to be 1% less. The potential difference between the two conductors, which are 1 cm apart, is thus 10 volts, which is 1% of the 1000 volts. Try to visualize the situation as in Fig. 13.5: The positive charge Q on the inner sphere induces a charge $-Q$ on the inner surface of the outer sphere, leaving the same old charge $+Q$ on the outer surface of the outer sphere. The charge on this outer surface is neutralized when $-Q$ is put on. To calculate the capacitance of the sandwich, we divide the charge Q by the 10-volt potential difference.

EXERCISE 7 Calculate it. But if we had chosen an outer-sphere radius of only 1.001 m—the two surfaces 1 millimeter apart—the potential of that sphere would have been only 0.1% less than the 1000 volts on the inner sphere, so the inner-to-outer potential difference would have been only 1 volt, or 0.1% of the 1000 volts. In that case the capacitance of the sandwich (charge divided by potential difference) would have been 10 times greater than before. A sandwich one-tenth as thick has 10 times the capacitance. This inverse proportionality of capacitance and plate separation is what we were trying to establish.

Let us repeat the argument algebraically: The charge Q that raises the inner sphere to potential V_{inner} is given by Eq. 1 as

$$\frac{Q}{V_{\text{inner}}} = C_{\text{sphere}} = \frac{r}{k_e}$$

That charge Q raises the potential of the outer sphere (radius $r + d$) somewhat less:

$$\frac{Q}{V_{\text{outer}}} = \frac{r + d}{k_e}$$

Now solve the above equations for the two potentials. The potential difference

$$V_{\text{inner}} - V_{\text{outer}} = k_e Q \left(\frac{1}{r} - \frac{1}{r + d}\right) \qquad \text{(Eq. 3)}$$

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[4] Read this *after* Sec. 13.5.

becomes proportional to the separation d if $d << r$, as we have assumed. Here is the algebra:

$$\frac{1}{r} - \frac{1}{r+d} = \frac{d}{r(r+d)} \quad \text{exactly}$$
$$\cong \frac{d}{r^2} \quad \text{approximately}$$

The capacitance of the sandwich is

$$C_{\text{sandwich}} \equiv \frac{Q}{V_{\text{inner}} - V_{\text{outer}}} = \frac{1}{k_e}\frac{r(r+d)}{d} \quad \text{exactly}$$
$$\cong \frac{1}{k_e}\frac{r^2}{d} \quad \text{approximately}$$

Recalling that the area of a sphere is $4\pi r^2$, we can write this capacitance in terms of the area:

$$C_{\text{sandwich}} = \frac{1}{4\pi k_e}\frac{\text{area}}{d} \qquad \text{(Eq. 2)}$$

This result can be taken over verbatim to the parallel plates. First we know that the capacitance of a parallel-plate sandwich is proportional to the area of the plates. Think of it as a bite taken out of a large-sphere sandwich. Then we know it is inversely proportional to the plate separation d. Equation 2 gives us the proportionality constant.

EXERCISE 8 Living Cell

Now we are in a position to calculate the capacitance of a cell membrane. We can verify the statement made in Sec. 13.3 that it does not take many ions passing through the membrane to cause noticeable changes of the potential of the inside of the cell. A typical cell may be one-hundredth of a millimeter (= 10 microns) in diameter. Assume that it is roughly spherical. Then its area is 3×10^{-10} m^2 (= 300 microns^2). If the membrane is 10 nm thick (= 10^{-8} m), the capacitance is

$$C = \frac{1}{4\pi k_e}\frac{\text{area}}{\text{thickness}} = \frac{3 \times 10^{-10}\ \text{m}^2}{4\pi\ (9 \times 10^9\ \text{meter/farad}) \times 10^{-8}\ \text{m}}$$
$$= 0.25 \times 10^{-12}\ \text{farads}$$
$$= 0.25\ \text{picofarads}$$

How many singly-charged ions does it take to nullify a 90-millivolt potential difference between inside and outside? It requires an influx of only

$$(90 \times 10^{-3}\ \text{volts})(2.5 \times 10^{-13}\ \text{farads}) = 2.25 \times 10^{-14}\ \text{coulombs}$$

worth of positive ions, or

$$\frac{2.25 \times 10^{-14}\ \text{coulombs}}{1.6 \times 10^{-19}\ \text{coulombs/ion}} = 1.4 \times 10^5\ \text{ions}$$

See Fig. 13.5.

Suppose that the potassium ion concentration inside is 0.15 moles per liter. Then in the cell volume of $5 \times 10^{-16}\ \mathrm{m}^3$ (= 500 $\mu\mathrm{m}^3$) there would be (HINT: 1 mole = 6×10^{23} ions = Avogadro's number.)

$$\frac{0.15 \times 6 \times 10^{23}\ \text{ions}}{\text{liter}} \times 5 \times 10^{-16}\ \mathrm{m}^3 \times \frac{10^3\ \text{liters}}{\mathrm{m}^3}$$
$$= 4.5 \times 10^{10}\ \text{potassium ions}$$

inside the cell. The addition of only ($1.4 \times 10^5 \div 4.5 \times 10^{10} =$) 3 ions per million already present, three parts per million, gives a 90-millivolt voltage swing.

But please don't think that 0.25 picofarads is a "small" capacitance on the scale of living things. Recall that the whole body has a capacitance of perhaps 100 picofarads.

EXERCISE 9 The cell, viewed not as a conductor-insulator-conductor sandwich, but just as a sphere 5 microns in radius, has a capacitance of only

$$\frac{r}{k_e} = \frac{5 \times 10^{-6}\ \mathrm{m}}{9 \times 10^9\ \mathrm{m/farad}} = \underline{\qquad\qquad}\ \text{farads}$$

That capacitance is of no biological significance.

Energy of a Charged Capacitor

If you want to add one elementary charge to a conducting surface charged up to a potential of 1000 volts, that requires an energy of 1000 electron volts. You have to do an amount of work equal to 1000 eV against the electrostatic forces repelling that charge from the surface. The capacitance of the surface determines how much the addition of that charge raises the potential of the surface. Take the human body: $C \approx 10^2$ picofarads. Adding a charge of 1.6×10^{-19} coulombs (= 1 elementary charge) raises the potential by

$$\frac{1.6 \times 10^{-19}\ \text{coulombs}}{10^{-10}\ \text{farads}} = 1.6 \times 10^{-9}\ \text{volts}$$

That doesn't make much of a dent in the 1000 volts we started out with. Nor will the next charge we add, which still requires just a hair over 1000 eV of work. Every charge we add requires a little more work than the previous one, because the potential of the surface is a little bit higher. If you want to double the potential, you have to double the charge. The last elementary charge brought in sees a potential of almost 2000 volts, and thus requires an energy of 2000 eV to overcome the electrostatic repulsion and get to the surface. The first elementary charge required 1000 eV; the last one required 2000 eV. How much energy is required to double the potential?

We have to sum the energies of all those elementary charges. Remember, the work required to add a charge to the surface is proportional to the charge already there. The average potential that has to be overcome in raising the surface from 1000 volts to 2000 volts is 1500 volts—the halfway potential. The work of charging is the product of the charge we add times this average potential. To change the potential of a surface with a capacitance of 10^{-10} farads by 1000 volts takes a charge of 10^{-7} coulombs (= 10^{-10} farads $\times$ 10^3 volts). So the total energy needed to go from 1000 to 2000 volts is

$$10^{-7} \text{ coulombs} \times 1500 \text{ volts} = 1.5 \times 10^{-4} \text{ joule}$$

(HINT: 1 volt = 1 joule/coulomb.) What about the energy to go from 0 to 1000 volts, to put the first 10^{-7} coulombs of charge on the surface? The average potential during that initial charging process is 500 volts, so the energy needed to go the first thousand volts is one-third the energy needed to go the second thousand: To double the potential requires quadrupling the energy.

Here is the argument again: To charge a capacitance C to a voltage V requires a charge $Q = CV$. The average potential during the charging is $\frac{1}{2}V$, so the electrostatic energy of the charged capacitance is $\frac{1}{2}QV$, or $\frac{1}{2}CV^2$: The energy is proportional to the square of the voltage.

Now it is easy to see where the spark you get when you share your charge with another person your size gets its energy! Before sharing, you had charge Q_o at potential V_o, with total electrostatic energy $\frac{1}{2}Q_oV_o$. After sharing, each of you has charge $\frac{1}{2}Q_o$ at potential $\frac{1}{2}V_o$, for an energy of $\frac{1}{8}Q_oV_o$ each, or a total energy of $\frac{1}{4}Q_oV_o$. Half the original energy gets lost somewhere. The spark is one way it could be dissipated, partly radiated away as light, partly as sound, mostly going into heat. Note that charge is conserved; electrical energy is not.

EXERCISE 10 When a 1 microfarad capacitor is charged to 1000 volts, its charge is 0.001 coulombs. Show the arithmetic. Then argue that the stored energy is 0.5 joule.

13.5 POINT CHARGE AND INVERSE-SQUARE FIELD[5]

The electric field has been introduced (Sec. 13.2) in the context of parallel conducting plates, because that geometry gives rise to a uniform field. Nevertheless, we all have the intuition that the further away you get from a charge, the less you ought to feel its influence. With the parallel plates, as long as you don't explore near the edges, that intuition might be accused of being deceptive: For a large plane of charge, the field does not fall off with distance. In this section, we study the nature of the field surrounding a single charge. This is not an academic exercise. Remember, charge is quantized: A proton has the smallest amount of positive charge found in nature. An electron has the smallest amount of negative charge. The magnitudes of these two opposite charges are equal, to phenomenal experimental accuracy, that is, to better than one part in 10^{20}.

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[5] Please postpone reading the proof of the parallel-plate capacitor formula (box) until after you have read about the inverse-square field.

Although a proton has what might be called a size (about 10^{-15} meters), it is so small that the abstraction of thinking of its charge as located at a "point" is tempting. Near the end of this section it will turn out that the field of a point charge is the same as the field outside a sphere of charge, so the abstraction has no harmful logical consequences. We won't worry whether a point charge consists of one elementary charge or a billion. But be prepared for a couple of pages of mathematical reasoning with no immediate application. Their purpose is to built your intuition about the nature of electrostatic interaction. Since that includes most of chemistry, no apology is offered.

Equipotential Surfaces Are Spheres[6]

A point charge is a nicely symmetrical object. It does not "face" in any particular direction. Suppose that another point charge is brought near. Let us call this second charge a "test charge" and suppose that both charges are positive. The strength of the repulsive force on the test charge depends only on how far away it is, not on "which way." All directions are equivalent. Consequently, the work done in bringing such a test charge from "far away" to a particular position near our point charge depends only on how far that position is from our point charge. It does not matter from what direction the test charge is brought in. So the equipotential surfaces associated with our point charge are spheres centered on it: a family of concentric spheres. It requires no work to move a test charge around on any given equipotential sphere. It does take work to move it to a smaller (higher potential) sphere. But please remember, these equipotential spheres are mathematical objects. There is nothing there to see or touch.

The mathematics comes in when we describe the spacing of the equipotential spheres. Unlike the equipotential planes between parallel plates, these are not uniformly spaced. The farther away you go from the point charge, the larger the spacing corresponding, say, to 1 volt. Here is the rule: If doubling the distance from the point charge gives a certain potential drop, then doubling again gives only half the drop. For example, in moving from a distance of 1 Å (= 10^{-10} meters) from a proton to a distance of 2 Å, the potential falls by 7.2 volts. In moving from a distance of 2 to 4 Å, the potential falls by 3.6 volts. Here is one way to put that into a formula:

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[6] For a point charge.

$$V(r) = k_e \frac{q}{r} \qquad \text{(Eq. 1)}$$

In words: The potential at a distance r from a point charge q is inversely proportional to r.

To be sure, Eq. 1 is just one way to express that rule. It specifies a particular choice of the "zero" of potential. We have said that such a choice is always arbitrary and made to suit our convenience. This particular choice leaves us with no value of r at which $V(r)$ is equal to zero. Another way to say that is, only if r is infinite is the potential equal to zero. This turns out to be a convenient choice. But note that any number added to the right-hand side of Eq. 1 gives a correct equation.

The universal constant k_e is measured to be

$$k_e = 8.99 \times 10^9 \ \frac{\text{volts meters}}{\text{coulomb}}$$

It is sometimes called the Coulomb constant. You have met it in Sec. 13.4, Capacitance, in the formula for the capacitance of the parallel-plate capacitor. Letting the electronic charge $e = 1.6 \times 10^{-19}$ coulombs, we can write

$$k_e e = 1.44 \times 10^{-9} \text{ volts meters} = 1.44 \text{ volts nm}$$

EXERCISE 11 Use Eq. 1 and this constant to verify the "example" above, those 7.2 and 3.6 volts.

EXERICSE 12 Use Eq. 1 to verify Eq. 13.4–1 for the capacitance of a spherical conductor.

EXERCISE 13 We are told that the sphere of radius 10 cm is the 1000-volt equipotential surface. (See Fig. 13.6.) We are not told the value of the charge that is the source of the electric field.

(a) Show that the potential at $r = 20$ cm is 500 volts.

(b) What is the potential at $r = 30$ cm?

(c) In going from $r = 10$ cm to $r = 20$ cm, the potential dropped 500 volts. In going from $r = 20$ cm to $r = 30$ cm, the potential dropped 167 volts. [Check your answer to part (b).] How much does the potential drop in going from $r = 30$ cm to $r = 40$ cm?

(d) How far do you have to go to drop from $V = 250$ volts to $V = 200$ volts? You can see how the spacing of equipotentials gets wider and wider as you move away.

(e) Why did you never have to calculate the value of the charge?

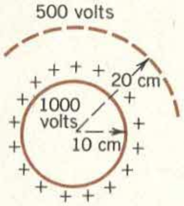

FIGURE 13.6

EXERCISE 14 Show that the Coulomb constant can be written $k_e = 2.3 \times 10^{-28}$ newton m^2/(elementary charge)^2.

Inverse-Square Field

In Chapter 11 we talked about how the light intensity at a distance r from a light bulb falls off with increasing r. A given amount of power gets distributed over an increasing area. The area of a sphere of radius r is $4\pi r^2$. So the power per unit area, power/$4\pi r^2$, falls off according to an inverse-square law. All we had to assume was that no energy gets lost on the way. But why talk about light in a section about electrostatics?

The electric field of a point charge has this same inverse-square falloff. The formula for the electric field strength at a distance r from a charge q is[7]

$$E(r) = k_e \frac{q}{r^2} \qquad \text{(Eq. 2)}$$

Inverse-square fields occupy such an important role in physics that many textbooks begin the study of electricity by announcing **Coulomb's Law:**

> The electrostatic force of repulsion (or attraction) between two point charges falls off as the inverse square of the distance between them.

From the study of astronomy we know that attraction between stars and planets also obeys an inverse-square law: Newton's Law of Gravitation. Both Coulomb's Law of electrostatic interaction and Newton's Law of gravitational interaction turn out to be very accurately inverse-square laws. For both laws, there are very precise experiments to distinguish whether the exponent of r is really -2.000000 or might perhaps be -2.000001. For the electrostatic interaction, the result is that the -2 is good to better than 10 decimal places. In Problem 47 we talk about the special feature of an inverse-square field not shared by any other type of r-dependence.

Electrostatic Induction

A phenomenon directly related to the inverse-square law is electrostatic induction. If a charge q is surrounded by a conducting shell of any shape, then a charge $-q$ is "induced" on the inner surface of that shell. That leaves a charge $+q$ on the outer surface. To test that, you could just connect the shell to a very large conductor and measure how much charge flows off.

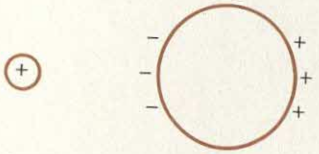

FIGURE 13.7

The other side of the same story is that charge located somewhere outside a conducting shell induces charge on the outside surface of the shell. On the sketch (Fig. 13.7) you see unlike charge induced on the near side while like charge is induced on the far side. Here the logic is more involved. A conductor is a material that has freely moving charges within it. Therefore, in electrostatics (i.e., in equilibrium) all points of a conductor are at the same potential. If they were not, those mobile charges would experience forces and would move under the action of those forces. In other words, the charge would redistribute until there were no forces, that is, until the entire conductor came to the same potential. Now you know that when no charge is moving, there is no electric field inside a conductor.

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[7] If you know calculus, you can differentiate Eq. 1 to obtain Eq. 2: A $1/r$ potential implies a $1/r^2$ field.

How about a conductor with a hole in it? A conductor with a cavity completely surrounded by conductor with no artificially introduced bound charges also has no electric field inside. For our conducting shell with some external charge (Fig. 13.4), the distribution of the induced charge on the surface is such as to just cancel out the electric field in the cavity. This phenomenon is called **electrostatic shielding.** Now you understand why sensitive electrical instruments are mounted in metal cases. Sometimes entire laboratories are protected from "electrical noise" due to nearby machinery by surrounding them with wire cages.

SUMMARY

An insulator can be charged by rubbing it with an insulator made of a different material. Potential differences of many thousands of volts can be achieved in this way.

When two conductors differing in potential by several thousand volts get close enough, a spark will jump between them.

Definition of the volt: The amount of work needed to bring one elementary charge to a place having a potential of 1 volt from far away is 1 electron volt (1 eV). By proportions,

1 volt = 1 joule per coulomb

1 elementary charge = 1.6×10^{-19} coulombs

1 electron volt = 1.6×10^{-19} joules

In the region between oppositely charged parallel conducting plates, the equipotential surfaces are a family of uniformly spaced parallel planes. The electric field in this region is uniform. That is, the force on a test charge is the same everywhere inside the sandwich. It falls off close to the edges.

Batteries: Different metals have different binding energies for electrons. When two different metals are put in contact with an electrolyte, separation of charge results. Such a metal-electrolyte-metal sandwich is called a *cell.* Household word: battery. When a battery is connected in a complete circuit, the chemical reactions occurring can drive a steady current around the circuit.

In a living cell the Cl^- ion concentration is lower than in the intercellular fluid outside. This concentration difference across the Cl^--permeable cell membrane is evidence that the inside is at a lower potential than the outside. In a nerve or muscle cell, this potential difference can be as much as 90 mV.

In an excitable cell, the permeability of the cell membrane to Na^+ ions is briefly enhanced, so the Na^+ concentration inside increases along with the electrical potential. This *action potential* lasts only a few milliseconds. The voltage spike is the basis of the EKG, EMG, and EEG.

Large conductors have larger electrical capacitance (ratio of charge to voltage) than small ones. For a sphere, the capacitance is directly proportional to the radius.

The capacitance of parallel plates is proportional to their area and inversely proportional to their separation.

The work needed to charge a capacitor with charge Q to a potential V is $\frac{1}{2}QV$.

The electric field of a point charge falls off inversely as the *square* of the distance from the charge. Such an inverse-square field is described by a potential $V(r) = k_e(q/r)$. The magnitude of the electric field vector is given by

$$E(r) = \frac{k_e q}{r^2}$$

For a + charge, the E-field is directed away from the charge. For a − charge, the E- field is directed toward the charge.

The field outside a charged conducting sphere is the same as if the charge were all located at the center.

KEY TERMS

Triboelectricity Charging of an insulator by rubbing another insulator against it.

Conductor A material in which electric charge moves freely.

Insulator A material in which electric charge tends to remain localized. Even high electric fields give rise only to small electric currents.

Elementary charge The charge on one electron (negative) or the charge on one proton (positive). The smallest charge found in nature.

Ion An atom or molecule that has become charged by gaining or losing one or more electrons. Thus the H^+ ion is the proton; the H^- ion is a proton with two electrons.

Ionize To cause the formation of ions. An electric spark in air causes electrons to be torn from the N_2 and O_2 molecules. The passage of fast charged particles through matter usually results in ionization.

Volt Unit of potential, defined as 1 joule per coulomb.

Electron volt Unit of energy. The work done in moving 1 electronic charge through a potential difference of 1 volt. 1 eV = 1.60×10^{-19} joules.

Potential Potential energy per unit charge.

AC Alternating current. In the United States, the frequency is 60 Hz. In Europe, 50 Hz is standard.

Amp The ampere is the unit of current, defined as 1 coulomb per second.

Equipotential surface An imaginary surface that is the locus of points having equal potential. When no current is flowing, the surface of any conductor is an equipotential surface.

Ground The largest conductor in the vicinity. Usually that is the earth, but it may be the chassis of the car or the metal case of the television set.

Electric field This term really has two meanings:

1. A region in which a charge would experience a force.
2. A vector whose magnitude is the force per unit charge, in the direction of the force on a positive test charge.

Field line A line having the direction of the electric field at every point. Field lines start on positive charge and end on negative charge. Where the field gets weaker, they diverge; where the field gets stronger, they converge.

Electronegativity If one substance is more electronegative than another, that means it requires more energy to remove an electron from it. Sulfur is more electronegative than sodium.

E.m.f. Pronounced "eeyemef." A source of electrical energy. Also, the open-circuit voltage of such an energy source. There are chemical e.m.f.'s and magnetic e.m.f.'s. The word *electromotance* is also used (rare).

Bioelectricity The various electric phenomena in living things. The potentials appearing across the membranes of living cells.

Active transport The movement of ions in living membranes in the direction of higher concentration, that is opposing diffusion.

Diffusion potential Electric potential associated with a concentration gradient of an ion species.

Permeability A measure of how easily a given molecular species passes through (a membrane). Usually expressed in moles per second per square centimeter per unit partial pressure difference. (See also Chapter 15 for another meaning, related to magnetic properties.)

Sodium–potassium pump The mechanism of active transport in living cells. Potassium is pumped in and sodium is pumped out. The inside of the cell achieves a negative potential relative to the outside.

Action potential A brief electric potential pulse across the membrane surrounding an excitable (i.e., nerve or muscle) cell. It is associated with the depolarization of the membrane as its sodium permeability is briefly enhanced by several orders of magnitude.

Capacitance Quotient of the charge on a conductor divided by the potential needed to maintain that much charge on it. Evidently, larger conductors have larger capacitance. The unit is the farad = 1 coulomb per volt.

Coulomb's Law The force of repulsion (attraction) between two point charges is proportional to the product of the charges and inversely proportional to the square of their separation:

$$F = k_e\, q_1 q_2 / r^2$$

with

$$k_e = 9.0 \times 10^9 \text{ newton } \frac{\text{meters}^2}{\text{coulomb}^2}$$

Electrostatic induction The mechanism by which a conductor acquires a charge by virtue of the proximity of charge of the opposite sign.

Electrostatic shielding Surrounding an instrument with a metal case or screen in order to keep out unwanted electric fields.

Dielectric An insulating material. A material with very small electrical conductivity. The application of an electric field to a dielectric polarizes the material.

Dielectric constant Usually called κ (kappa). The proportionality constant relating the electric field outside to the electric field inside a dielectric: $E_{\text{outside}} = \kappa E_{\text{inside}}$. More precisely, this is the relation between the components of the electric field normal to the surface. The dielectric constant of the vacuum is 1. The dielectric constant of air is 1.0006. The dielectric constant of water is 80.

Dielectric breakdown At very high fields, electrons may be torn off molecules, suddenly converting a dielectric into a conductor.

Dielectric strength The highest electric field that does not cause dielectric breakdown.

Faraday One mole of electric charge. Avogadro's number of electronic charges = 96,500 coulombs.

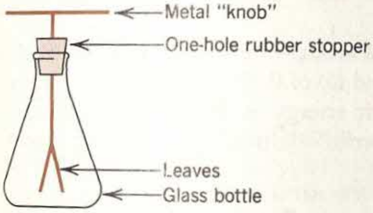
REVIEW QUESTIONS

- The shiny knob at the top of an AA flashlight battery is at a potential ______ volts higher / lower than the metal disk at the bottom.
- If two conducting plates 1 millimeter apart have a potential difference of 15 volts, the *E*-field in the space between them is ______ volts per meter.
- In the region between those plates, the force on a test charge 0.1 mm from the + plate is ______ times the force on a test charge twice as far from the + plate.
- If the two plates are kept connected to the 15-volt battery while the plates are brought 1% closer, the *E*-field between them increases / decreases by ______%, and the charge on each plate increases / decreases by ______%. What about the capacitance?
- Zinc is more electronegative than silver. In a silver–zinc battery the terminal connected to the silver electrode is marked + / −.
- Inside a living cell, the chloride concentration is lower than outside the cell. Evidently, the electric potential inside is higher / lower than outside.
- If the radius of a metal sphere were increased by 1%, its capacitance would increase / decrease by ______%. Explain.
- A small metal sphere is charged positively so that the potential at a distance of 1.00 meter from its center is 100 volts. What is the potential 1 cm further? What is the *E*-field at r = 1.00 m? How much larger / smaller is the *E*-field 1 cm further out? ______%
- To charge a 10-picofarad capacitor up to 100 volts requires a charge of ______ coulombs. This takes ______ joules of work.

PROBLEMS

✶ **1. Electron Volt** To move one elementary charge (1.6×10^{-19} coulombs) through a potential difference of 1 volt requires an amount of work equal to 1 electron volt.
(a) To move two elementary charges through a potential difference of 1 volt requires _______ eV of work.
(b) To move one elementary charge through a potential difference of 2 volts requires _______ eV of work.

✶ **2. The Faraday** Avogadro's number is 6.02×10^{23}; that is the number of molecules in a mole. Avogadro's number of electrons has a charge called 1 **Faraday.** Express that in coulombs.



PROBLEM 8
Electroscope.

✶ **3. Parallel Plates** Parallel plates 10 cm apart are kept at a potential difference of 90 volts.
(a) To move an electron from the + to the − plate requires _______ electron volts to work.
(b) To move an electron 1 cm away from the + plate toward the − plate requires _______ eV of work.
(c) To move an electron from the 10-volt equipotential to the 20-volt equipotential requires _______ eV of work.
(d) If you call the negative plate 0 volt and the positive plate +90 volts, how far from each plate is the 45-volt equipotential?
(e) Where is the 10-volt equipotential surface?

✶ **4. Parallel Plates.** (Refer to Problem 3.) What is the electric field strength in the region between the plates?

✶ **5. Parallel Plates.** (Refer to Problem 4.)
(a) How much work is done in moving an electron from the 45-volt equipotential to the 46-volt equipotential?
(b) How far apart are those two equipotential planes?
(c) What is the force on an electron at the 45-volt equipotential?
(d) Compare the quotient of *a*/*b* with *c*. (HINT: Make sure you express them in the same units.)

✶✶ **6. Speed of Electrons** How fast is a 20,000-eV electron traveling?

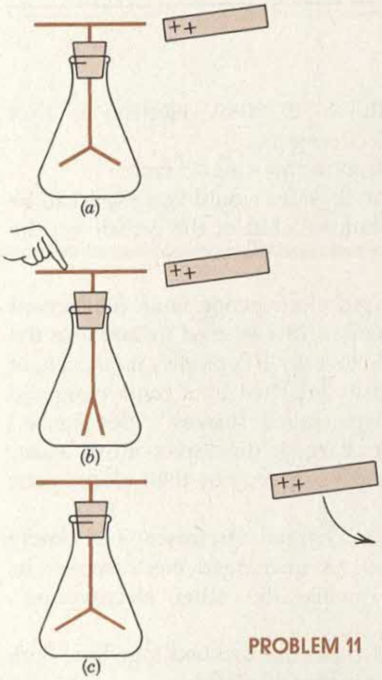
✶✶ **7. Parabolic Path** A 20,000-eV electron is shot horizontally from the electron gun.
(a) How much does it drop in traveling 25 cm?
(b) If you lay your TV on its side, would you expect to be able to detect the gravitational shift of the picture on the screen? Argue.

✶✶ **8. Electroscope** An electroscope is an instrument used to detect static charge. (If it is used to measure the charge, it is called an *electrometer*.) Typically, it consists of a conducting knob or plate attached by a conducting rod to a pair of gold-leaf flaps, called "leaves." (See figure.) When the conductor is charged, the leaves move apart, that is, they are raised against gravity by their electrostatic repulsion.
(a) When the knob of a charged electroscope is briefly touched to the knob of an uncharged electroscope, its leaves collapse partly, while the other electroscope's leaves rise. Explain.
(b) Two charged electroscopes are touched together. Both pairs of leaves fall. How is this possible?
(c) As those two charged electroscopes' knobs approach each other, the leaves of both start to collapse even before they touch. Explain. (HINT: If you separate them without allowing them to touch, they both return to their previous deflections.)

✶✶ **9. Two Electroscopes** Two identical electroscopes are charged with + charge. A has twice the charge of B. Now they are touched together. Explain why one collapses partly, one deflects further. (HINT: Touching the two conductors together brings them to the same potential.)

✶✶ **10. Two Different Electroscopes** A big electroscope is given a small + charge; a small electroscope is given a large + charge. Describe what happens when they are touched together.

✶ **11. Charging by Induction** A rod carrying a + charge is brought close to the knob of an electroscope, without touching. The leaves diverge.
(a) Explain how it is that the leaves carry a charge even though the electroscope as a whole is electrically neutral.
(b) Without removing the rod from the proximity of the knob, you touch the knob with your hand. The surface of your skin is a reasonably good conductor, so this amounts to grounding the electroscope. Argue that it now has a net



PROBLEM 11

negative charge, even though the leaves are collapsed.
(c) You remove your hand. Nothing happens. But when the rod is removed, the leaves diverge. Explain.

✱✱✱ **12. Benjamin Franklin** He was a Philadelphia printer, U.S. ambassador to France, and an inventor. He also made a number of fundamental discoveries about electricity. He introduced the concept of an "electric fluid" (positive charge) and its absence or deficit (negative charge). His convention is followed to this day: Electrons carry negative charge.

(a) What if he had made the opposite choice? Name some familiar things whose labels would be changed. Don't forget batteries!

(b) What if a third kind of charge should be discovered? What properties would it have? How would it interact with the + and − charge? This question is intended to exercise your imagination.

Benjamin Franklin (1706–1790).

✱✱ **13. Parallel-Plate Capacitor**

(a) The charge on a parallel-plate capacitor is kept constant while the plate separation is halved. What happens to the potential difference?

(b) The potential difference between the parallel plates is kept constant while their separation is halved. What happens to their charge?

✱✱ **14. Energy of a Parallel-Plate Capacitor** (see Problem 13.) In parts *(a)* and *(b)* of Problem 13, what happens to the stored electrostatic energy as the plate separation is halved? In each case, write down where the energy goes or comes from.

✱✱ **15. *E*-Field Between Parallel Plates** Since the electric field between oppositely charged parallel plates is uniform, the potential difference V is simply the product of the E-field times the plate separation d. (Work = force × distance.) We now derive the relation between the E-field and the charge on the plates. The capacitance $C = Q/V$ is given (Eq. 13.4–2) by $C = \text{area}/4\pi k_e d$.

(a) Prove that the charge per unit plate area is

$$\frac{Q}{\text{area}} = \frac{V}{4\pi k_e d} = E/4\pi k_e \quad \text{or} \quad E = 4\pi k_e \left(\frac{\text{charge}}{\text{area}}\right)$$

(b) How much charge per unit area is needed to give a field between the plates of 10^6 volts/meter?

(c) Does this answer depend on the plate separation?

✱✱ **16. Gauss' Law.** (Refer to Problem 15.) The formula "charge per unit area $= E/4\pi k_e$" can be rewritten $E \times \text{area} = 4\pi k_e \times \text{charge}$.

(a) Recall where that came from by deriving it for the surface charge on a charged metal sphere, using Coulomb's Law. For a nonuniform E-field, the product $E \times$ area has a meaning only for an "average" E-field. This average is defined by dividing the surface into small subareas area$_i$, each with respective local field vector $\mathbf{E}_i$ having a component $E_{i\perp}$ perpendicular to the surface there, and forming the sum of products $\Sigma E_{i\perp} x$ area$_i$. This expression is called *electric flux* through the surface. (See Problem 47.) For a closed surface, the formula

flux through surface = $4\pi k_e$ × total charge inside

is called Gauss' Law. It is stated here without proof.
(b) Turn the argument around and use Gauss' Law to prove that the *E*-field between oppositely charged parallel plates is uniform.

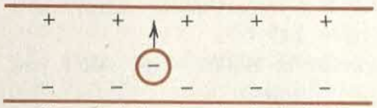
✶✶ **17. How Close Can the Plates Be?** In Problem 15, the relation charge/area = $E/4\pi k_e$ is derived for the surface charge of parallel plates with field E between them. Dry air becomes unstable against dielectric breakdown (sparking) near field strengths of 3×10^6 volts/meter.
(a) What is the maximum surface charge per unit area a parallel-plate capacitor can store without danger of sparking?
(b) With a 3000-volt battery, how closely can the plates be spaced?

✶ **18. Weight of an Electron** The mass of an electron is 0.911×10^{-30} kg. Calculate the strength of a (downward) electric field just sufficient to support an electron against the force of gravity.

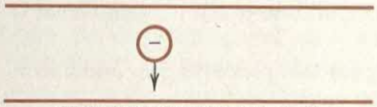
✶ **19. Electron Gun**
(a) What is the acceleration of an electron in an electric field of 1000 volts per meter?
(b) Parallel plates located 1 cm apart are kept at a potential difference of 10 volts. What is the acceleration of an electron released just outside the negative plate?
(c) There is a hole in the positive plate, so the electron comes shooting out into the field-free region outside the plate sandwich. How fast is it traveling upon emerging? (HINT: You know its acceleration and that it traveled 1 cm from rest. An energy argument uses the same logic: You know the (constant) force and the distance through which that force acted. The kinetic energy is equal to the work done.)

✶✶ **20. Electron Gun in the TV** In a color TV, the electron gun may have a voltage as high as 20,000 volts. Suppose that the TV tube has an *E*-field-free region of 25 cm from the gun to the screen.
(a) If you lay the TV on its back, so the electrons have to travel uphill, will they still get to the screen?
(b) How much additional voltage would the gun have to supply to cancel the effect of gravity on the electron beam? Are you surprised the answer is so small?

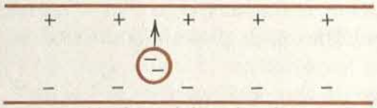
✶ **21. Millikan Oil-Drop Experiment** If the weight of an oil drop is known, then a measurement of the electric field needed to support it against gravity constitutes a measurement of the electric charge on the drop (see figure). The charge on the drop can be changed by ionizing the air between the horizontal plates. The experimenter watches the drop through a telescope and adjusts the voltage between the plates. The result is that the charge always changes by integral multiples of *e*. This is the prototype of the experiments that yielded the number 1.6×10^{-19} for the electronic charge *e*.
(a) A drop has mass 1.0×10^{-12} kg. If it carries one electronic charge, what field strength is needed to support it against falling?
(b) Is it true that if it carries two electronic charges, half the field strength is needed to support it? Argue!



Charged oil drop rises in electric field.



Oil drop falls if field is turned off.



Doubly charged drop rises faster.

PROBLEM 21

✶✶ **22. On Capacitance** When you share your charge with another person having the same electrical capacitance, each of you ends up with half the charge. The two of you, holding hands, have all the charge you had originally but only half the voltage. Viewed as one conductor, your total capacitance is twice each of your individual capacitances. To be sure, this is only approximately true, since the presence of the other person nearby may change the electric field near you, and vice versa.
(a) What happens when you share your charge by touching a conductor having twice your capacitance? How does

the charge distribute? Suppose that you start out at 10,000 volts and you touch two uncharged friends who are holding hands. At what potential will the three of you be now?
(b) You let go, and go back to the generator and get charged up to 10,000 volts again. Your hand-holding friends keep the charge you gave them. Then you return and touch them again. At what potential will the three of you end up?
(HINT: Remember, for each conductor, the ratio of charge to potential is constant. That constant is called its ____________________.)

✶✶ **23. Unequal Sharing of Charge** The metal dome of an electrostatic machine has a capacitance equal to one-ninth that of your body.
(a) If you touch it, what fraction of its charge do you remove? What fraction of its charge does it keep? (HINT: Assume that the two capacitances simply add while they are touching. $C + \frac{1}{9}C = ?$)
(b) The dome was originally at 10,000 volts. After you touch it, it comes to ________ volts.
(c) After you let go, the machine is turned on again and charges up to 10,000 volts again. You still have the charge you got from it the first time. You touch it again. What is your potential now?
(d) Argue that if you repeat this process again and again, you will end up at 10,000 volts.

✶ **24. Capacitors in Parallel** One plate of a 10-picofarad parallel-plate capacitor is connected to the + terminal of a 300-volt battery. The other plate is connected to the − terminal.
(a) Argue that the charge on the positive plate is 3×10^{-9} coulombs, with an equal and opposite charge on the negative plate.
(b) Two identical 10-pF capacitors are both connected in the same way across the 300-volt battery. The total charge on the two positive plates is 6×10^{-9} coulombs. Argue that when two capacitors are connected *in parallel,* their capacitances add.

✶✶ **25. Capacitors in Series** Both capacitors in the previous problem have a potential difference of 300 volts between their plates. They are now disconnected from the battery without being discharged, and reconnected with the + plate of one to the − plate of the other: in *series.*
(a) Argue that the potential difference between the two outside plates is now 600 volts, even though the charge on each is still 3×10^{-9} coulombs.
(b) Calculate the capacitance of the series combination.
(c) Argue that when two identical capacitors are connected in series, the capacitance of the combination is half the capacitance of each.
(d) The two inside plates are really one conductor. Even though the charge is separated, the net charge on this conductor is zero. What happens when the two capacitors are carefully disconnected? Do they retain their charge?
(e) If the two free ends are connected together, we usually see a spark. Explain what happens.

✶✶ **26. Capacitors in Series** If a capacitor C_1 is connected in series with a capacitor C_2, prove that the capacitance of the series combination is $C_1C_2/(C_1 + C_2)$. (HINT: The charge on each is the same, but the voltages add.)

✶ **27. Parallel-Plate Capacitor** Parallel conducting plates 1 millimeter apart are connected across a 90-volt battery.
(a) The *E*-field between the plates is ________ volts/meter.
(b) If the plate separation is halved to 0.5 mm, the *E*-field inside becomes ________ volts/m.
(c) Doubling the *E*-field at the plate surface implies doubling the charge on the surface. Therefore, the capacitance of the thinner sandwich is ________ times that of the original sandwich.

✶✶ **28. Parallel-Plate Capacitor; How It Scales** The capacitance of a parallel-plate capacitor is proportional to the plate area and inversely proportional to the plate separation. If all the dimensions are doubled, length, width, and thickness, then area is quadrupled but the plate separation is doubled. The capacitance is multiplied times ________.

✶ **29. Parallel-Plate Capacitor Formula**
(a) Calculate the capacitance of a parallel-plate capacitor with plate area 1 m^2 and plate separation 1×10^{-4} m.
(b) If the plates were made of a very thin foil and the separator of paper, how small a package could the whole thing be rolled up into? ________ cm^3. (HINT: Neglect the volume of the foil. Argue that rolling it up does not change the capacitance.)
(c) You see why it is desirable to use thin paper. What do you think limits how thin a paper can be used?

✶✶ **30. Energy Stored = $\frac{1}{2}QV$**
(a) Calculate how much work is done if a 1-microfarad capacitor is connected across a 12-volt battery.
(b) Argue that connecting that capacitor across a 24-volt battery would result in doing 4 times as much work; that is, the stored energy in a capacitor is proportional to the square of the voltage.

(c) Argue that a 1% increase in battery voltage would result in a 2% increase in stored energy.

✶ **31.** $C = Q/V$
(a) How much charge is on a 0.01 microfarad capacitor connected across a 9-volt battery?
(b) How much voltage is needed to give a 0.01-microfarad capacitor a charge of 0.18 microcoulombs?
(c) How would you connect two-9-volt batteries to obtain such a voltage?

✶ **32. Series or Parallel?** Two 0.01-microfarad capacitors are connected to a 9-volt battery in such a way that each one stores a charge of 0.045 microcoulombs.
(a) What is the voltage across each capacitor?
(b) How are they connected? Sketch.
(c) What is the total capacitance of the capacitor combination?

✶ **33. Capacitors in Parallel** You have a drawer full of 0.01-microfarad capacitors. Using one 9-volt battery, how could you store a charge of 9 microcoulombs?

✶✶ **34. Capacitors in Series and Parallel** When a 0.01-microfarad capacitor is connected across a 9-volt radio battery, the energy stored in the capacitor is 0.405 microjoules.
(a) Verify.
(b) If 10 such capacitors are connected in parallel across the battery, how much total energy is stored?
(c) If all 10 of these capacitors are now carefully disconnected from the battery and connected + to − in series, argue that the voltage between the ends of the chain is 90 volts.
(d) How much energy is now stored in the combination? Is the formula $\frac{1}{2}QV$ still applicable?

✶✶ **35. Dipole Moment** If two point charges $+q$ and $-q$ (equal and opposite) are located a distance a apart, the *dipole moment* of this charge configuration is a vector having magnitude qa directed from $-q$ to $+q$. Similarly, if a spherically symmetric distribution of positive charge q is displaced a distance a from a similar distribution of negative charge $-q$, the resulting charge configuration has a dipole moment qa in the direction of the positive displacement.
(a) Argue that the hydrogen chloride (HCl) molecule must have a permanent dipole moment.
(b) The dipole moment of the HCl molecule is 3.4×10^{-30} coulomb meters, and the H^+ is 0.13 nm from the Cl^-. What fraction of $e = 1.6 \times 10^{-19}$ coulombs is the effective charge of the proton? Your small answer tells you that the electron is still quite close to the proton, shielding its charge.

✶✶ **36. Dielectrics** Materials that do not conduct electricity well are usually called *insulators*. Another word is **dielectrics.** In general, any material that is neither a metal nor an electrolyte is called a dielectric. When such a material is subjected to an electric field, some + charges are displaced in the direction of the *E*-field, some − charges are displaced in the opposite direction. We say the material is electrically *polarized*. The surfaces across which the *E*-field is applied thus get a surface charge.
(a) Draw a sketch to prove that the effect of the surface charge is to decrease the measured electric field inside the material; that is, the *E*-field due to the surface charge opposes the applied field in the bulk of the material.
(b) Suppose that the *E*-field is applied through parallel plates with a battery across them, with the dielectric material sandwiched between. Also, the surface charge on the dielectric is $\frac{4}{5}$ of the charge on the adjacent plate. Argue that the measured *E*-field inside the dielectric is then only $\frac{1}{5}$ of the applied *E*-field.
(c) The ratio of the applied *E*-field outside to the measured *E*-field inside is called the *dielectric constant* of the material. Here the dielectric constant is 5. Argue that if the dielectric constant of a material is 10, the surface charge is $\frac{9}{10}$ of the charge on the adjacent plate.
(d) If the surface charge is $\frac{2}{3}$ of the plate charge, what is the dielectric constant of the material?

✶✶ **37. *E*-Field Inside a Dielectric.** (Refer to Problem 36.) Argue that if the dielectric constant is κ, the surface charge is a fraction $1 - 1/\kappa$ of the charge on the adjacent plate.

✶✶ **38. Capacitor with Dielectric** Problem 37 shows that for given plate separation, inserting a dielectric between the plates increases the capacitance. Argue that inserting a dielectric multiplies the capacitance times κ. (HINT: Remember, the voltage across the capacitor is the field strength times the thickness. You can either argue on the basis of a constant voltage, in which case the charge on the plates is increased, or on the basis of a constant charge, in which case the voltage across the plates is decreased.)

✶✶ **39. Nonlinear Dielectrics** For most dielectrics, the dielectric constant is independent of the strength of the applied field. This property is commonly referred to as *linearity*, because the measured *E*-field inside is proportional to the applied *E*-field. But at very high field strengths, linearity fails. This may occur in two ways. Either the material may suffer dielectric breakdown, meaning it suddenly becomes a good conductor. Electrons, previously localized, are torn loose by the high field, resulting in an electron avalanche. Or the polarization may tend to *saturate,* that is, to approach some maximum asymptotic value.

(a) The NaCl crystal has a cubic structure, consisting of close-packed planes with equal numbers of Na^+ ions and Cl^- ions. In an applied electric field, the sodium ions tend to be displaced one way, the chloride ions the other way. Assuming that the displacement cannot be greater than the plane-to-plane separation, argue that the polarization of solid sodium chloride has to saturate at high-enough fields.

(b) The density of sodium chloride is 2.16 grams/cm^3. Its molecular weight is 58.5; that means 58.5 grams per 6.02×10^{23} NaCl units. Use these data to calculate the volume per NaCl unit. The spacing of adjacent crystal planes is 2.2×10^{-10} meters. Each ion has charge $(+ \text{ or } -)1.6 \times 10^{-19}$ coulombs. Use these data to estimate an upper limit for the polarization of sodium chloride. Express in coulombs per square meter of surface charge. (HINT: This calculation requires a good bit of geometrical intuition. Your answer will be higher than the polarization actually attainable, because **dielectric breakdown** occurs first. The electric field is strong enough to tear electrons out of the chemical bonds even when the displacements of the ions are much less than the plane-to-plane separation.)

(c) Use the formula (from Problem 15) charge per unit area $= E/4\pi k_e$ to calculate the field due to the surface charge you calculated for part (b). Express in volts per meter and be properly shocked by its huge magnitude. Remember (previous problems) that the depolarizing field is *less* than the applied field. (HINT: Dielectric breakdown occurs in clean air around 3×10^6 volts/meter. Solids tend to have much higher **dielectric strength.**)

★★ **40. How Electric Polarization Occurs** If an atom is placed in an electric field, the symmetry of its electric charge distribution is spoiled. The protons will tend to be displaced in the direction of the *E*-field; the electrons will tend to be displaced the other way. That is polarization. At small field strengths, the displacements (and hence the polarization) will be proportional to the applied *E*-field. If a molecule is placed in an electric field, two other things can happen. The relative positions of the atoms can distort; that is, chemical bonds can stretch. Also, if the molecule has a dipole moment, the whole thing can rotate.

(a) Argue that if a molecule has a dipole moment in the absence of *E*-field, polarization will involve both distortion and rotation.

(b) Argue that if a molecule has no dipole moment in the absence of *E*-field, polarization at small field strengths involves only the distortion mechanism.

(c) The water (H_2O) molecule has a large dipole moment. This is because of the 105° angle between the two O-H bonds. Draw sketches showing the three ways in which water polarizes: bond stretching, angle getting smaller, and rotation of the whole molecule.

NOTE: Water has a very high dielectric constant: 80.

(d) How do you think argon gas polarizes? (HINT: Argon is a monatomic gas. The dielectric constant of liquid argon is 1.5.)

★★ **41. *E*-Field in a Cell Membrane** The potential difference between the inside and outside of a living cell might be about 70 millivolts (outside +, inside −).

(a) Suppose the cell membrane is 9 nm thick. Calculate the strength of the *E*-field in the membrane.

(b) The cell has a volume of 1000 μm^3, a surface area of 600 μm^2, and a membrane capacitance of 3 picofarads. (HINT: 1 μm^3 = 1 cubic micron.) How much charge has separated across the cell membrane to account for those 70 mV? How many K^+ ions would have to diffuse out of an electrically neutral cell to account for this much surface charge?

(c) The intracellular potassium ion concentration might be 145 mM. In a volume of 10^3 μm^3 there would then be 9×10^{10} K^+ ions. Verify. (HINT: *M* means moles per liter; Avogadro's number is 6.02×10^{23}. The point is that over a million potassium ions represents less than one in every 60,000 inside.)

★ **42. Coulomb's Law** The inverse-square law of electrostatic repulsion (attraction) can be written

$$F = \frac{k_e q_1 q_2}{r^2}$$

where q_1 and q_2 are the values of the two point charges, r their distance apart, and k_e the coulomb constant $= 9.0 \times 10^9$ newton meters^2/coulomb^2.

(a) Find the force of repulsion between two point charges of 1 microcoulomb ($= 10^{-6}$ C) each, located 1 meter apart.

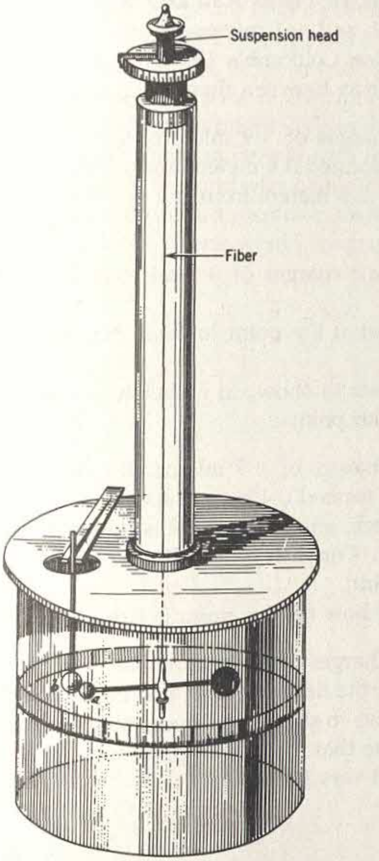
(b) Find the force of attraction between the proton and the electron in a hydrogen atom. Their average separation is 0.5×10^{-10} m.

★ **43. Hydrogen Atom**

(a) Find the force of attraction between a proton and an electron 1 Å ($= 10^{-10}$ m) apart. (HINT: Start by calculating the electric field 1 Å from the proton.) The force on the

electron is that E-field times the charge on the electron. Show that the answer is 2.3×10^{-8} newtons. Which way? Remember that the electron is negative.
(b) In case you think this is a small force, compare it with the electron's weight. Its mass is 0.911×10^{-30} kg. The electric force is ______ times the weight.
(c) Calculate the electron's acceleration. [Answer: 2.53×10^{22} m/sec^2] Why didn't you have to take the weight into account?
(d) At a distance of 10 Å, the attractive force would be only ______ times as strong.
(e) The hydrogen-atom radius is 0.5 Å. Calculate the force when the electron and proton are that close. (HINT: Did you get the same answer to part (b) of Problem 42?)

✱✱ **44.** If the distance between the electron and proton is increased by 1%, the attractive force is decreased by ______ %.



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PROBLEM 45 Coulomb's torsion balance—He used it in 1785 to arrive at the inverse-square law.

45. Coulomb Balance (See drawing) The two charged spheres are at center-to-center distance $r = 5.00$ cm. Each has mass $m = 1.00$ gram and charge $q = 1.00 \times 10^{-8}$ coulomb. The torque due to electrostatic repulsion is counterbalanced by the torsion fiber on which the rotor (spheres, beam, and mirror) is suspended.
(a) Calculate the repulsive force using Coulomb's Law,

$$F = k_e \frac{q_1 q_2}{r^2}$$

(b) Beside the electrostatic interaction, there is also a gravitational interaction—an attraction—between the two spheres. What fraction is this of the strength of the Coulomb repulsion? Answer in the form "1 part in ______." The Law of Gravitation is given in Problem 46.

✱✱ **46. Universal Law of Gravitation** When Coulomb discovered the inverse-square law for electrostatic forces, Newton's inverse-square law for gravitation was already known. The planets are attracted to the sun by a $1/r^2$ force. The strength of the force is proportional to the mass of the planet.

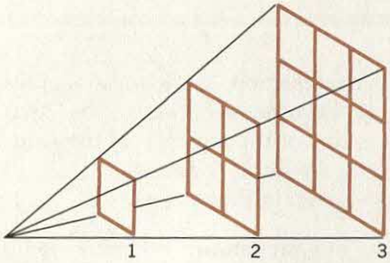
$$F = G\frac{M_{\text{sun}} m_{\text{planet}}}{r^2}$$

The constant G is a universal constant, valid for the gravitational attraction between any two objects having mass. It was determined by measuring the gravitational attraction between two metal spheres. Result:

$$G = 6.67 \times 10^{-11} \text{ newton m}^2/\text{kg}^2$$

(a) Calculate the gravitational attraction between a 1-kilogram lead sphere and a 0.1-kilogram gold sphere with center-to-center distance $r = 0.25$ m.
(b) The earth's radius is 6.37×10^6 m. The gravitational attraction between the earth and a 1-kg mass at its surface is 9.8 newtons. (You know that from Chapter 1.) Use the Law of Gravitation to calculate the mass of the earth. How close did you get to 6×10^{24} kg?
✱✱✱(c) Can you think of any other way to "weigh" the earth?

✱✱ **47. Electric Flux** The film "Coulomb's Law" (Eric Rogers, Physical Science Study Committee, Educational Services, Inc., 1960) asks you to imagine a "butter gun" used for coating toast uniformly with melted butter (see figure). The same amount of butter that would hit one square of toast 1 foot from the gun would hit four squares 2 feet away. So the thickness of the butter layer at 2 feet would be only one-quarter of the 1-foot value. Three feet away, that amount of butter would hit nine squares; so the layer would be one-ninth as thick as at 1 foot away. The thickness of the butter coating obeys an inverse-square law. The only assumption we needed for this conclusion is that the butter travels in straight lines and that the butter is "conserved." No butter is created or destroyed in flight.



PROBLEM 47 One square, four squares, nine squares . . .

There is a name for the analogous thing "emitted" by an electric charge that gives rise to an inverse-square field. What is it that goes out in straight lines and is conserved? We call it electric flux. Roughly speaking, flux is the product of field strength times the area it intersects normally. We can now say that Coulomb's Law (inverse-square law for electrostatic forces) is equivalent to the statement that the electric flux through *any* closed surface surrounding a charge is independent of the size and shape of the surface and simply proportional to the charge.

(a) Knowing the formula for the area of a sphere, use Eq. 13.5–2 to prove that the flux originating at a charge q passing through a sphere of radius r centered on q is given by $4\pi k_e q$.

(b) Show that the flux per unit area is equal to the perpendicular component of the field strength.

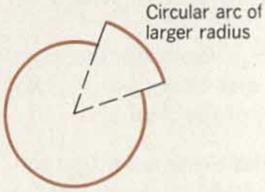
✱✱✱ **48. Pimple on the Sphere—Gauss's Law** In Problem 47, imagine a hole in the sphere. All the flux that would hit the area of the hole now goes farther and hits a larger sphere, radius R. The area it hits is the area of the hole multiplied times R^2/r^2.

(a) Show. The surface is closed by a surface generated by radial lines from the edge of the hole to the larger sphere. We have put a pimple on the sphere. Evidently, the flux hitting the pimply sphere is the same as the flux hitting the original sphere.

(b) Argue that the pimple-building argument can be extended to construct an arbitrary closed surface.

(c) The statement (in Problem 47) that the flux through any closed surface containing charge q is equal to $4\pi k_e q$ is called Gauss's Law. Argue that we have proved it from Coulomb's Law.

(d) Now reverse the argument: Prove Coulomb's Law starting from Gauss's Law.



PROBLEM 48 Pimple-on-the-sphere "proof" of Gauss' Law.

✱ **49. Electric Fields Add as Vectors** Charges of +1 microcoulomb and −1 microcoulomb are located 0.2 meters apart. Use Coulomb's Law to find the electric field strength halfway between them.

✱ **50.** Charges of +9 microcoulombs and −1 microcoulomb are located 0.4 meters apart. Find the *E*-field at a point located 0.3 meters from the + and 0.1 meters from the −.

✱ **51.** Two charges of +1 microcoulombs are located 0.2 m apart.

(a) Argue that at the point halfway between them, the *E*-field is zero.

(b) Use arrows to show, in a sketch, the direction of the *E*-field at other points.

✱✱ **52.** Charges of +9 microcoulombs and −1 microcoulomb are located 0.4 meters apart.

(a) On a sketch, show that there is one point where the *E*-field is zero. Convince yourself that there really is only one such point.

(b) Calculate how far this point is from the + charge.

✱✱ **53.** Charges of $+q$ and $-q$ are at a distance d.

(a) Calculate the field strength at their midpoint.

(b) Use arrows to show the direction of the *E*-filed at other points. Argue that there is no place where the field is zero until you get very far away.

★★ **54. Vector Addition of *E*-Fields** Charges of +9 microcoulombs and +1 microcoulombs are located 0.4 m apart. Show that there is one point where the *E*-field is zero. How far is it from the +9 charge?

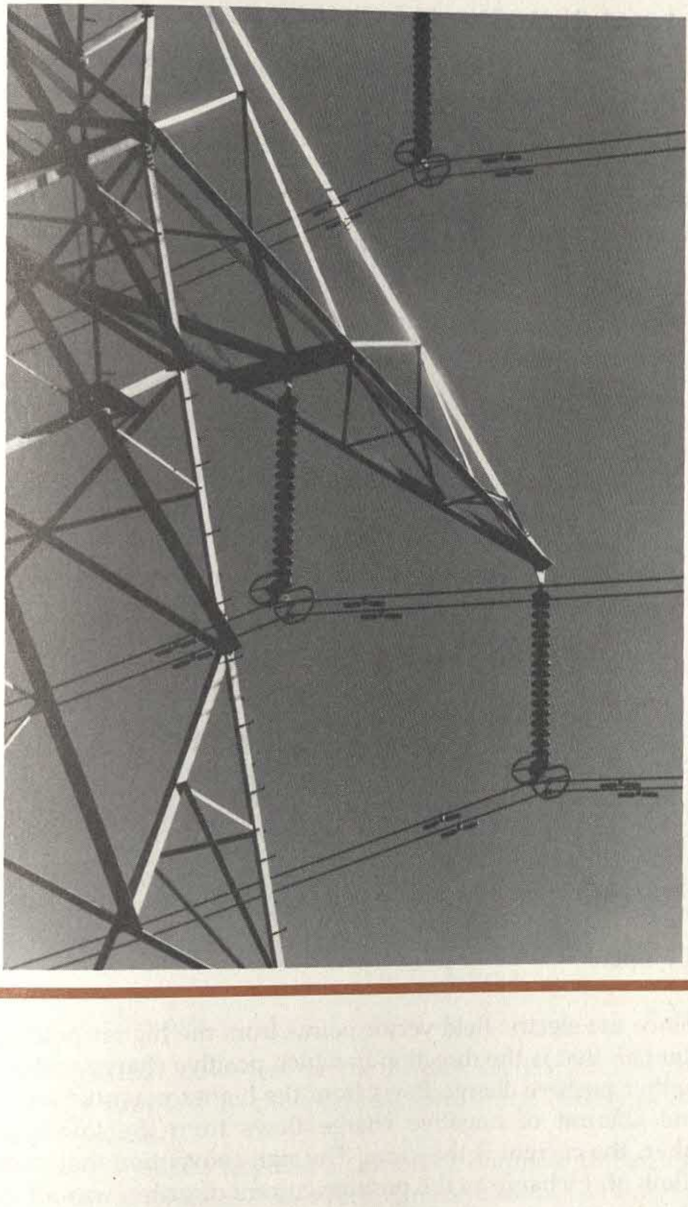
★★ **55. Energy of an Ionic Bond** Estimate how much work has to be done to tear apart a sodium chloride molecule. The Na^+-to-Cl^- separation is roughly 0.25 nm. Express work in eV.

NOTE: $1\ \text{eV} = 1.6 \times 10^{-19}$ joules. Your estimate will be on the high side, since we have not told you anything about the ion-ion repulsive forces. Compare Problem 39. Note that the Na^+-to-Cl^- separation is larger in the molecule than in the crystal.

☐ QUESTION FOR FURTHER THOUGHT

α. **Ozone Layer** In a lightning discharge, some ions become energetic enough to knock oxygen molecules (O_2) apart into their constituent oxygen atoms. Atomic oxygen may then latch on to other O_2 molecules to form O_3 molecules, called *ozone*. That is the gas you smell after lightning has struck, and often near high-voltage machinery. Ozone gives up that extra oxygen atom easily, making it a good oxidizing agent, hence also a germicide. It is highly corrosive in the lung. The breakup of ozone can be catalyzed by the fluorocarbons used in some spray cans and by components of jet-aircraft exhaust. This is why there is a worry about the depletion of ozone in the upper atmosphere. That ozone layer acts as an ultraviolet filter for sunlight. Without it, the incidence of skin cancer would increase, especially among people with fair skin.

Can you think of reasons why ozone constitutes a larger fraction of the upper atmosphere than of our atmosphere down here? After all, O_3 is heavier than O_2. You might think the ozone would concentrate at lower altitudes!



High-voltage power-transmission-line tower showing cables and insulators.

14

ELECTRIC CURRENT

SOME QUESTIONS THIS CHAPTER WILL ANSWER

According to Ohm's Law, how does the current in a circuit vary with the applied voltage?

Two resistances *R* connected in series have a net resistance of 2R. / $\frac{1}{2}$R.

What is the resistance of a light bulb marked "60 watts"?

Building codes specify a minimum diameter for the copper wire used in home electrical wiring. Why?

Given a battery, a voltmeter, and an ammeter, how would you connect them to measure the resistance of a wire?

A lie detector works by applying voltage between electrodes on the skin. Yet it never gives the subject an electric shock. Why not?

A shock from an electric fence is painful but causes no burns. Why not?

Why should the power lines near an EKG be shielded?

When you buy an electric drill, it has a three-prong plug. Why this expense, when you know the drill will work with only two terminals?

14.1 THE AMPERE

In Sec. 13.4, Capacitance, we talked about charge flowing from one conductor to another when the two make contact. Indeed, charge will flow if the two conductors are at different potentials. The rate at which charge leaves one conductor and enters the other is called the *electric current*. Current is measured in coulombs per second. That unit was defined in Sec. 13.1:

$$1 \text{ ampere} = \frac{1 \text{ coulomb}}{\text{second}}$$

Since the electric field vector points from the higher potential to the lower potential, that is the direction in which positive charge will tend to flow. But whether positive charge flows from the higher potential to the lower, or the same amount of negative charge flows from the lower potential to the higher, the current is the same. The sign convention that chose the direction of flow of + charge as the positive current direction was arbitrary, of course. But it is internationally followed. Some people wish that the opposite convention had been adopted. They point out that most wires are made of copper, and what moves is the electrons. The negative charges move, not the positive. Our choice of positive current direction is opposite to the direction the electrons move. If you want to argue in favor of the accepted convention, you might want to talk about current in a salt solution. In solution, the NaCl is ionized. The Na^+ ions travel from + to −. The Cl^- ions move from − to +. Both motions contribute to the current, of course. (See Problem 6.)

14.2 DISSIPATION AND RESISTANCE

When current flows, there is usually dissipation of electrical energy into heat. We used the term *dissipation* when we talked about the conversion of mechanical energy into heat by friction. Here also it is a frictional mechanism that governs how much current flows for a given potential difference. Without exploring the mechanism further, we are going to assume a particularly simple current-voltage law. But first, it will be useful to see how the power dissipation depends on current and voltage.

When one elementary charge flows from a region at 1000 volts to a region at 990 volts, it loses 10 electron volts of potential energy. Suppose that 10 eV was all work done against those frictional forces. Then the transit of one elementary charge down the "potential hill" results in the dissipation of 10 eV of energy. Suppose that happens continuously. For instance, suppose a current of 1 ampere (= 1 coulomb per second) flows through a potential difference of 10 volts. Then every second an energy of 10 joules (= 1 coulomb × 10 volts) is dissipated. That is a power dissipation of 10 watts (= 10 joules per second). Evidently, the rate at which electrical energy is lost is proportional to the current as well as to the potential drop each charge experiences:

$$\frac{\text{energy loss}}{\text{time}} = \frac{\text{charge}}{\text{time}} \times \frac{\text{energy loss}}{\text{charge}}$$

$$\text{power} = \text{current} \times \text{potential drop}$$

$$P = IV$$

That formula is not a law of physics. It is simply a consequence of the definitions of the words *current*, *potential*, and *power*. To give an example of its use, consider the common flashlight. Most flashlight batteries are dry cells marked 1.5 V. Suppose that a 1.5-volt battery drives 1 amp of current through the bulb of the flashlight. Then the power being drawn from the battery is

1 amp × 1.5 volts = 1.5 watts

Each elementary charge that flows out of the battery carries 1.5 electron volts of energy, energy supplied by chemical reactions in the battery at the rate of 1.5 joules every second.

In that example, the 1.5-volt battery had a current of 1 ampere through it. We must now ask, what determines how much current does in fact flow? If the battery is not connected to anything, very little current flows. The two electrodes, the little knob at the top marked "+" and the shiny flat disk at the bottom marked "−," are just out in the air. Air is a very poor conductor, unless it is ionized, as in a spark. A tiny current may leak between the electrodes on the paper or paint surfaces of the battery. This is especially likely if a film of moisture has developed. Even then, the current will be much less than a microamp (10^{-6} amperes). For practical purposes, it is

Size D, C, and AA dry cells. All three are 1½-volt cells. What does the larger size buy? Lower internal resistance and, hence, the possibility of supplying more current.

negligible: There is no current flow in an isolated dry cell unless it is connected in a *circuit*. In a flashlight, that is done by "closing the switch," establishing a conducting path between the electrodes (see Fig. 14.1). Open up a flashlight. You will see the metallic tip of the bulb, making contact with the positive electrode of the battery. What glows when the switch is "on" is a thin tungsten-wire filament. The filament runs between that tip and the metal body of the bulb, which may be threaded to screw into the socket. Turning the switch to "on" establishes electrical contact between the socket and the negative electrode of the battery.

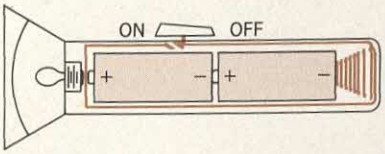


FIGURE 14.1 Flashlight, showing switch in the OFF position.

The only part of the flashlight that gets hot is the filament. So this is where most of the power is dissipated. It is the nature of the filament that determines how much current flows. A long, thin filament draws less current than a short, thick filament. You might think of the charge as a viscous liquid and the filament a tube through which it flows. If the filament is too long or too thin, too little current flows, and the filament does not get hot enough to glow. If it is too short or too fat, a lot of current flows, but the power is dissipated in other parts of the circuit. The tungsten filament in a flashlight bulb is so thin, you may need a magnifier to see it.

In a proper flashlight the filament gets hot enough so that it radiates white light. Most of the 1.5-volt potential drop is across the filament, rather than across the switch or one of the contacts. We say that most of the **resistance** of the circuit is in the filament. Indeed that is the point of constraining the current to go through such a narrow path there. The rest of the circuit has wider, more highly conducting pathways. In our example above, the bulb draws a current of 1 ampere. We say that its resistance is

$$\frac{1.5 \text{ volts}}{1 \text{ amp}} = 1.5 \text{ ohms}$$

DEFINITION

$$\text{resistance} = \frac{\text{voltage drop}}{\text{current}}$$

$$1 \text{ ohm} = \frac{1 \text{ volt}}{1 \text{ amp}}$$

What if the battery voltage were 1% greater? Then you might guess that the current drawn by the bulb would also be 1% greater. That guess would assume that the current is proportional to the voltage drop. In other words, the resistance does not depend on the voltage drop. In many situations this is such a good approximation that it has a name: **Ohm's Law.** Remember, a couple of paragraphs ago we said we would "assume a particularly simple current-voltage law." This approximate proportionality of the current and the voltage is that simple law.

When should you believe Ohm's Law? How good an approximation is it? It turns out to be really bad for hot flashlight bulbs, and quite good for cold ones. This means that it is good when the current is small. For then, a small increase in current does not heat up the bulb appreciably. The law is bad when the current is large enough to make the filament glow, even the low-temperature reddish glow. For then an increase in current makes it glow hotter, and the resistance increases. In technical jargon we say that a hot filament is a *nonlinear* circuit element. If the current is further increased, at one point the filament melts in one place and breaks. The bulb burns out. Here the nonlinearity becomes extreme: A tiny increase in voltage makes the current fall to zero!

The message of the previous paragraph is that Ohm's Law is not a great law of physics. It is just a good approximation at small currents. In the next few pages it will be used in order to develop your intuition about the size of currents and voltages in practical situations. After that, you will be able to answer such questions as when 10,000 volts is safe and when 1 volt is dangerous.

14.3. PRACTICE WITH OHM'S LAW

EXERCISE 1 The resistance of a 100-meter-long copper wire 1 square millimeter in cross section is 1.7 ohms. Verify that resistance of a 1-meter-long piece of the same wire is 0.017 ohms. (HINT: If 100 strands of this wire were made into a 100-meter-long cable, its resistance would also be 0.017 ohms: It would be equivalent to a 100-meter-long wire 1 cm^2 in cross section.) The resistance of a wire is proportional to its length and inversely proportional to its cross-sectional area; the proportionality constant is called the **resistivity** of the metal. Verify that the resistivity of copper is 1.7×10^{-6} ohm cm. (HINT: You need to multiply the resistivity by length and divide by area to get the resistance in ohms.)

Copper, of course, is one of the best room-temperature conductors. If you want a metal of much lower **conductivity** (higher resistivity), try an alloy: Nichrome (nickel-chromium) wire has a resistivity of 1.08×10^{-4} ohm cm, which makes it suitable for the heating elements of toasters and electric irons.

SAMPLE EXERCISE Some commercial nichrome wire (Number 29) comes 0.286 millimeters in diameter, or 0.000642 cm^2 in cross section. How long a piece would you use to make a 100-watt heating element operating on a 115-volt line?

SOLUTION: To dissipate 100 watts with a potential drop of 115 volts, we need to draw a current of 0.87 amp (= 100 watts/115 volts). This requires a resistance of

$$\frac{115 \text{ volts}}{0.87 \text{ amp}} = 132 \text{ ohms}$$

One meter of this wire has a resistance of

$$\frac{1.08 \times 10^{-4} \text{ ohm cm} \times 10^{2} \text{ cm}}{0.000642 \text{ cm}^2} = 16.8 \text{ ohms}$$

so we need

$$\frac{132 \text{ ohms}}{16.8 \text{ ohms/meter}} = 7.86 \text{ meters}$$

of this wire for our heating element.

There are other ways to accomplish the same goal. (See Fig. 14.2.) For example, we could use 10 wires in parallel. Then each wire would have the full 115 volts across it but would only have to dissipate 10 watts. Each would then draw 0.087 amp (= one-tenth the current). This requires a resistance of 1320 ohms (= 10 times the resistance) and a length of 78.6 meters (= 10 times the length). With 10 such wires, we need 100 times as much wire this way. It sounds like an uneconomical alternative. Of course, each of the 10 parallel wires carries less current than in the one-long-wire design, so it does not get as hot. In a practical design problem there is usually an optimum temperature. To save metal, you would use the thinnest wire compatible with the mechanical requirements. You would find experimentally what current kept it at the desired temperature. Then you would calculate, just as above, the length of wire which would draw that current from a 115-volt line. Your power requirement then determines how many parallel wires are needed. In an electric blanket, for example, one can use such fine wire; because such a low temperature is desired, many wires can be in parallel. That is why a break in one wire does not put the whole blanket out of commission.

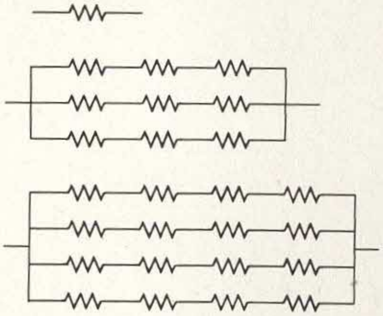


FIGURE 14.2 All have the same resistance.

14.4 RESISTANCE IN SERIES AND PARALLEL

Saying that the resistance of a wire is proportional to its length is equivalent to saying that if resistances are connected in series, their resistances add (see Fig. 14.3). For example, in the nichrome-wire sample problem discussed earlier, the 7.86-meter wire with the 132-ohm resistance could perfectly well be pieced together out of two 3.93-meter halves each having a resistance of 66 ohms. Each half would have half the potential drop across it: 115 volts/2 = 57.5 volts. Of course, they would have to be pieced end to end, that is, in series. If instead you connected the two 66-ohm pieces in parallel across the 115-volt line so that each had the full 115-volt drop across it, each would draw

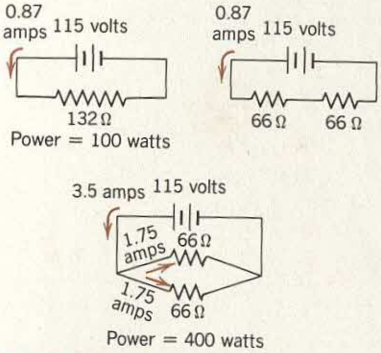


FIGURE 14.3 Resistances in series versus resistances in parallel.

$$\frac{115 \text{ volts}}{66 \text{ ohms}} = 1.75 \text{ amps}$$

This is twice the current above, and the total line current would be the sum of those two currents, 2 × 1.75 amps = 3.50 amps, or 4 times the current above. This says that the net resistance of the parallel pair is 33 ohms, or half the resistance of each wire. Instead of dissipating 100 watts as before, the power dissipation would now be 400 watts, with 200 watts dissipated in each 66-ohm wire.

Here is the general argument, presented algebraically. Two resistances R_A and R_B connected in series have resistance $R_A + R_B$. See Fig. 14.4(*a*). If the series combination is connected across a line voltage V, it will draw a current (same current through each)

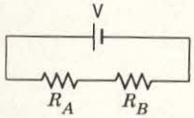
$$I = \frac{V}{R_A + R_B}$$

so that the voltage drops will be

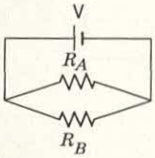
$$V_A = R_A I = \frac{R_A}{R_A + R_B} V$$

and

$$V_B = R_B I = \frac{R_B}{R_A + R_B} V$$



(a) Resistances in series.



(b) Resistances in parallel.

FIGURE 14.4

EXERCISE 2 Verify that $V_A + V_B = V$.

With resistances connected in parallel (Fig. 14.4(*b*)), there is the same voltage across each, so they draw currents

$$I_A = \frac{V}{R_A} \quad \text{and} \quad I_B = \frac{V}{R_B}$$

respectively, resulting in a total line current

$$I = I_A + I_B = V\left(\frac{1}{R_A} + \frac{1}{R_B}\right)$$

From the definition of the net resistance, $R = V/I$, it follows that

$$\frac{1}{R} = \frac{1}{R_A} + \frac{1}{R_B}$$

It is easy to see how to extend this to any number of resistances connected in parallel. Evidently, the resistance of any parallel combination is less than that of any of the component resistances.

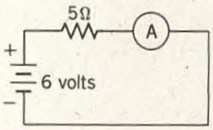


FIGURE 14.5

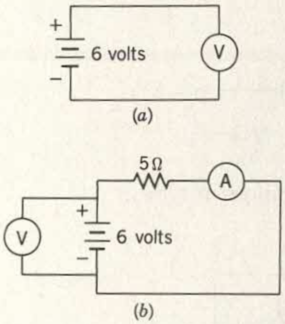


FIGURE 14.6

EXERCISE 3 If the current drawn from a 6-volt battery is 1 milliamp, show that the resistance of the whole circuit is 6000 ohms.

EXERCISE 4 A resistor marked 5 Ω is connected across the 6-volt battery (Fig. 14.5). The ammeter *A* in series with the resistor registers a current of 1.0 amp. Argue that the series resistance of the battery plus the meter is 1 ohm.

EXERCISE 5 A voltmeter *V* connected directly across the 6-volt battery registers 6.00 volts (Fig. 14.6 *a*). The same voltmeter connected as in *b* registers 5.90 volts, while the ammeter registers 1.0 amp.

(a) Argue that the ammeter has a resistance of 0.9 ohms.

(b) What is the resistance of the battery? (HINT: The voltmeter's resistance is very large—thousands of ohms for a good one. So your assumption that the battery current is 1.0 amp is valid.)

Galvanic Skin Resistance (GSR)

Body fluids are not as good conductors as metals. Their resistance is millions of times higher. This lower conductivity characterizes ionic as compared with electronic conductors: The mobility of ions in liquids is less than that of electrons in metals by a factor around 10^{-3}, and their concentrations are less by another factor of 10^{-3}. Still, compared to "insulators" like skin or wood or glass, the lymph and the blood are excellent conductors. That is why we can measure the electrical resistance between the inside and the outside of an area of skin noninvasively. The idea is to make contact to the outside at two places and use the body fluids as a low-resistance connection between the two inside areas (see Fig. 14.7). The measured resistance is essentially the sum of the resistance of each contact area, that is, twice the resistance of each electrode area, since current has to flow in through one and out through the other: The two are in series. To show that the resistance of the wet path inside is indeed neglibible, it suffices to attach an electrode to each wrist, for example. Then measure the current drawn by a small battery voltage, say 1 volt. Electrodes 1 cm^2 in area will draw, typically, currents on the order of 10 microamperes (= 10^{-5} amps), corresponding to a circuit resistance of 10^5 ohms. Values between 20 kiloohms and 1 megohm are common. To attach the electrodes in such a way as to make good electrical contact, cover a well-cleaned area of skin with a salt paste. If one of the two electrodes is now shifted so both are on the same wrist, the circuit resistance will show no appreciable change. This proves that the conduction between the electrodes takes place under, not on, the surface of the skin. A favorite place to attach electrodes for skin-resistance measurement is to two fingers of the same hand.

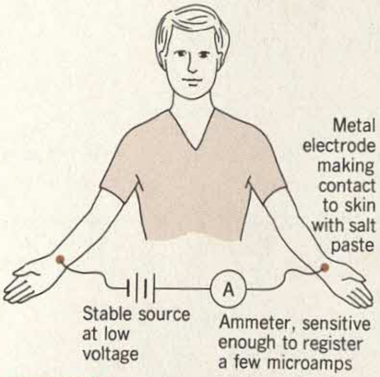

Metal electrode making contact to skin with salt paste
A
Stable source at low voltage
Ammeter, sensitive enough to register a few microamps

FIGURE 14.7 Measuring galvanic skin resistance (lie detector).

A polygraph (lie detector) plotting five different functions: skin resistance, pulse rate, blood pressure, and the upper and lower respiratory rates.

The remarkable feature of such measurements is that the skin resistance fluctuates from one minute to the next. It can even double. The fluctuations are clearly related to the emotional state of the subject. In particular, anxiety lowers the skin resistance. The obvious (and accepted) interpretation is that anxiety is correlated with perspiration. Having the pores of the skin full of salty sweat offers lower-resistance paths to the current. Indeed, galvanic skin resistance (GSR) measurements are the mainstay of the "lie detector" or **polygraph.** The art of polygraphy consists partly in correlating these with other indications such as pulse and breathing characteristics, muscle tension, blood pressure, and blood flow in specific organs.

How safe are such measurements? In living beings, membrane potentials are of the order of millivolts. Might not the imposition of a 1-volt potential difference between distant parts of the skin carry with it danger of interference with the regulatory functions the membrane potentials serve? One of these functions is that of keeping the heart beating! Here is where our understanding of the series circuit is needed. That 1-volt drop between the two skin electrodes occurs over the thickness of the skin at the positive electrode, the lymph and blood between the two electrode locations, and the skin at the negative electrode, all three in series. Suppose that each patch of skin has an outside-to-inside resistance of 50-thousand ohms, and the resistance between the two inside patches is 5 ohms. Then only one part in 20,000 of the voltage (5×10^{-5} volts) is dropped over the fluid connection. That may be a path 1 meter or so in length, while the skin is a fraction of a millimeter thick. The point is that the resistivity of skin is thousands of times greater than that of blood or lymph. WARNING: Never touch a live wire to an open wound!

14.5 ELECTRICAL INJURY

It is not voltage that hurts; what matters is current, or, to be more precise, current density. A current of half a milliampere across the skin is near the threshold of perception. The threshold of pain is above 1 milliamp. Nevertheless, a current of 20 microamperes flowing directly through the heart may be enough to induce ventricular **fibrillation.**[1] That is indeed the most likely mechanism of fatal electrical injury: A current of 100 to 200 milliamperes through electrodes on the chest can trigger fibrillation. The reason for the discrepancy between the 20 μA through the heart and the 100 mA (= 5000 times as much) through the skin is that there are, evidently, about 5000 paths of comparable conductance in parallel with the heart to shunt off (i.e., divert) most of the current.

Many household and industrial accidents associated with electric shock arise from involuntary body movements in response to the current, for example, falling off a ladder or dropping a heavy object. Some of these accidents occur only because the shock is a surprise. But the application of electric potential to a nerve or muscle can cause an involuntary muscle contraction. The so-called "cannot-let-go" current is between 6 and 20 mA. Above this threshold, muscles may "freeze" so that the victim cannot let go of a "hot" wire he has grabbed. Breathing may stop for the duration of the current, for the same reason, so that an apparent electrocution may be technically asphyxiation. Currents of several hundred milliamperes or more can cause cardiac arrest, burns, and unconsciousness. There can also be nerve damage, probably because of local temperature rise from ohmic dissipation. Autopsies of legally electrocuted criminals (tens of amperes) reveal actual tears in nervous tissue.

What voltage, then, is "safe"? The answer depends on the resistance of the circuit. If the victim has 1 cm^2 of skin contact with a conductor and very dry skin, that may have a resistance as high as 1 megohm (= 10^6 ohms), and he won't even feel the current from a 115-volt plug:

$$I = \frac{V}{R} = \frac{10^2 \text{ volts}}{10^6 \text{ ohms}} = 10^{-4} \text{ amps}$$

If, on the other hand, the victim is sitting in a full bathtub, that is a large-area contact of perhaps 500 ohms resistance. Remember, the resistance of the through-the-skin path is inversely proportional to the area. With 500 ohms, the 115-volt line will draw over 200 mA and may electrocute him. In the bathtub, even 10 volts, which would draw 20 mA (= 10 volts/500 ohms), could be lethal through asphyxiation or drowning. WARNING: Don't drop the telephone (48 volts dc) into the tub.

Voltages above 240 volts are dangerous because the high electric fields across the skin can cause dielectric breakdown. This is a sudden decrease of resistance analogous to the ionization of the air in lightning, and can result in severe burns. Once the skin has been damaged in this way, a low-resistance path for current flow is established. This is the reason for the seemingly elaborate precautions against accidental shock built into household wiring in Europe, where 250 volts is standard line voltage. In the United States, with our 115-volt house wiring, we tend to be more cavalier, and most of us live to tell about shocks from frayed insulation or reversed ground wires.

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[1] Not surprisingly, a defibrillator is a device that gives the heart a brief high-voltage dc shock to restart normal rhythm.

What about high-frequency alternating currents, that is, currents that reverse direction thousands of times per second? Jargon is "kilohertz R.F currents"—R.F. stands for "radio frequency." They are not as dangerous as dc or the standard U.S. 60-Hz ac, largely because they do not tend to cause fibrillation. The cattle prod is a device for administering an **R.F.** shock of several hundred volts that lasts for milliseconds. It inflicts only local pain, but is totally irresistible. The electric fence is a periodically pulsed version of the same thing. Antiprowler weapons with two point electrodes operate on the same principle. In these devices a capacitor is charged to a low voltage. It is discharged through a transformer (discussed in Chapter 16) called a *Tesla coil,* which puts out a high R.F. voltage (see Sec. 12.5). It may be useful to go through a sample calculation of the energy such a discharge can deliver in order to appreciate why these devices are relatively safe despite their high voltages. WARNING: Don't misunderstand; they hurt!

A cattle prod might operate on three $1\frac{1}{2}$-volt flashlight batteries in series, which charge up a 2-microfarad capacitor to 4.5 volts. Recall from Sec. 13.4 how to calculate the electrical energy stored in the capacitor:

$$\begin{aligned}\text{energy} &= \frac{1}{2}QV = \frac{1}{2}CV^2\\ &= \frac{1}{2}(2 \times 10^{-6}\text{ farads})(4.5\text{ volts})^2 = 2 \times 10^{-5}\text{ joule}\end{aligned}$$

That is the energy it takes to raise 1 gram 2 millimeters here on earth. It is roughly the kinetic energy of a flea just before landing. Even if the pulses this weapon emits have an amplitude of a thousand volts, and therefore give rise to danger-level current amplitudes, these last such a short time that they cannot do appreciable heating: No burns.

Recall now those sparks between the hand and the door knob (Secs. 13.1 and 13.4) after the body has been charged up triboelectrically, say by shuffling across a nylon carpet. Why don't they cause burns? Is it because they don't actually discharge very much energy? Again, we can calculate it. In dry air, it requires an electric field of about 3×10^6 volts/meter to start a spark, that is, to ionize the air sufficiently. A 1-centimeter spark is not uncommon and would correspond, if the E-field of 3×10^6 V/m were uniform, to a potential difference of $3 \times 10^6\text{ V/m} \times 10^{-2}\text{m} = 3 \times 10^4$ volts. Because the finger tip is not a plane surface but has a curvature, the field near it is actually not uniform, but more like that of a sphere with a radius of curvature of perhaps 2 millimeters. A field of 3 million volts per meter then requires that the finger tip be at a potential of (for a sphere, $V(r) = rE(r) = 2$

$\times\ 10^{-3}$ m $\times\ 3 \times 10^{6}$ V/m =) 6×10^{3} volts with respect to the surrounding conductors we call "ground"—in this case, the door knob. Six thousand volts! Wouldn't that kill you? With a body capacitance on the order of 10^{-10} farads, the charge stored on the body surface is

$$Q = CV = 10^{-10} \text{ farads} \times 6 \times 10^{3} \text{ volts} = 6 \times 10^{-7} \text{ coulombs}$$

and the electrostatic energy is

$$\begin{aligned}\text{electrostatic energy} &= \tfrac{1}{2}QV = \tfrac{1}{2}CV^2 \\ &= \tfrac{1}{2} \times 10^{-10} \text{ farads} \times (6 \times 10^{3} \text{ volts})^2 \\ &= 1.8 \times 10^{-3} \text{ joule}\end{aligned}$$

That amount of energy could vaporize about 1 microgram of water. In practice, you can see the light of the spark but can not feel its heat. The prick you feel on the finger tip is the effect of a high current density for a fraction of a millisecond.

14.6 CAPACITIVE COUPLING AND LEAKAGE CURRENTS*

With alternating current (ac) it is possible for current in one circuit to induce current in another circuit not connected to it. In other words, there is no conducting path—no "electrical connection"—between them. One way this can happen is by *electrostatic induction,* also called capacitive coupling. Suppose that the capacitance between two insulated wires is 10 picofarads. A potential difference of 100 volts would charge that capacitance with 10^{-9} coulombs (= 10^{-11} farads $\times\ 10^{2}$ volts). With our standard American 60-Hz frequency, that potential would be reversed every $\frac{1}{120}$ of a second. So 2×10^{-9} coulombs have to flow in $\frac{1}{120}$ sec, corresponding to a current of one-quarter of a microamp (= 2×10^{-9} coulombs/$\frac{1}{120}$ sec.) This is an approximate calculation. It does not take into account the sinusoidal time behavior of the alternating current. But it gives a good idea of the size of currents to be expected from capacitive coupling with standard 115-volt power lines.

Such electrostatically induced *leakage currents* are not safety hazards, but they can completely swamp measurements of biovoltages like an **EKG**. The large voltage spike of an electrocardiogram may be around 0.5 mV high. With a circuit resistance of 0.1 megohm (see GSR, above) that means a maximum current of

$$\frac{5 \times 10^{-4} \text{ volts}}{10^{5} \text{ ohms}} = 5 \times 10^{-9} \text{ amps}$$

or $\frac{1}{200}$ of a microamp. That signal would be virtually invisible if there were $\frac{1}{4}$ microamp of "60-cycle pickup" in the circuit. This is why all power lines near an EKG have to be "shielded." They may be covered with a metal coating that is "grounded," which means kept at the same potential as the metal case of the instrument.

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*Difficult.

In Chpt. 16 we discuss *inductive coupling*, or electro*magnetic* induction, which can also be a source of leakage currents. Indeed, for 60-Hz AC, electromagnetic induction is usually more important than electrostatic induction. The housing of an electric motor always has such leakage currents due to inductive coupling with the wires inside. Values up to 20 milliamperes are common. A large vacuum-cleaner motor may have 100 mA of leakage current flowing through the metal housing of the vacuum cleaner. As the resistance between various points on the housing is on the order of microohms, the potential differences between points on the vacuum cleaner are fractions of a microvolt, so it would seem that one need not worry about shock hazard. But if these leakage currents flow from the housing to "ground" (any large conductor nearby), then the voltage to ground is the RI-drop in the ground connection. A "good ground" has a resistance of milliohms or less and is recognized safety practice. That is the rationale for three-prong plugs, now standard for electric hand tools. Note that when a ground connection comes undone ("open"), the tool works just as well but becomes a lethal hazard: Those 100 milliamps of leakage current could then flow to ground through the operator. Under normal conditions, this is not a problem, because the operator's resistance to ground is too great. Any electric device used in surgery is now required to have a ground-fault interrupter (GFI); this detects an open ground connection by monitoring any difference in leakage current at two points in the circuit.

14.7 GROUNDING AND SAFETY

There appears to be a mystique about the whole idea of an electrical *ground*. Well, you don't have to be a licensed electrician to be initiated. Grounding (the British say *earthing*) simply means bringing a conductor to the same potential as its conducting surroundings. The earth is a reasonably good conductor so that we treat it as all being at the same potential. It has a large-enough capacitance so that its potential does not change much when we add some charge to it. Our domestic water pipes make good electrical contact to it, so we consider these also to be at ground potential. Is that valid? Suppose that a heating pipe has a resistance of 100 microohms between living room and basement. It also carries a leakage current of 10 milliamps from some appliance that has been "grounded" to it. Then there is a 1-microvolt voltage difference ($= 10^{-4}$ ohms $\times$ 10^{-2} amps) between the "ground" in the living room and the "ground" in the basement. Treating those two "grounds" as being at the same potential is surely a legitimate practical approximation: One microvolt out of a line voltage of 115 volts!

Now suppose that some insulation became frayed on the appliance, and the full line voltage of 115 volts were applied to the living-room heating pipe. Of course, the fuse would blow or the circuit breaker would trip. That happens when more than 15 amps is drawn[2] from the line, which only takes a fraction of a second. But before the circuit is interrupted, the people in the living room would still be safe: Even someone touching the heating pipe and therefore directly connected to line voltage would draw negligible current, since all the objects he or she might be touching—floor, walls, furniture— come essentially to the same potential as the heating pipe. Of course, now the RI-drop between living room and basement might no longer be negligible. It depends upon how many amperes flow through that 100-microohm heating pipe to the basement. That is determined by the effective resistance of the power lines. A 100-ampere surge would make a 10-millivolt potential difference between living room and basement ($= 10^{-4}$ ohms $\times 10^2$ amps)—before the fuse blows, that is.

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[2] 15-amp fuses and circuit breakers are standard "code" in most U.S. localities. Electric stoves and dryers often use 30-amp fuses.

In some hospital situations there is a low-resistance electrical connection through the heart. A cardiac catheter with EKG electrodes is typical. Then even the millivolts associated with leakage currents can be lethal, and proper grounding is indeed a science.

SUMMARY

When electric current flows, the current direction is defined as the direction of motion of the + charge, that is, from the higher potential to the lower potential point of the circuit. The part of the circuit where the current flow is from lower to higher potential (e.g., the battery) is called the source of e.m.f.

The rate at which electrical energy is dissipated in a circuit element is given by the product of the potential difference across it times the current through it: Power = VI.

Units: 1 volt × 1 amp = 1 watt

The resistance of a circuit element is defined as the ratio of the voltage drop across it to the current through it: $R = V/I$. Accordingly, jargon for the voltage drop across a passive element is "RI drop." Some people prefer to say "IR drop."

A circuit element whose resistance is independent of the current through it is called ohmic. It is said to obey Ohm's Law.

Two equal resistances in parallel have half the resistance of each. Two equal resistances in series have twice the resistance of each.

The resistance between the inside and outside of the human skin might be approximately 10^5 ohms for a 1-cm^2 patch. Very dry, it could be more than twice that. Sweaty, it could be less than half that.

How you feel currents (approximately):

Threshold of perception: 0.5 milliamps

Cannot-let-go current (dc): 10 milliamps

Triggers fibrillation: 20 microamps through the heart or 100 milliamps through the chest

A "good ground": less than 10^{-3} ohms to a large conductor, like a water pipe or heating duct.

Leakage currents of 100 milliamps or more are common in the housings of electric motors.

Electric hand tools have their metal housings grounded using a three-prong plug.

NOTE: Many have plastic housings.

In U.S. homes, currents in excess of 15 amps cause the fuse or circuit breaker to interrupt the circuit.

KEY TERMS

E.m.f. Electromotive force; also called *electromotance* A voltage source in which some other form of energy (chemical, mechanical, or magnetic) is converted into electrical energy. Also, the open-circuit voltage of that voltage source.

Resistance Quotient of the potential differences across a circuit element divided by the current through it. The unit is the ohm = 1 volt per amp.

Ohm's Law States that the current through a circuit element is proportional to the potential difference across it. This is an approximation good for small currents through homogeneous materials. Junctions between different materials are notoriously *nonohmic*.

Resistivity A bulk property of a conducting material, equal to the quotient of the electric field divided by the current density. The unit is the ohm meter.

Conductivity The reciprocal of the resistivity. The unit is the siemens per meter = $\text{ohm}^{-1}\ \text{meters}^{-1}$.

Polygraph A lie detector using simultaneous chart recording of several biological measurements, including skin resistance and pulse rate.

Fibrillation Pathology of the heart in which it abandons its normal rhythm and begins rapid small-amplitude contractions.

R.F. Radio frequency. Refers to alternating currents in the frequency range from about 10^5 to 10^{10} Hz.

E.K.G. Electrocardiograph. A device for obtaining a chart recording of the electric potentials associated with the contractions of the heart. The electrocardiogram is the chart records themselves.

E.E.G. Electroencephalograph. A device for obtaining a chart recording of the electric potentials associated with brain activity. The electroencephalogram is the chart records themselves.

E.M.G. Electromyograph. A device for obtaining a chart recording of the electric potentials associated with muscle activity. The electromyogram is the chart records themselves.

G.F.I. Ground-fault interrupter. A safety device that detects an increase in the resistance of a ground connection on an instrument case and cuts off electric power to the instrument.

REVIEW QUESTIONS

- A 10-ohm resistance connected across a 1.5-volt battery draws a current of ________ amps, and dissipates ________ watts of power. A 20-ohm resistance draws half/twice the current and dissipates one-quarter / one-half / twice / 4 times the power.
- If two 1-ohm resistances are connected in series across a 1.5-volt battery, each one has 0.75 volts across it so that a current of ________ amps is drawn and their total resistance is ________ ohms.
- If two 1-ohm resistances are connected in parallel across a 1.5-volt battery, each one has the full 1.5 volts across it so that a current of ________ amps is drawn from the battery and their total resistance is ________ ohms.
- The reason you can't feel any current flowing into your skin in a GSR measurement is that: Current through the skin is less than 100 microamps / no current is flowing through the skin / the voltages across the skin are less than 1 microvolt
- Why is it dangerous to drop the telephone into the bathtub?

NOTE: It is the ringing circuit that has the 48 volts.

- A cattleprod is capable of administering shocks of several hundred volts. Why doesn't it cause burns?
- A color TV has voltages of tens of thousands of volts in it. Why do you not get a shock when you touch the metal chassis?
- When handling some power tools with 2-prong plugs you feel a slight tingling sensation. Explain why the tingling has a vibration frequency of 120 Hz. How does a 3-prong plug prevent this kind of electric shock?

PROBLEMS

✶ **1. Units of Energy, Voltage, and Charge** The volt may be defined as 1 joule per coulomb.

(a) How much work is done by a dry cell labeled $1\frac{1}{2}$ volts when a current of 1 amp is drawn for 1 second?

(b) Show that an electric iron that draws 5 amps from an outlet labeled "120 volts" for 1 second has converted 600 joules of electrical energy into heat.

(c) Would you call it a 600-watt appliance? Explain.

✶ **2. Flashlight** Every electron has a charge of -1.6×10^{-19} coulombs. When one electron travels around a two-cell flashlight circuit, how much work is done on that electron? Each cell is labeled $1\frac{1}{2}$ volts, so this is a 3-volt flashlight. [Answer: 3 electron volts. Now convert that to joules.]

✶ **3. Volts, Amps, and Watts** An electric iron intended for use on 115-volt house current is marked "300 watts."

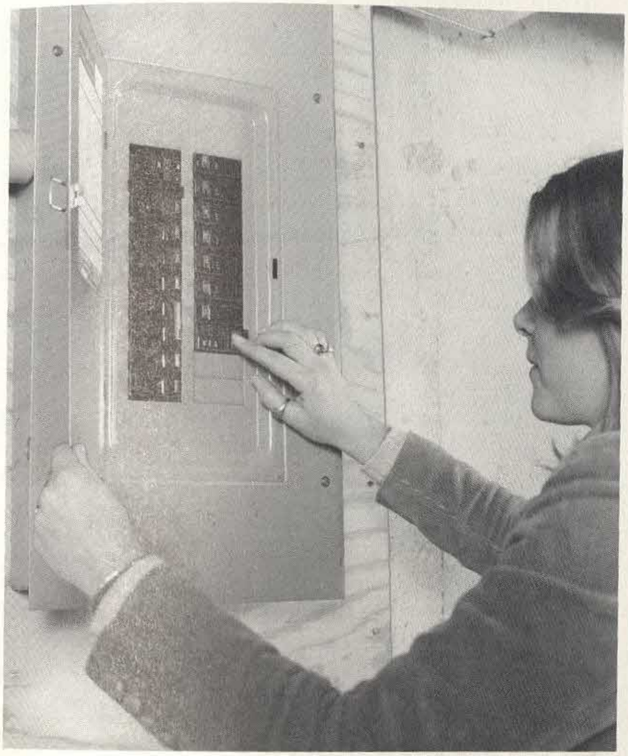
(a) How much current does it draw?

(b) What is its resistance?

(c) If one accidentally plugged it into the 230-volt line (twice the line voltage), how much current would it draw? How much power would it dissipate?

NOTE: This is often called a 220-volt line. Household electric plugs are made so that such confusions are not possible: The 115-volts plug just won't fit into a 230-volt receptable.

✶ **4. Fuses** 15-amp fuses and circuit breakers are pretty standard in U.S. homes; so is 115-volt line voltage.



PROBLEM 4 What is the maximum amount of power you can consume per fused line? Why is the electric oven on a special 230- volt line? Photo shows circuit breakers.

✶✶ **5. Electrolysis** Two electrodes are dipped into an ionic solution (like sodium chloride in water), and a constant voltage is applied between them. In the solution, the current is carried by ions of both sign. The Na^+ ions travel in the direction from anode to cathode (i.e., from + to − electrode); so do the H^+ ions. The Cl^- ions travel in the direction from cathode to anode (i.e., from − to + electrode); so do the OH^- ions.

(a) At the anode, a Cl^- ion gives up an electron. Why do you think you smell chlorine gas there? (HINT: Chlorine gas is Cl_2.)

(b) At the cathode, an H^+ ion takes on an electron. The neutral hydrogen goes off as bubbles of H_2 gas. When 6.02×10^{23} electrons have passed through the circuit, only half a mole of hydrogen gas has been liberated at the cathode. Why only half?

(c) Nonetheless, 1 mole of sodium hydroxide (caustic soda, NaOH) has been formed in the solution. Explain. (HINT: Write down the chemical reactions that take place.)

✶✶✶ **6. Current in an Electrolyte (Sodium Chloride)** In a salt solution being electrolyzed, 0.001 moles of Na^+ ions are neutralized at the cathode (negative electrode) and 0.001 moles of Cl^- are neutralized at the anode (positive electrode) every 100 seconds. The drift speed of the Cl^- is $\frac{3}{2}$ that of the Na^+, so 0.0004 moles of Na^+ leave the anode "compartment" and enter the cathode "compartment" while 0.0006 moles of Cl^- leave the cathode compartment and enter the anode compartment—every 100 seconds.

(a) Show that the current is 0.96 amps. (HINT: Avogadro's number is 6.0225×10^{23} per mole. The elementary charge is 1.6021×10^{-19} coulombs. Their product, 9.649×10^4 coulombs per mole is called the Faraday, also called "1 equivalent of charge.")

(b) What happens at the anode is that Cl_2 gas is liberated. Argue that the Cl^- concentration in the anode compartment decreases even though Cl^- is moving in. Be quantitative.

(c) What happens at the cathode is that H_2 gas is liberated. Argue that the Na^+ concentration in the cathode compartment is increasing while the Cl^- concentration there is decreasing. Be quantitative.

Some Simple Ohm's-Law Problems (Problems 7–14)

✶ **7.** A voltmeter connected across the terminals of an automobile battery reads 12.0 volts. When a light bulb marked "24W" is connected across the terminals, it draws 2 amps. The voltmeter still reads 12.0 volts.

(a) What is the resistance of the light bulb?

(b) A light bulb marked "36 W" is now connected across the battery. It draws 3 amps. What is its resistance?

(c) How much current would a 12-ohm resistance draw?

(d) How much current would a 24-ohm resistance draw?

✶ **8.** How much power does each bulb in Problem 7 consume? Convince yourself that the lower the resistance of the bulb, the more power it draws from the battery.

✶ **9.** A typical automobile storage battery has an open-circuit voltage of 12 volts and an internal resistance that is less than 0.1 ohms.

(a) Show that if the jumper cables are thick enough and the contacts are clean enough, an accidental short circuit can draw over 120 amps. (HINT: The obvious "short circuit" is caused by touching the metal contacts of the red and the black cables together while the other ends are connected to the battery.)

(b) Show that as long as that short remains, heat is being produced inside the battery at a rate of more than 1000 joules per second (more than 1 kilowatt). (HINT: Don't short your jumper cables, or you'll cook your battery. It may even explode. Even in a second or two of normal operation the contacts get warm. This tells you that the contact resistance is not really neglible! Don't touch the wrong clip to the body of the car.)

✶ **10.** A 10-ohm resistance is connected across the 12-volt battery.
(a) How much current does it draw?
(b) How much power does it consume?
(c) Two 10-ohm resistances are connected in parallel across the battery. Convince yourself that each one draws the same current as in part (a) so that the total current drawn from the battery is twice the value in part (a). What is the resistance of the parallel combination?
(d) How much power is now consumed?

11. The two 10-ohm resistances are now connected in series across the 12-volt battery.
(a) Convince yourself that each one has 6 volts across it. How much current is drawn from the battery?
(b) How much power is consumed in each one?
(c) If you multiply 12 volts times the current, the product is ________ watts. This is the power consumed in the series combination. Is it twice the answer to part (b)? Explain.

12. Three 10-ohm resistances are connected in series across the 12-volt battery.
(a) What is the voltage across each?
(b) Find the current, the series resistance, the power developed in each, and the total power consumed.
(c) Repeat for four identical resistances in series.

✶ **13.** Two 10-ohm resistances are connected in parallel, and this (5-ohm) combination is connected in series with another 10-ohm resistance.
(a) If the 12-volt battery is connected across this (15-ohm) combination, how much current is drawn? Sketch the circuit.
(b) How much power is developed in each resistance? How much total power? (HINT: One of the resistances consumes more power than the other two. Why?)
(c) Find the voltage drop across each resistance.
(d) Is the power developed in each resistance proportional to the voltage drop across it, or to the square of the voltage drop?

✶ **14.** Two 10-ohm resistances are connected in series, and a third 10-ohm resistance is connected in parallel with this (20-ohm) combination.
(a) The current through the series combination is half that through the single 10-ohm resistance. The power developed in each of the two resistances in series is
4 times / twice / half / one-quarter
that developed in the single 10-ohm resistance.
(b) Find the resistance of the circuit.

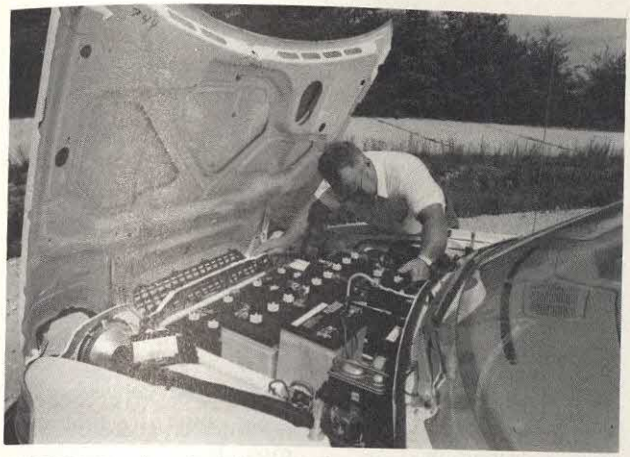
✶✶ **15. Automobile Battery** When a battery is "low," meaning uncharged, its internal resistance becomes appreciable. If you turn on the headlights, they are dim. With the lights off, a voltmeter still reads 12 volts. With the lights on, the voltmeter reading falls to 9 volts.
(a) Explain the 3-volt drop.
(b) With the lights on, the ammeter reads 1.5 amps. Argue that the internal resistance of the battery is 2 ohms.
(c) While 1.5 amps flows in the circuit, what is the resistance of the headlights.

NOTE: When the headlights are burning brightly, that is, when the battery voltage is its normal 12 volts, the headlight resistance is actually greater. Hot headlight filaments have a larger resistance than cool ones. Some people say that incandescent bulbs disobey Ohm's law. Others just say that their resistance is a function of the current they carry.

✶✶ **16. Charging a Storage Battery** Suppose that the battery charger has an open-circuit voltage of 14 volts and an internal resistance of 0.5 ohms.
(a) The charger is connected so as to recharge a "low" battery. Sketch the circuit diagram, assuming a battery voltage of 12 volts.
(b) Argue that a charging current of 4 amps flows if the internal resistance of the battery is much less than 0.5 ohms.
(c) What would happen if you connected the charger backward, that is, if by mistake you connected + to − and − to +?

✶ **17. Car Battery** A fully charged 12-volt automobile battery is guaranteed to deliver 60 ampere hours. That is, it can supply 1 amp for 60 hours, or 2 amps for 30 hours, and so on. You pull the key and leave the headlights on by mistake—two headlights, a 60-watt bulb in each. How long can you expect them to stay bright?

✶ **18. Electric Automobile** To drive a small car at a steady 45 miles per hour on a flat road requires about 6 kilowatts of power delivered to the transmission. For reasons of safety you don't want to use more than 96 volts (= eight 12-volt storage batteries in series).



PROBLEM 18 An electric car. The motor can be quite small and can develop large torques at slow speeds, so that no transmission is needed. But the batteries take up a lot more room than a gasoline tank!

(a) How much current do you need to draw?
(b) Your 12-volt battery charger works at 6 amps all night (= 8 hours) to charge your eight batteries. Assume that the whole process is 50% efficient; that is, half the energy that goes into the batteries is delivered to the transmission. How far can you drive in the morning?

★★ **19. Fuel Cell** A fuel cell is a source of e.m.f., which supplies electric power by oxidizing some fuel. Typically, the fuel is hydrogen gas, which is absorbed on the anode, while oxygen is absorbed on the cathode. The net chemical reaction is the oxidation of hydrogen to form water. The "ideal" e.m.f. is 1.23 volts. In practice, hydrogen fuel cells put out between 50% and 70% of the ideal voltage.

Suppose that you are designing an electric car to be powered by a hydrogen fuel cell. It requires an average of 0.75 million joules per mile to drive a medium-sized car on a level road at 44 miles per hour.
(a) How many moles of hydrogen gas does it take to go 200 miles? (HINT: 1 electron volt = 1.6×10^{-19} joules. Say each atom of hydrogen delivers 0.6 electron volts. There are two atoms in an H_2 molecule, and 6.02×10^{23} molecules in a mole.)
(b) Your tanks are designed to be filled to a maximum pressure of 2500 p.s.i. (= 170 atmospheres). How many 25-liter tanks does the car need to carry for that 200-mile trip? (HINT: Use the ideal-gas law (Chapter 6)).

★★ **20. Fuel Cell** Below is a list of the advantages fuel cells have over other devices for generating electric power (like steam turbines):

No moving parts.

High efficiency.

Easy start-up and turnoff.

Modular construction—Can be mass-produced.

No sulfur emissions, no pollution.

Here are some disadvantages:

Hydrogen is dangerous to have around; it explodes.

Unlike oil or gasoline, you can't just pour it in.

Room-temperature operation requires an expensive catalyst like platinum at the anode. To operate without a catalyst the temperature must be around 650°C.

Discuss why you think electric generating plants
automobiles are not powered by fuel cells yet.

★★ **21. Ni-Cd Batteries** Rechargeable batteries have been used to start automobiles ever since the hand crank went out of fashion. The rechargeable AA-size nickel-cadmium batteries that power pocket calculators became common in the 1970s. Their open-circuit voltage (e.m.f.) is 1.35 volts, and they can deliver 0.5 amps for about an hour after being fully charged.
(a) How much available energy is stored? Answer in joules.
(b) Does that sound like a lot? That much energy could bring about 5 cm^3 of water to a boil from room temperature. Compare the energy stored in one of these AA batteries with the energy content (heat of combustion) of the same volume of a liquid fuel like alcohol. Ethanol has a heat of combustion of 3.8×10^4 joules per cm^3. An AA battery is a cylinder ½ inch in diameter and 1½ inches long.

★★ **22. 115 Volts and Electrical Safety** Your blood is a good conductor. Your skin is a good insulator. For example, the resistance of the body fluids between finger and toe might be 10 ohms, while the resistance between inside and outside of 1 cm^2 of skin might be 10^5 ohms—at finger or toe.
(a) With your toe grounded, your finger touches a 115-volt wire. How much current flows through your body fluids? (HINT: To flow inside the body, current has to traverse two of those hundred-thousand-ohm skin barriers. Unless

you are sweaty, the surface of the skin is also a good insulator.)

(b) Why did you neglect the resistance of the blood in answering part (a)?

(c) Could you get more than 20 microamps through your heart this way? That is, is there danger of causing fibrillations? Explain. (HINT: This is tricky. There are paths from finger to toe that don't go through the heart.)

✶✶ **23. Can You Feel $1\frac{1}{2}$ Volts?** Even though you touch both terminals of a flashlight battery with your hands, you can not feel any current.

(a) Use the 10^5-ohm "patches" of the previous problem to explain why.

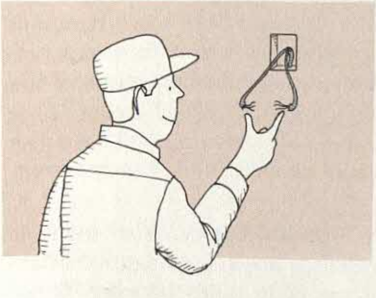
(b) If you touch both terminals with your tongue, you can feel the current. Are you feeling current along the tongue or through the tongue? How do you know current is indeed flowing along the surface of the tongue?

✶✶ **24. Quickie Voltmeter** An electrician likes to test the 115-volt ac line by wetting two fingers and briefly touching a finger to each side of the line (see figure). If he gets a shock, he knows there is voltage on the line.

(a) Suppose that each patch of wet skin has a through-the-skin resistance of 5×10^4 ohms. How much current flows through the fluids inside his fingers? Why did you neglect the resistance of those fluids (mostly blood)?

(b) Why is it essential that the fingers be on the same hand?

(c) Why is the electrician safer if he wets the fingers? (HINT: His feet may be grounded.)

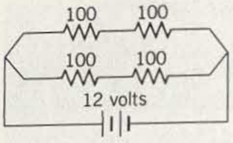


PROBLEM 24
Quickie voltmeter.

✶ **25. Resistances in Series and Parallel** You have a supply of 100-ohm resistors. Your 12-volt battery has an internal resistance of less than 1 ohm.

(a) Any one of the resistors connected across the battery draws 0.12 amps and dissipates 1.44 watts. Verify.

(b) Two of the resistors connected in series across the battery have a resistance of ______ ohms and draw ______ amps. Each one dissipates ______ watts.



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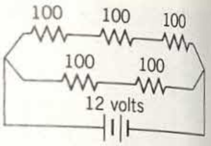

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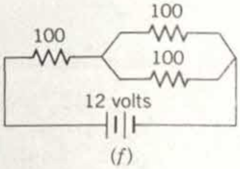


(*f*)

(c) Two of the resistors connected in parallel across the battery have a resistance of ______ ohms. Together they draw a current of ______ amps. Each one dissipates ______ watts.

(d)(e)(f) For the circuits illustrated, find the current and the power dissipated in each resistor.

✶✶ **26. Design Your Own**

(a) Show how to assemble four 100-ohm resistors in such a way as to have a net resistance of 40 ohms.

(b) Connected across the 12-volt battery, how much power is dissipated in each of the 100-ohm resistors?

✶✶✶ **27. Design Your Own** Show how to assemble four 100-ohm resistors in such a way as to have a net resistance of 60 ohms.

✶✶ **28. Kirchhoff's First Law** Suppose that there is a branch in a circuit so that the current can divide, some going into the right-hand branch, some going left. You conjecture that the current flowing into the branch point (fork, vertex, junction, node, . . .) is equal to the sum of the currents flowing out. Regardless of how many branches join at a point, your intuition tells you that the sum of the currents *in* is equal to the sum of the current *out*. The physical law you are intuiting is the *conservation of charge:* No charge is created or destroyed at the vertex. Good jargon is to say there are no sources or sinks of charge. Also charge is not stored up at the vertex. Here is an elegant way to state this "circuit theorem":

Kirchhoff's First Law: The algebraic sum of the currents flowing into any point on a circuit is zero.

(a) Review Sec. 14.4, Resistances in Series and Parallel, and point out where we needed that theorem.

(b) In Problem 25, Resistances in Series and Parallel, each circuit shown has two vertices. For each vertex, write down how much current flows in each branch.

⁎⁎ 29. Kirchhoff's Second Law

Statement: The algebraic sum of the voltages around any closed path in a circuit is zero.

It sounds like "what goes up must come down." The statement is a simple consequence of the definition of *voltage* as electric potential energy per unit charge. When you trace around a circuit and you have come back to the point where you started, you are back to the same potential energy. If a resistance is connected across a battery—a simple series circuit—the potential goes up as you trace − to + through the battery and goes down the same amount as you trace + to − through the resistance. The theorem starts to become useful when several circuit elements are connected in series and parallel combinations, especially if there is more than one e.m.f. in the circuit.

Trace through the circuits of Problem 25, Resistances in Series and Parallel, and show on a sketch how much the voltage goes up or down in each circuit element. Note that you can trace paths that don't go through a battery.

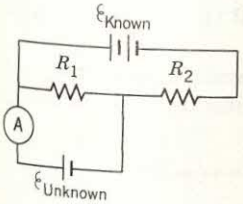
EXAMPLE: Part (d), the upper loop, going clockwise, we have −6V, −6V, +6V, +6V. Add them up and you get zero.

⁎⁎ 30. Potentiometer Using a voltmeter to measure an e.m.f. has the disadvantage that the meter has to draw some current in order to register. The best meters, of course, are the ones having the highest resistance. But it is possible to measure a voltage without drawing any current, using a circuit like the one shown in the figure. Essentially, it is a way of comparing two e.m.f.'s using well-calibrated resistors, without drawing current from the unknown.

(a) Suppose the unknown e.m.f. E_{unknown} is one-fourth the known e.m.f. E_{known}. If R_1 is one-third of R_2, show that no current will flow through the ammeter A.

(b) For arbitrary values, show that when no current flows through the meter,

$$E_{\text{unknown}} = \left[\frac{R_1}{(R_1 + R_2)}\right] E_{\text{known}}$$



PROBLEM 30 Potentiometer.

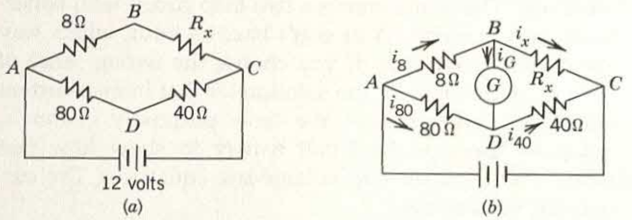
⁎⁎ 31. Wheatstone Bridge—Balanced The circuit shown in Fig. (a) is used for comparing resistances. One of the resistances is presumed unknown (say R_x), and one can be changed in small steps, as with a dial (say the one labeled 40Ω).

(a) Call the voltage at point A +12 volts, and the voltage at point C 0 volts (= ground). What is the voltage at B?

(b) What is the voltage at D?

(c) Convince yourself that the condition necessary for those two voltages to be equal is

$$\frac{8\ \Omega}{80\ \Omega} = \frac{R_x}{40\ \Omega}$$



PROBLEM 31 Wheatstone bridge.

Now connect a galvanometer or some other device sensitive to small currents between B and D as shown in Fig. (b). We say the bridge is *balanced* when no current flows in G. Here the balance condition makes $R_x = 4\ \Omega$.

(d) Write down the current in each branch at vertex A.

(e) Repeat for vertex B. Show that Kirchhoff's "current law" is satisfied at both branch points.

(f) Trace through the circuit in a number of different ways, writing down the voltage drop or rise in each circuit element. Convince yourself that Kirchhoff's "voltage law" is satisfied for each completed loop.

(g) How many different loops are there?

(h) If R_x has some value other than 4 ohms, the bridge is unbalanced and the current i_G through the galvanometer resistance R_G is no longer zero. Kirchhoff's current law at vertex B gives the condition

$$i_8 = i_G + i_x$$

At vertex D, the current law gives

$$i_{80} + i_G = i_{40}$$

But applying Kirchhoff's current law at vertices A and C is redundant. Show that it gives

$$i_8 + i_{80} = i_x + i_{40}$$

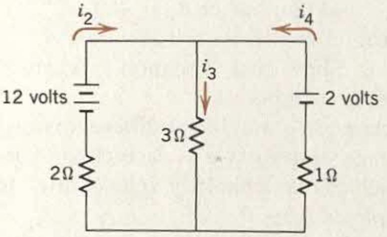
which can be obtained by adding the two previous equations.

(i) We see that a circuit consisting visibly of three loops, with four vertices and five named branch currents allows only two independent "current-law" conditions on these currents. To determine the five currents requires five equations, so we need three more. Write down three independent Kirchhoff's-voltage-law equations. Show that a fourth such equation, obtained by tracing around one of the other possible loops, is not an independent equation but could be derived from the others.

✶✶ **32. More Than One e.m.f.** You really need Kirchhoff's Laws to solve a circuit containing batteries in different loops. The figure shows a two-loop circuit with batteries in two branches. You don't have to intuit which way the currents will flow. If you choose the wrong sense of current in one branch, the solution for that branch current will come out negative! We have purposely chosen i_4 going *up* through the 2-volt battery to show how that works. Write down two voltage-law equations. The current-law equation is

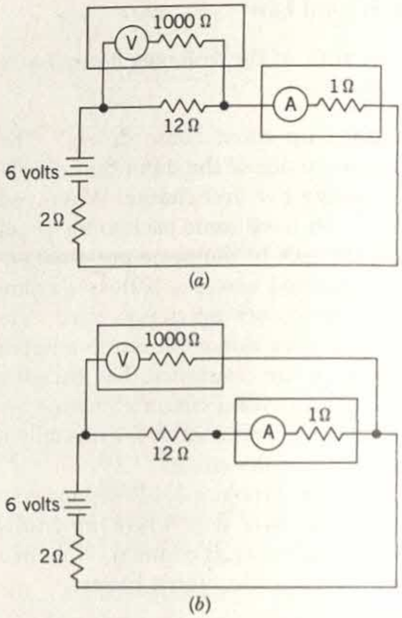
$$i_2 + i_4 = i_3$$

Use it to eliminate one of the three current variables, leaving two equations in two unknowns. Solve those any way you know how. Note that i_4 comes out negative.



PROBLEM 32 Two e.m.f.'s in one circuit.

33. Ammeter and Voltmeter To measure an unknown resistance, we want to know the voltage across it and the current through it. So the voltmeter should have a high resistance in order to shunt off as little current as possible. The ammeter should have a low resistance in order to have as small a voltage drop as possible. If the meters are connected as in *a* in the figure, the voltmeter (internal resistance 1000 Ω) does measure the voltage across the unknown, but the ammeter measures more current than goes through the unknown. Suppose that the "unknown" resistance is 12 ohms.



PROBLEM 33 (a) Voltmeter directly across unknown resistance. Ammeter measures current through the parallel combination. (b) Voltmeter across series combination of unknown and ammeter. Ammeter measures current through unknown resistance.

(a) The 12 Ω is in parallel with the meter's 1000 Ω. Verify that the parallel combination is 11.86 Ω, so that the whole circuit has a resistance of 14.86 Ω.

(b) What does the ammeter read?

(c) Multiply that current times the 11.86 to find what the voltmeter reads.

(d) (Voltmeter reading)/(ammeter reading) = ________, which is about 1% below the true resistance.

(e) Now suppose the meters connected as in *b* in the figure. The ammeter now reads the true current through the unknown, but the voltmeter reading now includes the voltage drop across the ammeter. Verify that the circuit resistance is 14.83 Ω and that the voltmeter reads 5.19 volts.

(f) Verify that the ammeter reads 0.40 A so that (voltmeter reading)/(ammeter reading) = 13.0 Ω. That is 8% higher than the true resistance.

(g) Repeat the two calculations with a 1200-ohm "unknown" to convince yourself that hookup *a* is not always better than *b*.

☐ QUESTION FOR FURTHER THOUGHT

α. After reading about "capacitive coupling," describe how to tell whether a circuit element that has a voltage across its terminals is an e.m.f. or a charged capacitor.

Electromagnet lifting scrap metal. This method leaves behind nonmagnetic materials like copper, cloth, and plastics.

15

MAGNETIC FIELD

SOME QUESTIONS THIS CHAPTER WILL ANSWER

How can a current-carrying wire be used to measure the strength of a magnetic field?

Why is a right-hand rule necessary in describing the force on a charge moving in a magnetic field?

How does an electric motor work?

Is the strength of the magnetic field 2 inches away from one of those bulletin-board magnets much greater or much less than 1 newton per amp per meter?

What configuration of current gives rise to a uniform magnetic field?

If the current in a uniformly wound coil is doubled, the strength of the magnetic field inside is multiplied times ______.

Explain why the self-inductance of a coil is proportional to the square of the number of turns.

The volt second is a unit of ______ ______.

The remarkable ability of migratory birds to navigate accurately over distances of thousands of miles has given rise to speculation that they have a built-in **magnetic compass.** Thus far, physiological research has failed to find any organ in birds—or in man, for that matter— that senses the direction of a steady magnetic field.[1]

Does that mean that **magnets** have no effect on us? Researchers working with large magnets have tried clamping the heads of human subjects between the pole faces and report no adverse effects. Monkeys in very high fields show slower reaction times. That kind of experiment is neither controlled nor specific enough to yield much useful information. But research on enzymes and on simple animals in magnetic fields is now fashionable. It may seem odd that the biological effects of steady and slowly varying magnetic fields are so little understood, while those of fields alternating around 10^{15} times per second are so familiar: These are the fields we perceive as visible light.

What we all know about magnets is that they attract iron: nails, thumb tacks, paper clips, pins—no, not brass screws. You might experiment with keys and coins and kitchen utensils if you have access to a little bulletin-board magnet. I use those to stick cartoons to the refrigerator door. (See Lab at your Desk No. 5 at the end of this chapter.) Most of us have a lot of intuition about how magnets behave. They have two "ends," conventionally marked N and S, respectively. When two magnets are brought close, like ends repel, unlike ends attract (see photo). If you break a magnet in half, each half is still a magnet. And we all know that the **magnetic compass** tends to point north—or perhaps one should say, one end points north and one end points south. The explanation of this phenomenon is that the earth is itself a magnet. That idea, schoolboy lore today, was not advanced until close to the year 1600.

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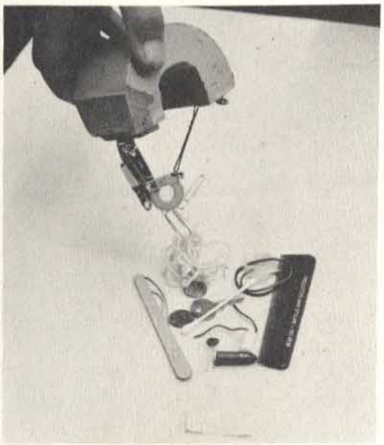
[1] Magnetotactic bacteria orient in the earth's magnetic field. Their built-in compass *has* been discovered.

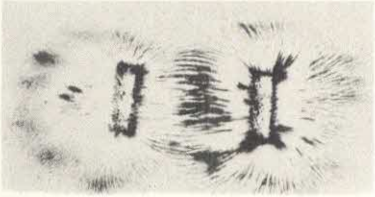
The connection between magnetism and electricity is a much more recent discovery (nineteenth century), and its popular understanding is generally less developed. The word *electromagnet* may evoke the idea of wrapping a nail with a current-carrying wire and, sure enough, the nail becomes magnetized. Most people vaguely associate that with how an electric motor works. There is an *armature*—a coil wound on an iron core. As it turns, the direction of the current in it is cleverly changed so that the interaction with another magnetic field results in a one-way torque.



Paper clips, thumb tacks, nail clipper, washer—all of these objects cling to the magnet and to each other when they are magnetized. The brass key, the copper wire, and the paper fastener appear quite unmoved.

15.1 MAGNETIC FIELD

The term *magnetic field* appears several times in the above paragraphs. There is that word *field* again. We had *gravitational* field in Chapter 1. You met *electric* field in Chapter 13. What is magnetic field? You may have seen a demonstration experiment in which iron filings were sprinkled on a piece of paper placed above the poles of a horseshoe magnet. The filings tended to align in a symmetric pattern, the lines diverging from the two poles (see photo). The word *field* is often used to denote the region of influence of a magnet, and such a pattern of iron filings shows where the influence is strong. The word is also used quantitatively, as a measure of the strength and direction of that influence: The iron filings tend to align parallel to the direction of the magnetic field, and the field strength determines the torque tending to align them. Here we have a vector field, something which has a direction and a magnitude (a strength) at every point in space, just like electric field and gravitational field. We might think of exploring magnetic fields with a small compass needle. We say small, because such measurements are meaningful only for a field that is pretty uniform over regions the size of our measuring device. If you mark an "N" on the end of the needle that points north in the earth's magnetic field, you can use it to point along the direction of any other magnetic field. The torque tending to turn it in that direction is proportional to the field strength.



The iron filings have oriented themselves parallel to the magnetic-field direction and are pulled into the strongest part of the field.

If we were to use such a torque measurement to define magnetic field strength, we would have to specify a "standard" compass needle. Another way to approach the definition arrives at convenient units without requiring any new standard. It uses the interaction of magnetic field and electric current. Here is how:

A straight wire carrying current in a direction perpendicular to a magnetic field experiences a transverse force proportional to the strength of the current and, not surprisingly, to the length of the wire:

$$\frac{\text{force}}{\text{length}} \propto \text{current}$$

The proportionality constant is defined to be the magnetic field strength B. For a current I we have that the force per unit length is

$$\frac{\text{force}}{\text{length}} = BI$$

The direction of that force is perpendicular both to the magnetic-field direction and to the current direction. Note that one more specification is needed. When you say perpendicular, you have to say right or left. One way to say it is this: Point the thumb of the right hand along the current and the forefinger along the field. Then the palm faces in the direction of the force. (See Fig. 15.1.) You might like to think of the forefinger as the arrow that points along the field, and the palm as pushing in the direction of the force. That prescription is called a **right-hand rule.**

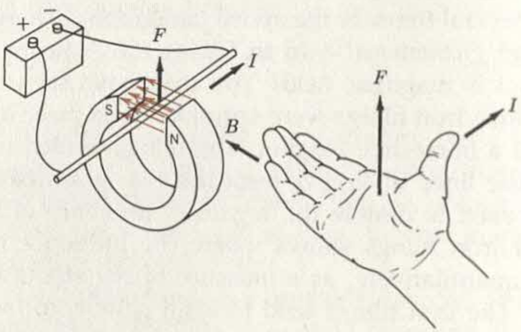


FIGURE 15.1 Right-hand rule for the force on a current in a magnetic field. The pointing finger points along the field; the thumb points along the current. Then the palm faces in the direction of the force. Mnemonic: Palm pushes . . .

We did not have such trouble defining the directions of the electric or gravitational fields. Forces there were parallel to fields, and the notion of right-or-left did not enter. When the force is perpendicular to the field, handedness raises its head. We tend to think of handedness[2] more in biological contexts. There is the difference between your right hand and your left hand. There is the way the shells of snails twist. There is the spiral structure of molecules of edible sugars. Here we find handedness in the description of the electromagnetic interaction. You might wonder why we apparently avoided any talk of right-or-left when we used the compass needle to define field direction. Of course, we needed to paint the "N" on the needle first and needed geographical notions to do that. If we had started with an electromagnet instead of a permanently magnetized needle, then which end was "N" would have depended on whether the coil was wound right-handed or left-handed.

A word about the units of B-field. We defined B as force per unit length per unit current so that the force is given by

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[2] "**Chirality**" is the technical name.

$$F = BIL \qquad \text{(Eq. 1)}$$

This means that B must be measured in newtons per meter per ampere. That unit is called the **tesla:**

$$1 \text{ tesla} = 1 \frac{\text{newton}}{\text{amp meter}}$$

One tesla turns out to be a huge field, usually attained only in large magnets that have to be wheeled around the lab. A smaller unit is the *gauss,* defined so that 1 tesla is 10,000 gauss. The earth's magnetic field in the United States is on the order of 0.6 gauss.

To get some intuition about the size of magnetic forces in the earth's field, try a current of 1 ampere:

$$\frac{\text{force}}{\text{length}} = \left(0.6 \times 10^{-4} \frac{\text{newton}}{\text{amp meter}}\right) \times (1 \text{ amp}) = 6 \times 10^{-5} \frac{\text{newton}}{\text{meter}}$$

To get a force of 1 pound this way takes almost 75 kilometers of wire. Better wrap it into a coil.

EXERCISE 1 What is the force on 10 cm of armature wire lying perpendicular to a magnetic field of 0.01 tesla and carrying 2 amps?
[Answer: 0.002 N]

EXERCISE 2 If the radius of that armature is 3 cm, and it has 600 turns, show that it can develop a torque of 0.036 Nm. That torque might be about right for turning an electric blender, for example.

(HINT: Torque = force times moment arm. Since the force is perpendicular to the wire, the moment arm is the radius.)

15.2 FORCE ON MOVING CHARGE

In developing the definition of B-field, we talked about current flow in a wire and the force on the wire when it is in a transverse B-field. But the wire was not essential; it was just a convenient way of having a current flow in a straight line. Current flowing in an ionic solution, or current flowing in an ionized plasma, or a beam of ions or electrons traversing a vacuum, all these will experience a force in a transverse magnetic field, a force given by that same law. The essential feature of this force is that a charge move across a B-field. Indeed, instead of formulating the magnetic force in terms of force per unit length of current, it might seem more fundamental to write down the force per unit charge. Since the current due to motion of a charge is proportional to that charge's velocity, the force is proportional to the velocity of the charge:

ELECTRIC MOTOR

In a simple dc motor the split ring commutator makes contact with the metal "brushes." As the rotor turns, the current in the armature is reversed, twice in each revolution. (See Problem 21.)

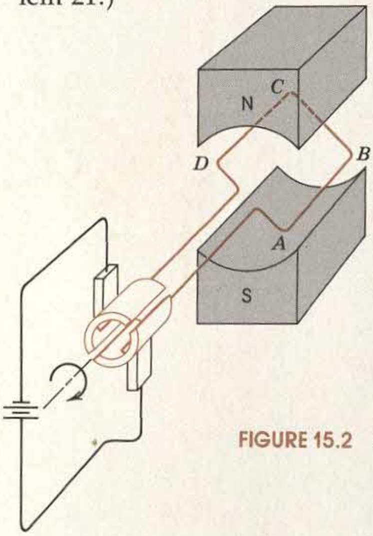

FIGURE 15.2 A simple dc motor.

In an ac synchronous motor the insulated rings are connected to the windings of the armature at opposing points. Since the current reverses twice each revolution, no commutator is needed.

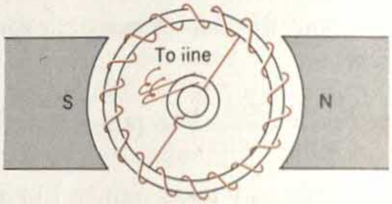

FIGURE 15.3 An ac synchronous motor.

$$\frac{\text{force}}{\text{charge}} \propto \text{velocity}$$

Again the proportionality constant turns out to be the magnetic field strength B. To see how this follows from our previous formulation in terms of current, write

$$\text{current} = \frac{\text{charge}}{\text{time}} = \frac{\text{charge}}{\text{length}} \times \frac{\text{length}}{\text{time}}$$

and note that length/time is velocity. Then you can get from our force-per-unit-length formula to a force-per-unit-charge expression:

$$\begin{aligned}
\frac{\text{force}}{\text{charge}} &= \frac{\text{force}}{\text{length}} \times \frac{\text{length}}{\text{charge}} && \text{(now substitute Eq. 15.1–1)} \\
&= \left(B \times \frac{\text{charge}}{\text{time}}\right) \times \frac{\text{length}}{\text{charge}} && \text{(divide out the charge)} \\
&= B \times \frac{\text{length}}{\text{time}} \\
&= B \times v
\end{aligned}$$

Repeat: force = (charge) × (field) × (velocity):

$$F = qBv \tag{Eq. 1}$$

Whether Eq. 15.1–1 or Eq. 1 is chosen as the definition of B-field is irrelevant. The two are completely equivalent.

EXAMPLE The electron beam in a TV tube might have a speed on the order of 8×10^7 m/sec. When such a beam moves in a direction perpendicular to the earth's magnetic field ($B = 0.6 \times 10^{-4}$ tesla), the magnetic force is given by the formula

$$\begin{aligned}\frac{\text{force}}{\text{charge}} &= 0.6 \times 10^{-4}\,\frac{\text{newton}}{\text{amp meter}} \times 8 \times 10^7\,\frac{\text{m}}{\text{sec}}\\ &= 4.8 \times 10^3\,\frac{\text{newton}}{\text{coulomb}}\end{aligned}$$

The units remind us that force per unit charge is what we have called electric field. An electric field of 4800 newtons/coulomb may seem like a lot until you realize that each electron has a charge of 1.6×10^{-19} coulombs so that the magnetic force on one electron is only

$$4.8 \times 10^3\,\frac{\text{newton}}{\text{coulomb}} \times 1.6 \times 10^{-19}\text{ coulombs} = 7.7 \times 10^{-16}\text{ newtons}$$

Exercise 4 shows that in traveling 10 centimeters—the length of a small television tube—across the earth's magnetic field the deflection of the electron beam would be just under 1 millimeter. This tells you that you might be able to detect the shift of the picture if you turn your TV set upside down. It also tells you that the deflecting magnetic fields that make the picture inside the tube have to be about two orders of magnitude greater than the earth's field.[3]

Cathode-ray tube (CRT).

EXERCISE 3 A certain TV tube has a 5000-volt electron beam. That means that each electron has an energy of 5000 electron volts (= 5×10^3 eV). Calculate the radius of its circular path in a B-field of 0.01 tesla. Here is one way:

The magnetic force is always at right angles to the B-field *and* to the velocity. That means the speed of the electron does not change, only its direction. Since this direction remains perpendicular to the B-field, however, the magnetic force remains at the same strength. The electron's acceleration thus remains at the same magnitude. The electron's path is a circle.

You already know that the acceleration of an object in uniform circular motion is v^2/R. If you know the mass m of the electron and its speed v, you can solve for the radius R. Here you don't know v directly, but you do know

[3] Television tubes do use magnetic beam deflection systems. Laboratory oscilloscopes and computer CRT (cathode-ray tube) displays use electrostatic deflection.

that $\frac{1}{2}mv^2 = 5000$ eV. You take it from there. Recall that $m = 0.911 \times 10^{-30}$ kg and 1 eV $= 1.6 \times 10^{-19}$ joule.

EXERCISE 4 In the example in the text the electron's speed is given as 8×10^7 m/sec, so it travels the 10 cm to the fluorescent screen of the tube in 1.25 nanoseconds. (Verify.) In the earth's B-field we calculated that it experiences a transverse force of 7.7×10^{-16} newtons, hence a transverse accelertion of 7.7×10^{-16} N/ 0.911×10^{-30} kg $= 8.5 \times 10^{14}$ m/sec^2. In 1.25 nsec the transverse deflection $\frac{1}{2}at^2$ is ______ mm.

If you don't like to remember formulas like $\frac{1}{2}at^2$, think about the transverse velocity. It starts from zero. In 1.25 nsec it gets up to 8.5×10^{14} m/sec^2 $\times 1.25 \times 10^{-9}$ sec $= 1.06 \times 10^6$ m/sec, perpendicular to the original direction. The average transverse velocity over the 1.25 nsec is half of that, so the transverse deflection is ______ mm. Aren't you glad that turned out to be a small fraction of the 10-cm distance to the screen? This means that the velocity vector has turned only through a small angle. If it had not turned out that way, we could not have used the 10 cm to figure the time of flight! Show that our approximation was off only 2 parts in 10^5.
(HINT: Pythagoras!)

What if the velocity v of the charge is not perpendicular to the magnetic field? Then the force per unit charge, instead of being Bv, is only $Bv_\perp$, where $v_\perp$ is the component of the velocity perpendicular to the field. Or, if you prefer, you can write $B_\perp v$, where $B_\perp$ is the component of the field perpendicular to the velocity. Writing

$$\frac{\text{force}}{\text{charge}} = Bv \sin \theta \qquad \text{(Eq. 1')}$$

avoids the subscript $\perp$ by introducing the angle θ that the B-field makes with the velocity.

To extend this same generality to our formula for the magnetic force on a length of current, we should write it

$$F = BIL \sin \theta \qquad \text{(Eq. 15.1-1')}$$

The direction of the force is, as before, perpendicular both to the field and to the direction of motion of positive charge (the current direction), with the sense given by the right-hand rule.

EXERCISE 5 Convince yourself that an electron whose velocity makes an acute angle with a uniform B-field travels in a spiral path.

(a) What is the direction of the axis of the spiral?

(b) As the electron proceeds along its spiral path, the acute angle: increases/decreases/remains the same

15.3 SOURCES OF MAGNETIC FIELD

The field concept, which we have used in connection with gravitational forces, with electrostatic forces, and now with magnetic forces, serves both an esthetic and a practical function. All these forces involve *action at a distance*, pushes and pulls where we don't readily see what is pushing or pulling. The field is our somewhat abstract substitute for hooks or chains or rubber bands. Having endowed our fields with the ability to exert forces, we define them in such a way that the field depends only on the object exerting the force and not on the object on which the force is exerted. For example, suppose that a charge q_1 exerts an electrostatic force on a charge q_2. We say that the electric field E_1, of which q_1 is the source, acting at the position of q_2, exerts a force E_1q_2 on the charge q_2. Because we know the law of action and reaction, we know that we could turn the description around. We could talk about q_2 as the source of the field E_2, saying that the force on q_1 is given by E_2q_1, and that the two forces are equal and opposite. Which description we choose is usually a matter of convenience. That is where the practical function of the field concept comes in. It was easy to describe the E-field in the region between two oppositely charged parallel planes, and thus the force on a small point charge placed in that region. It would have been terribly impractical to describe that force as a vector sum of pairwise forces between individual point charges.

In electrostatics, the source of an electric field is always electric charge. In planetary astronomy, the source of a gravitational field is always mass. What is the source of magnetic field? It would be tempting just to say "magnets." A couple of paragraphs further on you will see how such a glib answer could be misleading. What kinds of magnets are you familiar with? You know that bulletin-board magnets have an N-face and an S-face, and that with two magnets an N-face attracts an S-face and repels an N-face: Likes repel, unlikes attract, just as with electric charges.[4] Is it possible to isolate an "N," without an "S" to go with it? Is there such a thing as a magnetic **monopole**, analogous to an electric charge, where positive can be isolated from negative? Or do "N" and "S" always occur in pairs, as magnetic **dipoles**?[5] This question is still unsolved, but no magnetic monopole has yet been found. Theoretical physicists have been trying for years to explain why it can't exist. Would the existence of magnetic monopoles violate some law of physics? And if not, why don't they exist? It would be fun if, by the time this is printed, we knew more answers.[6]

If we don't have a monopole, do we at least have an elementary dipole, a "smallest magnet"? When we break a magnet in half, we end up with two magnets. Does that process ever end? Yes, it does. But this time the answer is not simple. The electron is itself a magnet, and every electron has the same magnetic dipole moment; that is, all electrons are magnets of the same strength. The magnetism of a piece of iron is, for practical purposes, the sum of the contributions of all the electrons in it. They may or may not all be aligned.

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[4] The gravitational force is different. There seems to be no such thing as gravitational repulsion.

[5] The Greek prefix *mono* means one; *di* means two.

[6] Here I write very carefully. Would my publisher ask me to delete this paragraph in order to avoid controversy?

The **magnetic dipole moment** of an electron turns out not to be the smallest in nature. The proton has an even smaller one. This nuclear magnetism is so much smaller than the electronic that as far as the bulk magnetic properties of materials are concerned, it can be neglected.[7]

In looking for the source of magnetic field, we have quite naturally asked questions on the microscopic level. In asking for the smallest fraction of a magnet that is still a magnet, we were led to the electron. But staying with macroscopic considerations, let us remember that a current flowing in a magnetic field experiences a force. By the action-reaction argument we conclude that the current is itself a source of magnetic field. Since a charge moving in a magnetic field experiences a force, the source of the field must experience an equal and opposite force. That "reaction" force is also a "magnetic force," so it must be ascribable to a magnetic field. The strength of that field depends only on the strength of the current, not on whether it is a current of electrons or protons or any other ions. *It is not the magnetic dipole moment (if any) of the charge carrier, just the motion of charge that is responsible for the magnetic field of a current.* The strength of the field is proportional to the strength of the current and, for most current configurations, falls off as you get farther away from the current. Here we give only a rough description of the magnetic field of a moving point charge.

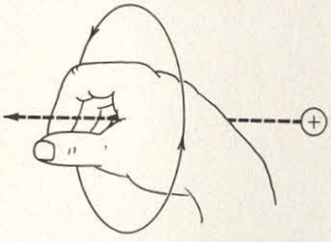

FIGURE 15.4 Right-hand rule for the sense in which *B*-lines curl around the direction of motion of a positive charge.

Direction? You guessed it! The *B*-field is perpendicular to the velocity of the charge. Here is the full prescription: Draw a line from the "source point," that is, from the moving charge, to the point at which you are going to test the field. That line is called the *radius vector*. The *B*-field is perpendicular to both the velocity vector of the charge and to this radius vector. Looking at the "lines" of the *B*-field, they are circles surrounding the line of motion of the charge. To describe the sense (right or left?), we again need a right-hand rule: Point the thumb of the right hand in the direction of motion of positive charge, that is, in the current direction. Then the way the fingers curl indicates the direction of the *B*-field surrounding the current. (See Figure 15.4.)

EXERCISE 6 Use this right-hand rule to find out which way the *B*-field inside a coil of wire will point.

[Answer: Let the fingers of the right hand curl in the direction of current in the coil. The thumb will point down the axis of the coil in the direction of the *B*-field (Fig. 15.5)]

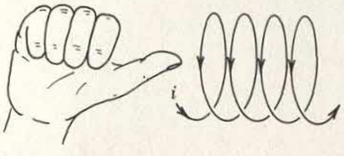

FIGURE 15.5 Right-hand rule for direction of *B*-field inside a coil carrying a current.

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[7] Nuclear magnetic moments are basic to a useful analytic technique called **nuclear magnetic resonance** (NMR) spectroscopy and to the medical diagnostic technique called magnetic resonance imaging (MRI). See Problem 34.

How the B-field falls off with increasing distance from the moving charge is a long and complicated story. Part of the story is told at the end of this chapter, in Problem 28, **Law of Biot and Savart.** But you might guess that here, again, there is an inverse-square law, and you would be right. For various current configurations one gets different field configurations. One special current configuration gives rise to a particularly simple field—a uniform field. A large uniform sheet of current, that is, an infinite plane of charge, moving with its velocity vector in the plane, has a magnetic field that is uniform, meaning that it does not fall off with distance from the plane. You can see that its direction will be parallel to the plane but perpendicular to the velocity vector. There is much similarity here to the electrostatic case. There, too, it is the exact inverse-square character of the field of a point charge that results in the uniformity of the field of a uniform plane of charge. You see why the inverse-square nature of the magnetic field of a moving point charge seemed worthy of mention.

Let's carry the analogy with the electrostatic case further. There we related the strength of the uniform E-field between two oppositely charged parallel planes to the charge per unit area of the planes. With such a sandwich, the region containing the E-field was confined. Of course, if we have a sheet of positive charge, we have to specify the position of the equal negative charge it induces. Putting all that negative charge into one sheet corresponds to many practical situations. Similarly, with a sheet of current going one way, we have to specify where the return path completes the circuit. We again choose a parallel plane, a **current sheet** with equal and opposite current some distance away. Then the magnetic field will be confined to the region between the two current sheets. We state without proof the relation between the strength of the current and the strength of the uniform magnetic field inside the sandwich:

$$B = 4\pi k_m \left(\frac{\text{current}}{\text{length}}\right) \qquad \text{(Eq. 1)}$$

where

$$k_m = 10^{-7}\,\frac{\text{newtons}}{\text{amp}^2}$$

In some books you will see $4\pi k_m$ written μ_o and called the **permeability of free space.** Equation 1 may remind you of the corresponding one for the electrostatic field between two uniform planes of charge (Problem 13.15). It introduces a new universal constant, k_m (m for magnetic). The analogy might be made more striking by writing, instead of current per unit length for a current sheet, the velocity times the charge per unit area for a plane of charge:

$$B = 4\pi k_m \times v \times \left(\frac{\text{charge}}{\text{area}}\right)$$

Don't forget: The direction of **B** is perpendicular to **v**.

UNITS

You may wonder what historical accident made the proportionality constant k_m such a nice round number in these units. Its electrical analogue k_e was something like 8.9874×10^9 meters per farad. It all comes from the definition of the ampere. Remember, 1 coulomb is 1 ampere second. The ampere was defined in such a way as to make the magnetic-field formulas look pretty, without regard to the electric-field formulas. There is another set of electrical units called e.s.u. (electrostatic units), used by many physicists and by some textbooks. It leans the other way, favoring the simplicity of the electric-field formulas and putting messy constants in the magnetic ones. Because voltmeters and ammeters are calibrated in the "practical" units (volts and amps) rather than e.s.u., the former have won out in recent years, but the controversy over electrical units has been passionately waged for decades. The system of units used in this book is called the SI, the **Système International.**

It might be useful to point out again that the product of a B-field times a velocity is a force per unit charge. Dimensionally, that is an E-field. If we recall that

$$1\frac{\text{newton}}{\text{coulomb}} = 1\frac{\text{volt}}{\text{meter}}$$

for the practical E-field units, we note similarly that

$$1\frac{\text{newton}}{\text{amp meter}} = 1\frac{\text{volt sec}}{\text{meter}^2}$$

for the B-field units.

EXERCISE 7 Prove.

NOTE: This unit of magnetic field is called the *tesla*.

15.4 INDUCTANCE

How are uniform magnetic fields realized in practice? In the last section you read about infinite current sheets, parallel planes with antiparallel currents. That doesn't sound very practical. The practical answer is: a long coil.

Long Coil

You can see why that works by going back and thinking about the infinite sandwich. It has more symmetry than meets the eye. If you rotate the sandwich about the field direction, the direction parallel to the planes and perpendicular to the current, you don't change the field inside the sandwich. Folding those current sheets around into a cylinder, shrinking them arbitrarily—no pleats, please—does not change the field inside. The shape of the cross section does not matter; it just has to be uniform. Any long uniform cylindrical coil has the same *B*-field as any other with the same current per unit length (Fig. 15.6). If you were going to wind the coil, you would probably say current *times* turns per unit length.

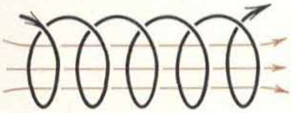


FIGURE 15.6 Lines of *B*-field in and near a coil.

Is the magnetic field really uniform? And is it really zero outside? Again, the answer is that the field falls off near the ends of the coil. This is similar to the behavior of the electric field in a parallel-plate sandwich near the edge of the sandwich. Well away from the ends, the magnetic field of a uniform coil is very uniform. As for field "leaking out," that is noticeable mostly near the ends.

We now prepare for the definition of a geometrical property of a coil that measures its ability to store magnetic energy. **Inductance** is something a big coil has more of than a little coil. Yet we said that current per unit length is all that matters to the *B*-field inside the coil. So before we define inductance, we should define the concept of magnetic flux. Let's write Eq. 15.3–1 in practical language, in terms of the number of turns of wire making up the coil:

$$B = 4\pi k_m \times (\text{no. of turns per unit length of coil}) \times (\text{current})$$

Now, what does a fat coil have more of than a thin coil? The product of the area enclosed by a turn of wire times the *B*-field inside is called the **magnetic flux** threading that turn. The idea is that flux is something that comes in one end of the coil and goes out the other, in the direction of the magnetic field:

$$\text{magnetic flux} = B \times \text{area}$$

So a coil of larger cross section has more flux passing through each turn. But that flux threads each turn. To get the flux threading the whole coil, we multiply by the number of turns:

$$\text{total flux through the coil} = (\text{no. of turns}) \times B \times \text{area}$$

Remember that the *B*-field in that expression is proportional to the current in the coil; therefore, the total flux is also proportional to the current. We call the proportionality constant the *inductance* of the coil, *L*:

$$\text{flux} = L \times \text{current} \qquad (\text{Definition of inductance } L)$$

Some people say self-inductance.

Exercise 8 shows that given the tightness of the winding, the inductance of a cylindrical coil of wire is proportional to the length of the cylinder and to its cross-sectional area, and hence to its volume. Exercise 9 shows that the inductance is also proportional to the square of the number of turns per unit length. A coil designer who needs large inductance will opt for the thinnest wire, that is, the largest number of turns per unit length, that will satisfy the current-carrying needs of the device he is designing without getting too warm.

EXERCISE 8

(a) A long coil with 5 turns per millimeter carrying a current of 4 amps has a magnetic field of 0.0251 T inside. Verify.

(b) If the radius of the coil is 5 cm, each turn threads a flux of 1.97×10^{-4} T m^2. Verify.

(c) Show that the flux unit, the tesla meter^2, could also be called the joule per amp, or the volt second.

NOTE: It has a name of its own, the **weber.** Hence the tesla is also called the weber per square meter.

(d) If the coil is 1 meter long, it will have 5000 turns and will thread a flux of 0.987 volt sec. Verify.

(e) Verify that the inductance of the coil is 0.197 volt sec/amp.

NOTE: This unit of inductance, the volt sec amp^{-1}, is also called the **henry.**

(f) Show that 1 henry = 1 ohm second = 1 joule/amp^2.

(g) Show that a coil twice as long would thread twice as much flux. What would its inductance be?

(h) Show that a coil having twice the cross-sectional area would thread twice as much flux.

(i) In what sense is it true that the inductance is proportional to the volume of the coil? That is, what has to be held constant?

EXERCISE 9 If that coil had 10 turns per millimeter instead of 5, the *B*-field inside would be doubled. But so would the number of turns for a 1-meter-long coil.

(a) What would its inductance be?

(b) In what sense is it true that the inductance of a coil is proportional to the square of the number of turns per unit length?

(c) In what sense is it true that the inductance is proportional to the square of the number of turns? Note that different quantities are held constant for the truth of the various proportionalities.

We said that the inductance of a coil describes how much magnetic energy is stored in the coil when a given current passes through it. You may prefer to say "stored in the magnetic field." We state without proof that the energy stored in an inductance is proportional to the square of the current:

$$\text{magnetic energy} = \tfrac{1}{2}LI^2 \qquad \text{(Eq. 1)}$$

Since the inductance of a coil is proportional to its volume and the *B*-field inside is proportional to the current, we can also say that the magnetic energy per unit volume is proportional to the square of the *B*-field (Problem 45):

$$\frac{\text{magnetic energy}}{\text{volume}} = \frac{\frac{1}{2}B^2}{4\pi k_m} \qquad \text{(Eq. 2)}$$

Those two formulas may remind you of electrostatics. The electrostatic energy stored in a capacitance C (Section 13.4) is proportional to the square of the voltage:

$$\text{electrostatic energy} = \tfrac{1}{2}CV^2$$

Also, the electrostatic field energy is proportional to the square of the E-field (Problem 49):

$$\frac{\text{electrostatic energy}}{\text{volume}} = \frac{\frac{1}{2}E^2}{4\pi k_e} \qquad \text{(Eq. 3)}$$

These formulas probably don't mean much to you at this point. But the conversion of electric to magnetic field energy, and vice versa, is the key to an understanding of electrical oscillations (Sec. 16.4) and thus, indirectly, of electromagnetic radiation (Sec. 16.5).

The word *inductance* is used to characterize any structure that stores magnetic energy when current passes through it. Accordingly, any conductor of electricity has inductance. In common usage people often say inductance instead of saying coil, although a better word might be *inductor*. An inductor is also called a *choke*. An analogous popular use of words occurs with *capacitance* and *capacitor*. The old-fashioned word for that is *condenser*.

Inductance as a Circuit Element

What happens if the current changes in an inductance? Since the energy stored in the inductance depends on the current, increasing or decreasing the current means getting energy into or out of the magnetic field. So an inductance tends to oppose current changes. This "inertia" is what makes coils useful circuit elements. Indeed, their magnetic energy is somewhat analogous to kinetic energy in mechanical systems. In other words, inductance in an electric circuit is analogous to mass in a mechanical system. The way in which inductance reacts to change in current is via induced e.m.f. Electromagnetic induction is the subject of the next chapter.

SUMMARY

How magnets behave.

Definition of magnetic field in terms of the force on a current element; the tesla.

Direction of the magnetic force; the right-hand rule.

Transverse force on a moving charge in a magnetic field.

Why $v \times B$ is like an electric field.

The curved path of a charge moving in a uniform magnetic field.

What if the field and the velocity are not perpendicular?

The magnetic moments of electrons and nuclei as sources of magnetic field.

Current as a source of magnetic field.

Magnetic field of a current sheet. Definition of k_m.

Magnetic field inside a coil.

Units of electromagnetic quantities. The SI.

Definition of magnetic flux.

Definition of self-inductance.

Inductance of a long coil.

Magnetic energy.

KEY TERMS

Biomagnetism Effect of magnetic fields on living things. Magnetic fields generated by living things.

Geomagnetism The science of the magnetic field of the earth. The study of what causes it.

Magnet Attracts things containing iron. Has two "poles" known as N and S. Like poles repel, while unlike poles attract. There are permanent magnets, and there is magnetization induced by an applied magnetic field.

Remanence Some induced magnetization is left over after the magnetic field is removed.

Magnetic compass The end of the compass needle marked N points north in the earth's magnetic field. A compass can be used to map any magnetic field.

Tesla (abbreviated T) Unit of magnetic field strength. In a field of 1 T, a transverse current of 1 amp experiences a force of 1 newton per meter.

Lorentz force Magnetic force on a moving charge, proportional to the velocity and to the component of the magnetic field normal to the velocity. Direction given by the right-hand rule.

Right-hand rule (a) If the thumb of the right hand points along the current and the forefinger points along the magnetic field, then the palm faces in the direction of the force. (b) If the thumb of the right hand points along the current, the fingers curl in the sense of the magnetic field that surrounds the current. (c) If the fingers of the right hand curl in the sense of the current in a coil, the thumb points along the magnetic field inside.

Commutator A switching device for changing the sense of a current twice (or 4 times or 6 times . . .) each revolution of a rotating machine. A simple commutator is a split ring (Fig. 15.7).

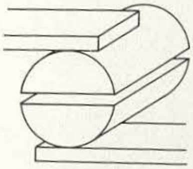


FIGURE 15.7 Commutator—showing split ring and brushes.

Chirality Handedness (right-handed? left-handed?)

Cyclotron orbit When viewed along the direction of an applied magnetic field, an electron moves in clockwise circles or spirals. The cyclotron frequency is $eB/2\pi m$.

Law of Biot and Savart A moving charge (or current element) is the source of a magnetic field whose direction is perpendicular to the velocity (current direction) and to the relative position vector **r**, in a sense given by the right-hand rule. Its strength is proportional to the velocity, and fall off with distance r according to a $1/r^2$ law.

Ampere's law The circulation of the B-field around a path threading a circuit is proportional to the current threaded. For the circular B-lines surrounding a straight wire carrying current I, the circulation along a circular B-line of radius r is $B \times 2\pi r = 4\pi k_m I$.

Current sheets A uniform sheet of current gives rise to a uniform magnetic field of strength given by

$$B = 2\pi k_m \times \text{current per unit width of sheet}$$

with $k_m = 10^{-7}$ henry/meter.

Permeability of free space The constant $\mu_o = 4\pi k_m = 4\pi \times 10^{-7}$ henry/meter $= 4\pi \times 10^{-7}$ newton/amp^2.

Permeability A measure of how easily a material is magnetized when subjected to an external magnetic field. For example, iron has a permeability around 1000. This means that an electromagnet containing iron acquires a magnetic field 1000 times as strong as the same electromagnet empty or filled with air.

Magnetic monopole A free "N" pole or "S" pole. A point where B-lines begin or end. No such entity has ever been observed.

Magnetic dipole An elementary magnet. No matter how small, a magnet has both an "N" and an "S" pole, so the B-lines are continuous.

Magnetic moment of the electron Also called the *Bohr magneton* $\mu_B = 9.27 \times 10^{-24}$ joules/tesla.

Nuclear magneton A unit of magnetic moment given by $\mu_N = (m_{\text{electron}}/m_{\text{proton}})\mu_B$. The magnetic moment of the proton is 2.8 μ_N.

S.I. Système International. System of units adopted by international agreements, based on the MKS system. The unit of current is the ampere. The unit of electric field is the volt per meter. The unit of magnetic field is the tesla.

Gaussian units System of units in which the electric and magnetic field have the same dimensions. For example, in Gaussian c.g.s., electric field is measured in statvolts per centimeter (= statcoulombs per cm^2) and magnetic field is measured in gauss. In Gaussian units, the formula for the magnetic force (Lorentz force, Eq. 15.2–1) on a moving charge has to be written $F = qB\, v/c$.

Magnetic flux Through an element of area ΔA, the flux of magnetic field is the product $B_\perp \Delta A$ of the component of B-field normal to that area times the area. For an extended area over which the magnetic field is not uniform, divide it up into small area elements and add the individual contributions.

Weber Unit of magnetic flux. 1 weber = 1 volt second = 1 tesla m^2.

Gauss's law The total magnetic flux through a closed surface is always zero. It follows that the flux threading a closed curve is independent of which surface bounded by that curve is chosen to define it. (Not discussed in the text.)

Inductance Or self-inductance. A measure of the ability of a structure to contain magnetic energy when it passes current. The ratio of the total magnetic flux linked by a circuit to the current in that circuit producing that flux. For a long coil of length L and cross-sectional area A uniformly wound with N turns of wire, the inductance is given by $L = 4\pi k_m N^2 A/L$, where $k_m = 10^{-7}$ henry/meter.

Henry Unit of inductance. An inductance of 1 henry links a flux of 1 volt second when it passes a current of 1 ampere.

Choke A coil used in a circuit to oppose changes in current. In alternating-current circuits, a choke (inductance) has an "inductive" impedance. (Not discussed in the text.)

NMR Nuclear magnetic resonance. When subjected to an external magnetic field, the magnetic moment of a nucleus exhibits a characteristic frequency at which it can absorb and re-emit electromagnetic radiation. NMR spectroscopy is a powerful technique in chemical analysis. See Problem 34.

ESR Electron spin resonance, or

EPR Electron paramagnetic resonance (same thing). In the previous entry, substitute the word *electron* for *nucleus*. Of course, ESR frequencies are several thousand times greater than NMR frequencies for the same magnetic field.

REVIEW QUESTIONS

- What is meant by a *vector field?*
- What is the relation between the direction of the magnetic field and the direction of the magnetic force on a current?
- What is the relation between the direction of the magnetic field and the direction of the magnetic force on a moving charge?
- If the *B*-field is reversed and the sign of the moving charge is also reversed, what happens to the force?
- An electron moving horizontally in a *B*-field directed downward is deflected to the right. How about a proton?
- How is an electron affected by a magnetic field directed along its direction of motion?
- Electric field lines begin and end on electric charges. Why don't magnetic field lines ever begin or end?
- The magnetism of iron is due primarily to the magnetic moment of the electron / iron nucleus.
- As the distance r from source to field point is increased, the magnetic field of a moving charge falls off as $1/r^2$. The magnetic field of a straight beam of moving charges falls off as $1/r$. How does the magnetic field of a large sheet of moving charge fall off with distance?
- The SI unit of E-field is the volt per meter. The SI unit of B-field is the tesla (= newton per amp per meter). What is the unit of E/B?
- A long coil is made by wrapping wire around a cardboard cylinder. If twice as much wire is wrapped around the same cylinder, the inductance of the coil will be: doubled/quadrupled

- If a fatter cylinder is used instead, so the length of wire wrapped is doubled but the number of turns stays the same, what happens to the inductance? (HINT: Don't let this one trick you. What happens to the area of a circle if the circumference is doubled?)

☐ LAB AT YOUR DESK 1. Magnetic Compass

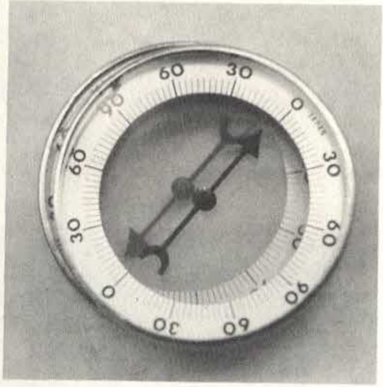
Do you have a rough idea which way is north? Check that your compass points where you think it should. Many compass needles have a dull end and a shiny end. Which one is "N"? As you move around the room, does the direction indicated by the needle change? Can you ascribe the changes to iron components in the furniture, floor or walls?

NOTE: Old buildings are often wood, stone, or brick. Tall modern buildings usually have steel skeletons.

If your compass needle is mounted so you can turn it on its side, you can use it as a "dip needle." How far off from horizontal is the earth's magnetic field in your locality?

☐ LAB AT YOUR DESK 2. Two Compass Needles

Observe the interaction as they are brought close. (a) Approach one compass to the other along a north-south line. (b) Now try an east-west line. (c) Lower one compass over the other. Explain your observations in terms of what you know about magnets.



Two compass needles interacting.

☐ LAB AT YOUR DESK 3. Compass and Bulletin-Board Magnet

(a) Use the compass to decide which face of the bulletin-board magnet should be labeled "N." (b) Approach the compass to the ceramic magnet from various directions, in order to map the magnetic field of the ceramic magnet. (c) Many of these ceramic magnets have a hole, but it is too small to get a compass inside. Without doing the experiment, could you decide which direction the field has inside the hole? (d) Invent a way to make a compass out of a bulletin-board magnet.

☐ LAB AT YOUR DESK 4. Bulletin-Board Magnets

They come in ring and disk shapes. Get hold of a couple of them and start experimenting. The first thing you notice is that they exert forces on each other. (a) Try turning one of them over. How does that affect the force it exerts on the other? (b) Try stacking them, one above the other. Does it matter if you turn one of them over? (c) Does it matter if you rotate one of them in its own plane? (d) Argue cogently that the N-S axis of these magnets is perpendicular to their plane. (e) Suppose somebody labeled the faces of one of your magnets N and S, respectively. How could you label the faces of your other magnet consistently? (f) If the "somebody" had labeled the magnet backward, argue that there is no experiment you could do with just the two magnets to discover the error. That is, argue that it requires one "correctly" labeled magnet to start the labeling of others. This shows that the choice of which is N and which is S is just a convention. (g) Argue that according to the accepted convention, the north magnetic pole of the earth would be labeled "S."

☐ LAB AT YOUR DESK 5. What Will a Magnet Pick Up?

(a) Experimenting with a strong magnet is more fun, but you can certainly develop your intuition with just one of those little bulletin-board magnets. With a strong magnet, there are essentially three different kinds of behavior shown by different materials:

Strongly attracted—ferromagnetic
Weakly attracted—paramagnetic
Weakly repelled—diamagnetic

If you take your objects out of a toolbox, you will find most of them ferromagnetic. Most tools contain iron, which is ferromagnetic. Rust is not. How about wire? Copper is paramagnetic, and unless you have a very strong magnet, you probably won't notice any attraction. The same goes for aluminum. When you move to the kitchen, most of the good edibles are diamagnetic, but again, you probably won't be able to notice the very feeble repulsion.

(b) Explanation: In paramagnetic materials, each individual atom is a magnet. The applied magnetic field tends to align the atoms parallel to itself. The parallel magnetic moments—that of the inducing magnet and that of the induced magnet—attract each other. In diamagnetic materials the atoms have no magnetic moment before the magnetic field is applied. The magnetism of the material is induced by the field; the induced magnetic moment is opposite to the applied field. So the inducing magnet and the induced magnet repel each other. In ferromagnetic materials, the atoms have magnetic moments and interact magnetically so that these moments tend to align throughout a sizable region, called a *magnetic domain*. These domains persist also in the absence of applied field. Then the domains are randomly oriented, so that the whole object may have no net magnetic moment. The application of a magnetic field tends to align the domains. Actually, the domains parallel to the field tend to grow, while the others shrink.

(c) If you have read in Chapter 9 about order, disorder, and entropy, you are in a good position to answer the following question: Why does a ferromagnetic material at finite (i.e., nonzero) temperature tend to lose its magnetization when the applied magnetic field is removed? Actually, different materials differ in their *remanence*, their ability to remember their magnetization for a long time. The bulletin-board magnets "remember" for years. An iron nail "forgets" in a fraction of a second.

PROBLEMS

✶ **1. Force on Current** What is the electromagnetic force on 1 meter of horizontal wire carrying a current of 1 amp in a vertical magnetic field of 0.75×10^{-4} tesla (= 0.75 gauss)?

✶✶ **2. Force on Current in the Earth's Field** In the northern United States, the direction of the earth's field is primarily vertically down. The angle the field vector makes with the horizontal, called the dip angle, is generally 70° or more. In a location where the field strength is 0.8×10^{-4} tesla (= 0.8 gauss), the vertical component of the field is 0.75 gauss. To find the force on a horizontal current, all that counts is the vertical component of the magnetic field.

(a) What is the sideways force on 1 meter of horizontal wire carrying a current of 1 amp?

(b) If the current flows northward, in what direction is the force?

(c) If the current flows eastward, in what direction is the force? (HINT: If the current flows eastward, the force is not entirely horizontal, as it was for the northward current.) Is the vertical component of the force up or down? Which way does the compass point? Are you surprised that the northward component of the earth's field is only about one-third of the field strength? cos 70° = ?

∗ 3. **Power-Line Hum** The earth's magnetic field points almost straight down if you live pretty far north. The author lives in northern Ohio, where it makes an angle of 78° with the horizontal. Here its strength is about 0.8×10^{-4} tesla.

(a) If a horizontal power line carries 50 amperes toward the north, what is the magnetic force on each meter of wire? Direction? Compare with the force due to a brisk east wind.

(b) If the current is alternating (ac) at 60 Hz, would you expect the wire to vibrate and emit a 60-Hz musical tone? Have you ever heard such a hum near a power line?

∗ 4. **Electron Beam in the Earth's B-Field** The TV screen is facing north. As you rotate the cabinet clockwise so the screen faces east, which way should the picture move on the screen? Is it the vertical component or the horizontal (northward) component of the magnetic field that gives rise to the deflection?

NOTE: TV manufacturers put magnetic shielding around the picture tube.

∗∗ 5. **Cosmic Rays** The earth's magnetic field is directed north at the equator, north and down in the Northern Hemisphere, north and up in the Southern Hemisphere.

(a) Which way is a cosmic-ray proton deflected if it approaches the earth moving vertically down at the equator?

(b) Would you expect more cosmic-ray protons to hit the earth near the equator or near the poles? Explain. What you are predicting is indeed observed and is called the **latitude effect.** Charged particles trapped several hundred miles above the earth's surface by the earth's magnetic field constitute the *Van Allen radiation belts.* They constitute a hazard to the electronics in our artificial satellites.

∗ 6. **Electron in a Uniform Magnetic Field** An electron moving at 5×10^6 m/sec at right angles to a uniform magnetic field of 0.001 T experiences a force of 8×10^{-16} newtons.

(a) Verify.

(b) Calculate its acceleration. (HINT: electron mass = 0.911×10^{-30} kg.)

(c) Set this equal to v^2/R to find the radius of the circle it describes.

(d) In going once around the circle it goes a distance $2\pi R$. How long does that take? In other words, what is the period of the motion?

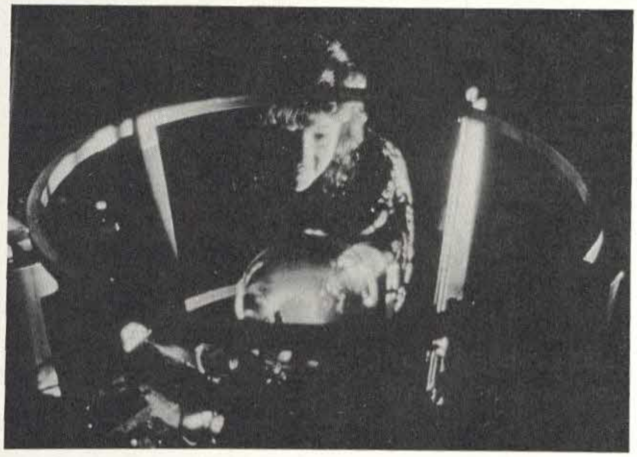
(e) What would be the period if the electron traveled twice as fast? (HINT: It is the *B*-field that is fixed, not the radius of the trajectory.)

∗∗ 7. **Electron in a Uniform Magnetic Field**

(a) How long does it take a 10-keV electron to travel once around its circular path in a uniform *B*-field of 0.0001 T?

(b) If the electron has a component of velocity parallel to the *B*-field, instead of a circle, its path will be a spiral. Suppose that the 10-keV electron is moving in a direction making a 2° angle with the magnetic-field direction. How far along the field direction does it move in one twist of the spiral? That distance is called the *pitch* of the spiral. (See photo below.)

∗∗ 8. **Ions in a Circular Path** An ion accelerator is designed so that protons traveling at 5×10^6 m/sec in the uniform magnetic field are kept in a circle 1.5 meters in radius.



The Bainbridge-Winch experiment shows electrons traveling in a circular path. The electron beam (center of photo) is made visible by a very low-pressure gas in the glass tube. The magnetic field that makes the beam curve is produced by current in the two parallel coils. In this photograph those are over one meter in diameter, in parallel (almost horizontal) planes one radius apart. (See also Problem 42, "Helmholtz coils.")

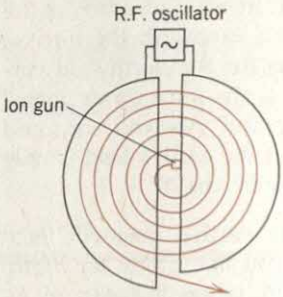
(a) Deuterons have the same charge but twice the mass. In order to travel in the same circle as the protons, deuterons would have to have ______ times the energy.
(b) Their frequency (= number of revolutions per second) would be ______ that of the protons.
(c) What data did you not need to answer these questions?

✱✱ **9. Ions in a Circular Path** More about the accelerator of Problem 8.
(a) How does the frequency of revolution depend on the energy?
(b) If you wanted the deuterons to go around at the same frequency as the protons, what would you have to do to the strength of the magnetic field?

✱✱✱ **10. Mass Spectrograph** The curvature of an ion beam moving in a magnetic field is inversely proportional to the momentum of each ion. If the ions have been accelerated in an electric ion gun, they all have the same kinetic energy. Even ions of different mass (but the same charge q) will have the same kinetic energy qV after having been accelerated through a potential difference V. In a uniform B-field the ion beam will break up into separate beams for each value of the mass m. A movable slit on the ion detector can sample the separate beams.
(a) The acceleration of a particle moving perpendicularly to a B-field at speed v in a circle of radius R is v^2/R. Set the Lorentz force qvB equal to the product of mass times acceleration mv^2/R to show how R depends on the momentum.
(b) Letting $qV = \frac{1}{2}mv^2$ = kinetic energy, show how R depends on m for given qV.
(c) With a 100-volt ion gun and a uniform magnetic field of 1.0×10^{-2} T, how far apart are the slit positions for $^4He^{2+}$ and $^3He^{2+}$? In other words, how much do the diameters $2R$ differ for the two ions? (HINT: 6.02×10^{23} helium-4 ions weigh 4 grams.)

✱✱ **11. Cyclotron** With a 1000-volt e.m.f., a proton can be accelerated to an energy of 1000 eV. The cyclotron is a device that permits a proton to "fall through" the voltage drop again and again and be accelerated to a multiple of the 1000-eV kinetic energy. It does this by using a magnetic field to bend the path of the protons to be almost circular. The voltage is varied sinusoidally in time at the same frequency as the revolution frequency of the protons.

The "dees" (note the D-shape in the figure) are the two semicircular halves of a cylindrical metal can, with a small gap between them. Acceleration takes place only while the protons are crossing the gap. When they are well within one of the dees, they are well shielded from electric fields. If a proton enters the gap while the full electric field is directed from no.1 to no.2, it gains 1000 eV in crossing the gap. When it has traveled half a period, it is starting to cross the gap the other way, and gets another 1000-eV kick, because the voltage has reversed by this time. Every complete cycle it gains 2000 eV. Of course, the "dees" have to be well evacuated, so the protons are not slowed by collisions with air molecules.



PROBLEM 11 Cyclotron. For positive ions, the magnetic field is directed out of the paper. The oscillator is tuned to the "cyclotron frequency" of the ion, so the electric field in the gap between the hollow metal "dees" reverses in the time taken by the ion to traverse the semicircle between gap crossings.

Set the magnetic deflecting force evB equal to the proton mass times the acceleration v^2/R for a circular path. Then the frequency $v/2\pi R$ is given by $eB/2\pi m$.
(a) Do the algebra to prove this.
(b) Find the oscillator frequency the cyclotron would use if the uniform B-field is 0.08 tesla.
(c) Evidently, the proton's path is not exactly circular. Every time it crosses the gap, it gets a kick and goes a bit faster. It is really a sort of growing spiral. Show that the relation between the radius R and the kinetic energy is

$$\frac{1}{2}mv^2 = \frac{\frac{1}{2}e^2B^2R^2}{m}$$

(d) How big a cyclotron (R = ?) is needed with B = 0.08 T to have a proton kinetic energy of 80×10^3 eV?
(e) Show that this requires 40 revolutions with our 1000-volt oscillator amplitude. Why not 80?

✱✱ **12. Cyclotron with Alpha Particles**
(a) If you substitute helium nuclei He^{2+} for the protons in the cyclotron (see Problem 11), and want to use the

same oscillator frequency, you would need a bigger / a smaller / the same B-field. Remember, the charge is double that of the proton, and the mass is quadruple the proton mass.

(b) For the same size (R) cyclotron, what energy He^{2+} would you obtain?

✶✶ **13. Hall Effect**

(a) An electron moving north at 0.2 m/sec in a magnetic field of 0.4 tesla directed down experiences a Lorentz force of ________ newtons directed east / west.

(b) The electron is in a solid slab of material in which a constant drift speed of 0.2 m/sec requires an electric field of 250 volts/meter, supplied by a battery connected across the ends of the slab. The transverse magnetic force causes the western edge of the slab to charge up positive, the eastern edge negative. OK? This charging is called the Hall effect. Show that the transverse electric field associated with this edge charge is 0.08 volts/m. This field is called the Hall field.

(c) A north-south field of 250 V/m added to a west-east field of 0.08 V/m defines a direction (the vector sum) just a small angle θ east of south. That angle θ, defined such that $\tan \theta = 0.08/250$, is called the Hall angle. Repeat:

$$\tan \theta = \frac{\text{Hall field}}{\text{applied electric field}}$$

Suppose that the smallest measurable Hall angle is 10^{-3} radians. What is the smallest B-field that would show the Hall effect with this sample and this battery?

(d) The sample is 5 mm long. What is the battery voltage?

(e) The sample is 1 mm wide? What is the Hall voltage?

✶✶ **14. Hall Mobility** Measuring Hall angle (see Problem 13) is a lot easier than measuring drift velocity. The quotient of drift velocity divided by longitudinal electric field is called a mobility:

$$\mu = \frac{v_{\text{drift}}}{E_{\text{applied}}}$$

In most materials the mobility is constant over a large range of applied E-fields. That is just another way of saying they obey Ohm's Law.

(a) Use the Lorentz force formula in the form $E_{\text{Hall}} = v_{\text{drift}}B$ to show that

$$\tan \theta = \frac{E_{\text{Hall}}}{E_{\text{applied}}} = \mu B$$

(b) For the data of Problem 13 find the mobility μ and express it in m^2/volt sec.

(c) If a 2-T magnet is available, what is the smallest mobility that will still give a Hall angle greater than 10^{-3} radians?

NOTE: The strongest magnetic field a conventional iron electromagnet can produce is not quite 3 tesla. With that field, all the magnetic domains in the iron are aligned: the iron is saturated.

✶ **15. Direction of the Magnetic Field of a Current** A horizontal wire carries current toward the west.

(a) What is the direction of the resulting B-field below the wire?

[Answer: South]

(b) Above the wire?

(c) North of the wire?

(d) South of the wire?

✶ **16. Circular Current** A horizontal ring carries current clockwise, as seen from above. What is the direction of the resulting B-field at its center?

✶ **17. Two Dipoles**

(a) Two horizontal rings, one above the other, both carry clockwise currents. Do they attract or repel? Explain.

(b) Two horizontal rings in the same plane both carry clockwise currents. Do they attract or repel? Explain.

NOTE: There is also a torque. If you experiment with bulletin-board magnets the torques are very obvious.

✶ **18. Two Parallel Currents** Two straight wires in the same horizontal plane carry current toward the north. The magnetic field due to the western one at a point on the eastern one is directed vertically down.

(a) Use the right-hand rule to check that.

(b) In what direction is the force exerted by the western one on the eastern one?

(c) In what direction is the force exerted by the eastern one on the western one?

(d) Is the law of action and reaction (Chapter 2) obeyed here?

✶✶ **19. Two Perpendicular Currents** Two positive charges are on a collision course: A is moving north; B is moving west.

(a) Use the right-hand rule to show that the magnetic force on B due to A is directed south.

(b) In what direction is the magnetic force on A due to B?

(c) Do these two forces obey the law of action and reaction?

NOTE: At this point you might be worried about the conservation of momentum. How can that be violated? The resolution of the apparent paradox is that the fields themselves carry momentum! This is not a simple story.

✶ **20. Force Between Parallel Currents**

(a) Use the formula $B = (2 \times 10^{-7}\ N/A^2)i/d$ to calculate the magnetic field at a distance $d = 10$ cm from a wire carrying a current $i = 25$ amps.

(b) Calculate the force on 1 meter of another wire carrying 25 amps in the same direction at a distance of 10 cm from the first.

(c) Argue that the force is attractive. (NOTE: If the currents were antiparallel, it would be repulsive.)

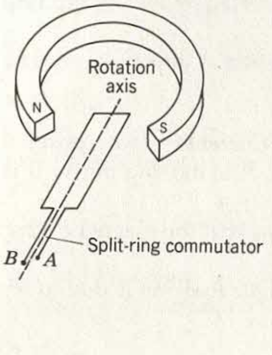
✶ **21. DC Motor** The sketch shows a simplified "electric motor."

(a) If current flows from A to B, argue that the torque on the wire loop would be such as to tend to turn the rotor clockwise.

(b) When it has turned one-quarter turn, there is no more torque. Prove in one sentence.

(c) If it turns past this (vertical) position, the way to get a clockwise torque is to have the current flow from B to A. The split-ring commutator of Fig. 15.2 shows a simple device for reversing the current direction every half-turn. Argue that having four or six wire loops in different planes and split rings with 8 or 12 sections would give more uniform torque. Draw a sample configuration.

(d) Instead of a permanent magnet, suppose that the source of the magnetic field were an electromagnet: an iron core wound with a "field coil." If the current source for the field coil is the same as that for the rotating coil (called the "armature"), argue that the motor would run also if the current were alternating (ac) instead of direct (dc).



PROBLEM 21 A dc motor.

✶✶ **22. Synchronous Motor** Kitchen clocks and phonographs have electric motors whose speed is exactly synchronized with the frequency of the alternating line voltage. In the United States that is exactly 60 periods per second (60 Hz).

(a) Use a diagram like that of Fig. 15.2 to show that the torque on the rotating coil (the armature) will be clockwise with the current having the sense shown (in on the right; out on the left).

(b) If that torque is allowed to turn the rotor one-quarter turn, argue that there will be no torque, regardless of the current.

(c) As the rotor turns past this neutral position, there will be clockwise torque on it only if the sense of the current is reversed. Argue that the rotor will tend to speed up if the frequency of the alternating current is slightly greater than the rotation frequency.

(d) Show that the rotor frequency will tend to synchronize with the frequency of the alternating current.

✶✶ **23. Magnetic Fields in a Storm** Lightning has large currents, and their magnetic fields are measurable. But they don't last very long. How about the motion of the clouds? Clouds usually carry electric charge. Is their magnetic field detectable? Suppose that a cloud covers an area of $10^4 \mathrm{m}^2$ and carries a charge of 10^2 coulombs distributed more or less uniformly. If the wind blows it at 2 m/sec, what magnetic field appears below the cloud? (HINT: This is only half a sandwich, so the 4π of Eq. 15.3-1 becomes 2π.) Are you disappointed that such a large charge makes such a small B-field? The speed of the wind is just very slow compared with the speed of ions in an electric discharge.

✶✶ **24. Magnetic Moment of the Electron** The magnetic dipole moment of an electron is $\mu_B = 9.27 \times 10^{-24}$ joules/tesla, called 1 Bohr magneton.

(a) The energy change in "flipping" an electron aligned *against* a B-field to alignment *with* the field is $2\mu_B B$. Calculate the energy change when an electron is flipped in a field of 10^{-2} T.

(b) Come back to this problem after reading Chapter 17 and learning about Planck's constant h ($= 6.626 \times 10^{-34}$ joule sec). The formula for the energy change $h\nu = 2\mu_B B$ can then be interpreted as a formula for the frequency of an electromagnetic radiation associated with the spin flip, $\nu = 2\ \mu_B B/h$. Put in the numbers and compare with Problem 34 ESR, NMR.

(c) Close up, the magnetic field of an electron can be considerable. The formula for the strength of the B-field a dis-

tance r away from a magnetic moment μ is $2k_m\mu/r^3$. The atoms of copper, for example, are about 2.5 Å($= 2.5 \times 10^{-10}$ m) apart. Calculate the B-field that far away from an electron.
(d) Have you ever seen an inverse-cube law before? For an inverse-square law (familiar!), when you double the distance r, the field is down to one-quarter of its former value. For an inverse-cube law, doubling r means the field is down to ________ of its former value. Now you can guess why one of those bulletin-board magnets only picks up thumb tacks when they are very close but then holds them very tightly.

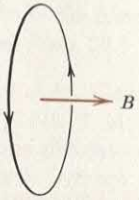
NOTE: The force $\propto 1/r^6$.

✶ **25. Magnetic Moment of a Circular Current** A ring of current I has a magnetic moment given by $\mu = I \times$ area of the ring.
(a) Use the formula $B = 2k_m\mu/r^3$ (Problem 24) to find the magnetic field on the axis of the ring 1 cm in radius carrying 1 ampere if the field point is 1 meter from the center of the ring.
(b) How many aligned electrons would have a total magnetic moment equal to that of the ring. (See Problem 24, Magnetic moment of the electron.)

NOTE: The $1/r^3$ formula is an approximation valid for r much greater than the diameter of the ring.

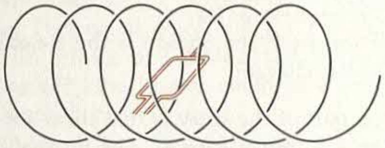
Field at the Center of a Circular Loop The formula is $2\pi k_m i/R$ for the B-field at the center of a loop of radius R carrying current i (see figure). Suppose that you have a loop 30 cm in radius.
(c) How much current is needed in order to cancel out the earth's field at the center if its strength is 0.6×10^{-4} tesla?
(d) Describe how the loop has to be positioned.

✶ **26. Magnetic Field of a Long Straight Wire** At a point a distance d from a wire carrying current i, the field strength is $B = 2k_m i/d$. An automobile battery might deliver 50 amps to the starting motor.
(a) Calculate the B-field 5 cm (about 2 inches) from the lead wire.
(b) Is an ordinary compass needle sensitive enough to detect this field?
(c) Some cars have a magnetic compass mounted near the steering wheel. Argue that you can/cannot believe the reading of such a compass while the starting motor is running.
(d) The dome light uses 2 amps or less. But those lead wires pass closer to the dashboard. Can turning the dome light on perturb the reading of the magnetic compass? (HINT: The horizontal, that is, northerly component of the earth's magnetic field is of the order of 2×10^{-5} newtons per amp per meter.)



PROBLEM 25

✶ **27. Magnetic Fields Inside an Electric Motor** Equation 15.3–1 tells us the field strength inside a current-carrying coil. A 1-amp current in a coil with 100 turns per centimeter ($= 10^4$ turns/m) gives rise to a field strength just over 10^{-2} newtons per amp per meter.
(a) Verify
(b) If the wire—including the insulating varnish—is 0.5 mm thick, how many layers of wire are wound on this coil?
(c) Suppose that the rotor is 5 cm long, has 100 turns of wire, and also carries 1 amp. Find the force on that bundle of wires in a B-field of 10^{-2} N/A m. Assume that the wires are perpendicular to the direction of the field.

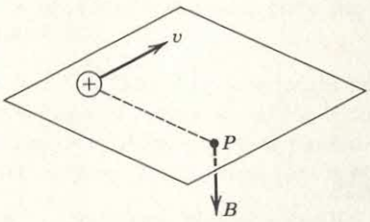


PROBLEM 27

✶✶ **28. Law of Biot and Savart** Here is part of the "long and complicated story" of the inverse-square law for the magnetic field of a moving charge. If the observer is located a distance r from the charge at right angles to its direction of motion, the magnetic field he or she measures is

$$B = k_m \frac{qv}{r^2}$$

where v is the speed of the charge (see the figure). The direction of this B-field is perpendicular to the velocity **v** and to the vector **r** connecting the charge to the field point. Its sense is that of advance of a right-hand screw turning from **v** to **r**.
(a) Find the magnitude of the B-field 1 meter off to the right from an electron moving at 10^6 m/sec.
[Answer: 1.6×10^{-20} newton per amp per meter.]



PROBLEM 28 Law of Biot and Savart. (Remember, the final "t" is silent in French.) The direction of the *B*-field of a moving charge is perpendicular to the velocity and perpendicular to the vector connecting the position of the charge (source point) to the field point. Sense given by right-hand rule.

(b) If the observer is so located that the radius vector **r** is not at right angles to the charge's velocity vector **v**, only the normal component of the velocity contributes. The formula becomes

$$B = k_m \frac{qv \sin \theta}{r^2}$$

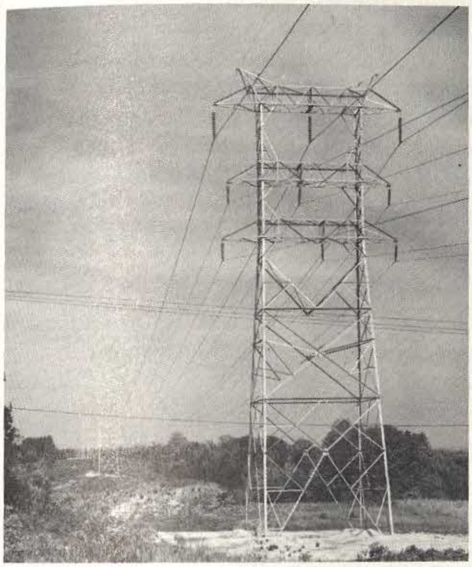
where θ is the angle the radius vector makes with the velocity vector. Show that the magnetic field vanishes directly in front of the charge, near or far.
(c) In what other direction from the charge is the *B*-field zero, even very close to the charge?

NOTE: The complicated part of the story is that these formulas are only a low-speed approximation. The magnetic field depends not only on where the charge is, but also on where it was a short time ago. Indeed, the field of an *accelerated charge* has a component that falls off as $1/r^2$ and a component that falls off as $1/r$ for large r. The latter is the important part of the *radiation field* and explains the propagation of radio waves, light waves, and X-rays, which are discussed at the end of Chapter 16.

✶✶ **29. Biot and Savart and a Speed Limit** One meter off to the right of that electron moving at 10^6 m/sec is another electron also moving at the same speed in the same direction.
(a) Use the formula for the force on a moving charge in a magnetic field to find the magnetic force on this electron. If the electrons are moving north, in what direction is the force?
(b) Compare this magnetic force with the Coulomb repulsion. What is the ratio of the two forces?
(c) How fast would you expect the two electrons have to be moving in order for the electric and magnetic forces to be equal? Note that your calculation involved the ratio k_e/k_m (see Sec. 13.5) and did not involve the 1-meter separation at all.
(d) If you were traveling along with the electrons, you would observe only an electrostatic repulsion. Since there would be no observed motion, there would be no magnetic attraction. It may not surprise you that an observer in the laboratory frame-of-reference measures a different force. But it would really grate on your intuition if what you observe as a repulsion—electrons getting futher apart in time—the laboratory observer were to observe as an attraction—electrons coming closer in time. Formulate this "intuition" as a law limiting the speed of electrons.
[Answer: The universal speed limit is 3×10^8 m/sec. You will read all about that in Chapter 20, Relativity]

✶✶ **30. Ampere's Law** The magnetic field of a long straight wire carrying a current I, measured at a point a distance d from the wire is given by the formula $B = 2k_m I/d$.
(a) Find the magnetic field 1 meter away from a 1-ampere current.
[Answer: 2×10^{-7} T]
(b) The *B*-lines are circles surrounding the wire. Prove that the circulation of the *B*-field once around a *B*-line is independent of which *B*-line you choose. [HINT: Circulation $= B \times 2\pi d$ = (strength of the *B*-field) × (length of the *B*-line).]
(c) A certain electron beam consists of 10^{10} electrons per millimeter moving at 2×10^6 m/sec. What is the magnetic field 1 meter away?
(d) Argue that the magnetic field 2 meters away is exactly half as strong.
(e) The Law of Biot and Savart (see Problem 28) is an inverse-square law. Each electron of the beam contributes a *B*-field that falls off as the inverse *square* of its distance r from the field point. Contrast the contribution of an electron 1 m from the field point (the closest electron) with its contribution 1 microsecond later. (HINT: Remember the $\sin \theta$ factor in the $B + S$ Law.)
(f) Repeat this calculation with the field point 2 meters from the beam.
✶✶✶(g) How is it possible that the field due to all the electrons of the beam falls off as $1/d$ when the Law of $B + S$ has a $1/r^2$ fall-off? (HINT: Distinguish the r in $B + S$ from the d in the formula above and don't forget the $\sin \theta$ in $B + S$.)

✶ **31. Under a Power Line** The popular press has begun to worry about the magnetic fields near those high-



High-voltage power line.

voltage electric lines that carry electric power from a generating plant to a distant city. Might they have harmful biological effects? Voltages around half a million volts are not uncommon. If tens of megawatts of power are to be transmitted, we are talking about currents of tens of amperes.
(a) Calculate the magnetic field strength 25 meters below a wire carrying 10 amperes. (HINT: $B = 2k_m I/d$.)
(b) Compare with the strength of the earth's field. Remember that those power-line fields oscillate at 60 Hz, so it is difficult to assert that their effect must be negligible simply because they are so much weaker than the constant field we live with all the time. This is still a controversial subject!

★★★ **32. Gaussian Units** The SI unit of magnetic field is the *tesla*. In the so-called Gaussian units, the unit of B-field is the gauss ($= 10^{-4}$ tesla) and the unit of charge is the statcoulomb. With q in statcoulombs, Coulomb's Law can be written $F_{es} = q_1 q_2/r^2$; that is, the constant k_e is equal to 1. This is correct with F in dynes and r in centimeters. In SI units, the formula for the force on a charge moving across a B-field is $F = qvB$. If you want to rewrite that formula in Gaussian units, show that it would take the form

$$F = \left(\frac{1}{c}\right) qvB$$

where the constant c is 3×10^{10} cm/sec. (HINT: $(k_e/k_m) = c$. See Prob. 29.)

★★ **33. Gaussian Units** In the e.s.u. system, the units of E and B, although they have different names, have the same dimensions. If you come across a book or a journal article written in e.s.u., the formula for the force on a moving charge will contain a proportionality constant c having the dimensions of a velocity. Our Eq. 15.2–1 will be written

$$F = q\,\frac{v}{c}\,B \qquad \text{(Eq. 15.2–1-esu)}$$

That velocity c, which is the conversion factor between the two sets of units, has an important interpretation. It turns out to be a universal speed limit, a concept central to the Special Theory of Relativity.

Going back to electrostatics, Coulomb's Law is written

$$F = k_e\,\frac{q_1 q_2}{r^2}$$

With charge measured in coulombs, the constant k_e is measured to be 8.9874×10^9 newton m^2/coulomb^2. With charge measured in statcoulombs (= e.s.u.), k_e is unity by definition, provided that F is in dynes and r in centimeters. That is, $k_e = 1$ dyne cm^2/statcoulomb^2.
(a) Two equal point charges 1 cm apart repel each other with a force of 1 dyne. How many statcoulombs in each charge? How many coulombs? (HINT: 1 newton $= 10^5$ dynes, 1 m $= 10^2$ cm.)
(b) Show that the ratio of the coulomb to the statcoulomb is 2.9979×10^9 according to this measurement.
(c) To use Eq. 15.2–1, $F = qvB$, the B-field should be expressed in tesla, the charge in coulombs, and F and v in MKS units. To use Eq. 15.2–1-esu, the B-field is expressed in gauss ($= 10^{-4}$ tesla), the charge in statcoulombs, and F and v in cgs units. Using the results of part (b), show that the speed c comes out 2.9979×10^{10} cm/sec.

You might be worried about a quantity like B-field having different *dimensions* depending on the choice of units. This unusual situation arises because the *definition* of B-field is actually different in the two sets of units. Compare Eqs. 15.2–1 and 15.2–1-esu.

★★ **34. E.S.R., N.M.R.** Like a spinning top, an electron has an intrinsic angular momentum, called *electron spin*. By virtue of its magnetic dipole moment, an electron in a magnetic field experiences a torque. Like a spinning top, it precesses: Its angular momentum turns around the field direction. The stronger the field, the faster the

Patient inside an MRI (magnetic resonance imaging) machine that shows the image of internal organs on the computer screen. The second screen permits studying the images made earlier.

precession. In fact, the precession frequency is proportional to the field strength B:

$$\nu = \left(28.0 \times 10^9 \frac{\text{Hz}}{\text{T}}\right) \times B$$

If the electron is bathed in an electromagnetic field oscillating at that frequency, it can absorb energy from the alternating field and get rid of it into its surroundings. Most effectively, it emits this energy in the form of sound (ultrasound). For an electron in a solid, for example, the rate at which the electron can absorb and re-emit energy is a measure of its "coupling" to the crystal lattice. Strong coupling means a short "spin-lattice relaxation time," also known as T_1.

(a) A typical value for T_1 might be 0.01 sec. How many precession cycles does an electron perform in 10^{-2} seconds in a B-field of 10^{-2} T?

[Answer: 2.8 million] That means the resonance is very sharp. Essentially, it means that a change of B-field of one part in 2.8 million would be detectable. This is what makes electron spin resonance (ESR) such a useful tool. It can measure local magnetic fields with phenomenal accuracy.

(b) Many atomic nuclei also have spin and also have magnetic moments. For a proton, the formula for the nuclear magnetic resonance (NMR) frequency is

$$\nu = \left(42.58 \times 10^6 \frac{\text{Hz}}{\text{T}}\right) \times \text{B}$$

To what frequency must the oscillator be tuned to observe the NMR of hydrogen nuclei (called proton resonance) in an applied magnetic field of 10^{-2} tesla?

(c) A typical variation of the B-field inside a solid might be 2×10^{-4} T. What percent accuracy must be achieved in a frequency measurement in order to detect such a variation?

[Answer: 2%] In practice, NMR is capable of a thousand times greater accuracy than that. Hence its usefulness in chemical analysis, by distinguishing the chemical environment of the resonating protons. See Problem 24, Magnetic Moment of the Electron.

✶✶✶ 35. **Stable Magnetic Fields Using NMR** A proton resonance apparatus can detect B-field changes of one part in 10^5. Invent a "feedback" method of stabilizing the current in an electromagnet to that accuracy.

NOTE: Such servomechanisms are used to regulate the power supplies in commercial NMRs.

✶✶ 36. **Solenoid**

(a) Calculate the magnetic field well inside a cylindrical coil 20 cm long and 3 cm in diameter uniformly wound with 10,000 turns of wire carrying 1 ampere.

(b) What information given did you not need?

(c) Viewed as current going around in a circle, we are talking about 10,000 amps here. Even with copper wire, that coil is going to get hot. What design considerations would help achieve high fields? Not surprisingly, very high magnetic fields are achieved by very short current pulses.

✶ 37. **Field of a Current Sheet** Just to practice using the formula

$$B = 4\pi \times 10^{-7}\ \text{N/A}^2 \times (\text{current per unit length})$$

find the field strength between two parallel sheets of current carrying 10^3 amps per meter in opposite directions.

NOTE: The formula is for infinite sheets. That means it will yield a good approximation for points that are not near the edge of a sheet. Nothing is said about the distance between the sheets. For finite sheets, that distance has to be small compared with the width in order to approximate infinite sheets.

✶✶ 38. **Field Inside a Coil** The current-sheet formula holds for the field inside a uniform coil, as long as the field point is not too close to the end of the coil.

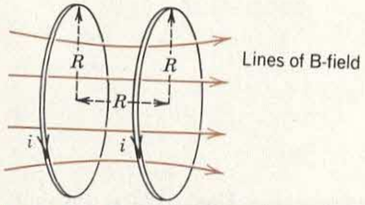
(a) Use it to find the *B*-field inside a cylindrical coil 50 cm long, 10 cm in diameter, wound with 1000 turns of wire carrying a current of 1 amp.
(b) Argue that a coil twice as long (1 m) with 2000 turns would have the same field strength inside if it carried the same current.
(c) How would doubling the diameter of the coil affect the *B*-field inside?

✶ **39. Inductance of a Coil** Calculate the self-inductance of the coil of Problem 38 for all three situations:
(a) Original coil.
(b) Twice as long.
(c) Double the diameter.

✶✶ **40. Self-Inductance of a Long Coil** A coil 0.50 m long has 1000 turns.
(a) Calculate the *B*-field inside it when a 1-amp current flows.
(b) If the cross-sectional area is 6.0×10^{-4} m^2, calculate the magnetic flux per turn.
(c) The total magnetic flux threading the coil is 1000 times that. Set total flux equal to *Li* and calculate the self-inductance.
(d) A coil having the same shape but twice as many turns (i.e., 2000) would have a self-inductance ______ times as great.

✶✶ **41. Iron Core** When carrying a current of 1 amp, a uniformly wound coil has a *B*-field of 1.0×10^{-3} tesla inside. A core made of a magnetic material is placed inside, and the *B*-field strength in the core is 2.0×10^{-1} tesla. Suppose that the self-inductance of the empty coil is 20 mH.
(a) Argue that the self-inductance with the core is 4 H.
(b) If the core fills only half the volume of the coil, argue that the self-inductance is roughly 2H.

✶✶ **42. Helmholtz Coils** When a rather uniform *B*-field is desired but a uniformly wound coil is impractical, an expedient is to use two identical rings of current in parallel planes located exactly one ring radius apart (see figure). This is not only cheaper, it also allows the hands to get in. The formula for the field near the middle is $8.992\, k_m I/R$, where I is the total current in each ring and R is the radius.
(a) Suppose that you need a field of 0.01 T and your rings are 30 cm in radius, how much current must each ring carry?
(b) In practice, this is achieved by having several hundred windings in each ring. If the power supply can supply a maximum of 1 amp, how many windings are needed in each ring? (See coils in photo of Bainbridge–Winch experiment in Section 15.2.)



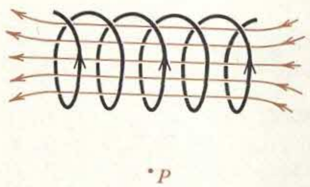
PROBLEM 42 Helmholtz coils. Their planes are one radius apart. In the region midway between them, the magnetic field is remarkably uniform.

✶✶✶ **43. Lines of Force—Gauss's Law** One way to get a picture of a vector field is with a field map. In principle, an arrow can be drawn at every point in space having the direction of the field at that point. A field map consists of a selection of such arrows. A curve drawn such as to have the direction of the field at each point is called a "line of force."

Of course, a field map is three-dimensional, and a picture in a book is just some kind of a two-dimensional projection. So the picture of the field of a bar magnet (photo in Section 15.1) is at best suggestive. What you realize immediately is that the lines diverge where the field gets weaker and converge where the field gets stronger. If all magnetic lines that are started are continued, it turns out that the density of lines (number of lines per unit area) is everywhere proportional to the field strength. That is a powerful theorem, called **Gauss's Law.**

NOTE: Gauss's Law holds for magnetic, for electric, and for gravitational fields. Please do not think that it holds for all vector fields!
(a) The field well inside a uniformly wound, current-carrying coil is uniform. Argue that lines of force cannot cut the coil except very near the ends.
(b) Magnetic lines of force always close; that is, they have to bite their own tails. That is not true for electric lines of force; some do, some don't. The electric lines of force that don't close have to start and stop somewhere. What do they start and stop on?

✶✶ **44. Flux** A map of the *B*-field of a short coil of current-carrying wire appears below. The *B*-lines are close together inside and further apart outside. This corresponds to a strong field inside, falling off rapidly at each end of the coil. For a nonuniform field, the definition of



PROBLEM 44

flux requires a limiting process. The surface through which the flux is to be defined has to be divided into small area elements. The magnetic flux through each area element is the product of its area times the perpendicular component of the *B*-field there. The flux through the whole surface is the sum of those products. More precisely, it is the limit of that sum as the area elements are taken smaller and smaller. Such a limit is called an integral, in this case a surface integral.

(a) Use a symmetry argument to show that the flux through a closed drum-shaped surface closely surrounding the coil is zero.

(b) Here is a beautiful property of magnetic fields that gives detailed information about how they fall off away from their sources: The flux through *any* closed surface is zero. This is called Gauss's Law. Use it to argue that the *B*-field at the point *P* points in the opposite direction to the *B*-field inside the coil. What surface will you use in your argument? Note that the *B*-field at *P* is much weaker than the field inside. In the text we said that it is zero.

✶ **45. Magnetic Energy** The energy stored in an inductance *L* carrying a current *i* is given by the formula $\frac{1}{2}Li^2$.

(a) Calculate the magnetic-field energy of a 20-mH coil carrying 2 amps.

(b) If the current is doubled, what happens to the energy?

(c) If the current is increased by 1%, what happens to the energy?

✶ **46 Magnetic Energy—Dimensional Analysis** You know that the magnetic energy stored in a coil depends on its self-inductance *L* and on the current *i* it carries. Assume an expression of the form constant × $L^p i^q$. Use dimensional analysis to show that the exponents *p* and *q* have to have the values 1 and 2, respectively.

✶✶ **47. Self-inductance** The self-inductance of a long uniformly-wound coil is given by the formula:

$$L = 4\pi k_m N^2 \text{ area/length.}$$

Derive this formula, following the logic of Sec. 15.4. (HINT: Exercise 9.)

✶✶ **48. Magnetic Energy Density** Someone says, "The self-inductance of a coil is proportional to its volume."

(a) Argue that this is true provided that the number of turns per unit length is fixed.

(b) Use Problem 47 with the formula $\frac{1}{2}Li^2$ to derive Eq. 15.4–2.

✶ **49. Field Energy Density** The electric field near the surface of the earth is about 100 volts per meter: The earth is negative relative to the upper atmosphere. Compare the electrostatic field energy per unit volume (Eq. 15.4–3) with the magnetic field energy per unit volume (Eq. 15.4–2), using a typical value for the earth's *B*-field, like 0.8×10^{-4} T.

✶✶ **50. Magnetic Energy Storage** Electric generating plants have to be large enough to handle peak-hour current loads. It would reduce capital costs if, instead of building bigger generators, one could generate power at a uniform rate and store the energy for several hours, to be used later at the high-demand times of the day. Storage batteries offer one way to effect energy storage, but they have great losses. Superconducting (zero-resistance) electromagnets offer a nonlossy way. One of the problems in developing such a technology is containment.

(a) Argue that the magnetic force on the wire in a current-carrying coil is directed radially outward, that is, it tends to expand the coil. (HINT: All you need is the right-hand rule.)

(b) Since the energy stored in an inductance *L* is $\frac{1}{2}Li^2$, storing a lot of energy requires high currents. High currents means large magnetic fields. But large fields destroy superconductivity. Suppose that we have a wire that can remain superconducting up to 5 tesla. That tells us the maximum energy stored per unit volume,

$$\frac{B^2}{8\pi k_m} = \frac{(5T)^2}{(8\pi \times 10^{-7}\ N/A^2)}$$

$$= \underline{\qquad\qquad} \frac{\text{joules}}{\text{m}^3}$$

(c) Now use the uniform-coil formula

$$B = 4\pi k_m \times \text{current per unit length}$$

to calculate the current. You will need to know how tightly the coil is wound. Take a coil with 100 turns per meter. Show that the maximum allowable current is 4×10^4 amps.

(d) Don't think that is a small current. Each meter of wire experiences an outward force of

$$Bil = 5\text{ T} \times 4 \times 10^4\text{ A} \times 1\text{ m} = \underline{\qquad\qquad}\text{ newtons}$$

That is 20 tons.

(e) Converting that to the stress (= outward force per unit area) on the sides of the coil, we have

$$\text{stress} = \frac{\text{force}}{\text{area}} = Bi \times (\text{turns per unit length})$$

$$= \frac{5\ \text{T} \times 4 \times 10^4\ \text{A} \times 10^2}{\text{m}} = \underline{\qquad\qquad} \frac{\text{N}}{\text{m}^2}$$

That is 2000 tons per square meter and corresponds to a pressure of 200 atmospheres. You can see that the coil, if it is big enough to be useful, would have to be buried in solid ground!

★★ 51. **Superconducting Coil for Energy Storage** In Problem 50 you multiplied by n, the number of turns per unit length of coil, to find the maximum permissible current. Later you divided by n to find the stress, that is, the pressure on the walls containing the coil. That seems to say the pressure p is independent of n. We now prove that formally. The stored energy density $u = B^2/8\pi k_m$ is determined as soon as the maximum allowable B-field is specified. The force on one turn of wire carrying current i is $Bi \cdot 2\pi r$, where $2\pi r$ is the circumference of the coil. The force per unit area of coil is the force per turn divided by the area per turn,

(a) Pressure $p = \dfrac{Bi2\pi r}{2\pi r/n} = \underline{\qquad\qquad\qquad}$ (simplify)

(b) The field strength is given by $B = 4\pi k_m in$. Solve this for i, and substitute in part (a) to see how p is related to u. (HINT: Check to see that p and u have the same dimensions. Note that p is a force per unit area, while u is an energy per unit volume.)

☐ QUESTIONS FOR FURTHER THOUGHT

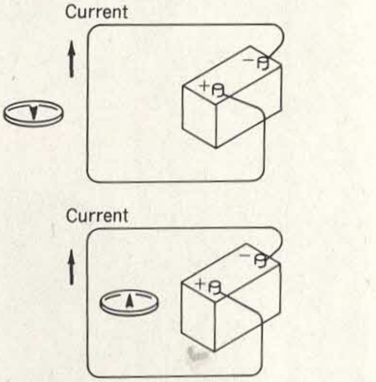


FIGURE 15.8 Magnetic field of a straight wire carrying a current explored by magnetic compass.

α. **Alternating Current** Oersted discovered the magnetic effects of current by bringing a compass needle near a battery "shorted" with a wire. Why doesn't a compass needle deflect when placed near a lamp cord in your home?

β. **Hysteresis** Some materials remember their history. A material that exhibits magnetic hysteresis has a magnetization that depends not only on the magnetic field it is experiencing at the moment, but also on the fields it experienced some time ago. If you magnetize a piece of steel and then remove the applied magnetic field, the steel will not retain all of its magnetization, but it will not lose it all either. To demagnetize it completely, you have to apply a magnetic field in the opposite direction.

The graph (Fig. 15.9) shows how a typical magnetic material (like steel) behaves as the applied field is increased. The magnetization saturates (curve 1). Curve 2 shows what happens to the magnetization as the applied field is then decreased and later reversed. Again the magnetization saturates. Notice that curve 3 (applied field decreasing and reversing again) does not retrace curve 1. Indeed, curves 2 and 3 are mirror images.

(a) What ordinary symmetry property of the material is displayed by this mirror symmetry of the hysteresis loop?

(b) Different magnetic alloys (mixtures of metals) have different shapes of hysteresis loops. Some saturate at low values of B, some at high. Some retain a lot of magnetization when the field is removed, some retain little. If you were designing an alloy to use for a magnetic computer memory, what shape hysteresis loop would you strive for?

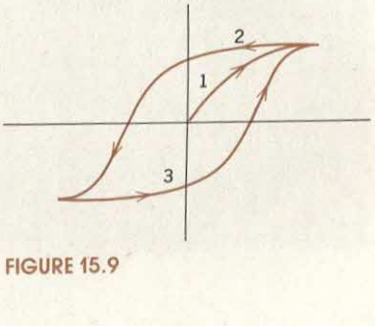


FIGURE 15.9

γ. **Bird Navigation** We still don't know how the birds do it, but we know that the magnetic forces in biology are tiny. The earth's magnetic field is about 10^{-4} tesla or less. The electric current in a nerve might be a picoampere, 10^{-12} amps.

So 1 centimeter of conducting nerve perpendicular to the field experiences an electromagnetic force of

$$\left(10^{-4}\,\frac{\text{N}}{\text{Am}}\right) \times (10^{-12}\text{ A}) \times (10^{-2}\text{ m}) = \underline{\hspace{2cm}}\ \text{newtons}$$

A bird weighing 1 or 2 newtons is not likely to be sensitive to such a small force. How about the motional e.m.f. associated with that picoampere current? The transverse force on each electron (charge e) having a drift speed $v = 10^{-4}$ m/sec is evB, corresponding to an electric field $evB/e = vB = (10^{-4}\text{ m/sec}) \times (10^{-4}\text{ N/A m}) = \underline{\hspace{1.5cm}}$ volts/meter. Over 1 centimeter of nerve, that would amount to a transverse voltage of 10^{-10} volts, which is clearly too small a voltage to be perceived as a signal. Voltages across cell membranes are in the millivolt range, not fractions of a nanovolt. Indeed, the voltages associated with random thermal motion of electric charge have average magnitudes around 25 millivolts at body temperature. Signals smaller than that are "in the noise."

We are led to conclude that if a bird uses the earth's magnetic field for navigation, it must have a permanent magnet (a magnetic compass) somewhere in its body. The author is not aware that the anatomists have found such magnets in birds. At the time of writing, the mechanism of bird orientation is still somewhat mysterious.

8. **Some Features of the Magnetic Field** Remember the "topology" of the electric fields we studied in electrostatics. All E-lines started on + charges and ended on − charges. Magnetic field lines, on the other hand, don't start or stop anywhere. They always close. For example, the B-lines around a long straight wire carrying a current are circles surrounding the wire.

(a) The convention we have chosen for the direction of the B-field is that outside a magnet, the field lines run from N to S. This convention agrees with the direction in which a compass needle points. Which way do the field lines run inside a magnet?

(b) The *electric flux* through a sphere surrounding an electric charge is directly proportional to the charge. You may recall that this is a direct consequence of Coulomb's Law, that is, of the inverse-square behavior of the E field. Give a plausibility argument for the theorem that the *magnetic flux* through a sphere surrounding a magnet is always zero. The theorem is actually more general: The B-flux through any closed surface vanishes.

(c) What can you say about a closed surface through which there is no E-flux?

(d) The theorem of part (b) says that there is no magnetic analogue to electric charge. In jargon: There is no magnetic monopole. If someone were to claim to have discovered a particle that is a magnetic monopole, what experiments would you suggest to confirm the discovery?

NOTE: This is intended as an open-ended question, not having a single correct answer. See also Sec. 21.6.

Hoover Dam on the Colorado River with (artificial) Lake Mead. Inside the dam are huge water turbines for electric power generation.

16

ELECTROMAGNETIC INDUCTION

SOME QUESTIONS THIS CHAPTER WILL ANSWER

State Faraday's law of electromagnetic induction.

The starting motor of a car draws less current from the battery when it runs faster. Explain.

Doorbells are designed to operate around 16 volts. What should be the turns ratio of a step-down transformer used to power a doorbell from a 115-volt line?

What advantage is there in using high ac voltages for transporting electric power over long distances?

Argue that a capacitor in series with an inductor has a resonance frequency.

Any electric charge is a source of electric field. A moving charge gives rise to magnetic field as well. An accelerated charge gives rise to a changing magnetic field. Explain why an accelerated charge emits electromagnetic radiation.

By age 10 I had seen many electric motors. I knew they had to be "plugged in" to run. One day I asked my uncle how electricity is made. He worked in a factory with lots of rotating machinery. He said electricity comes from generators. Vaguely I knew that had to do with rotating machinery. "An electric generator is just like an electric motor, but backwards," my uncle said. "Turning it makes the current flow, instead of the current making it turn." Later on, I learned about the induction coil (spark coil). Our car had one, to make voltages high enough to spark the spark plugs. After that I found out about transformers, and that they "step up" voltages in the television set and "step down" voltages in the power lines outside the house.

These disconnected facts may be familiar household wisdom to you, too. In any case, they are all the cultural background you need for this chapter.

In Chapter 13, Electrostatics, the concept of *electric field* was introduced in situations in which charge was stationary. There the source of *E*-field was always charge. Chapter 15, Magnetic Field, introduced the magnetic field, in situations where the source of the *B*-field could be charge in motion or it could be magnetic dipoles associated with electrons or atomic nuclei. In the present chapter, you will meet electric fields that arise because magnetic field is changing nearby. That is called electromagnetic induction. In the same way, some magnetic fields can be ascribed to electric fields changing nearby. In other words, when they are changing, there is an intimate connection between electric field and magnetic field. The way the one field changes in space is related to the way the other field changes in time. The ability of empty space (yes, vacuum!) to support electromagnetic waves (light waves, radio waves, etc.) is the direct consequence of this intimate connection. So is the fact that all those waves travel in free space at the same speed.

Section 16.1 introduces the description of the interaction between fields by describing in "field language" the motion of charge in a magnetic field. You may notice that Sec. 16.1 really belongs in the previous chapter. It does not bring in any new phenomena.

16.1 MOTIONAL E.M.F.

One device used to measure the speed of blood flow in an artery places the artery between the pole faces of a magnet and measures the voltage induced across the artery. That voltage is proportional to the flow velocity and the strength of the magnetic field. We can see why if we use our knowledge of the force on a charge moving in a magnetic field.

The blood has some ions, which are responsible for its electrical conductivity. As a positive ion is swept along in a direction perpendicular to the magnetic field, it experiences a magnetic force. That force is in the direction perpendicular to both its velocity and the *B*-field, in the sense given by the right-hand rule. The buildup of positive charge on one side of the blood vessel, and consequently of negative charge on the opposite side, can be measured as a difference of electric potential between the two sides (Fig. 16.1).

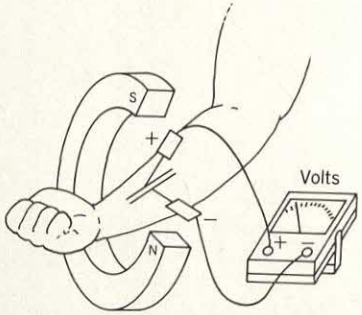


FIGURE 16.1 Measuring speed of blood flow in an artery. The magnetic field is perpendicular to the flow direction. The measured transverse voltage is proportional to the speed.

To calculate the magnitude of that potential difference, make the assumption that all the blood flows at, say, 0.1 m/sec across the entire cross section of an artery 0.01 m in diameter. This is actually not very realistic, as the speed close to the walls is much slower than near the middle of the blood vessel. But it will serve to explain the principle of the measurement and to get rough results. A powerful magnet might have a field of 1 tesla (= 1 newton/amp meter = 10^4 gauss) between the pole faces. That is more than 10,000 times as strong as the earth's magnetic field. Remembering that the force on a charge moving in a magnetic field is given by the product

$$F = qvB \qquad \text{(Eq. 1)}$$

of the charge times its velocity transverse to the field times the *B*-field, we can write the force per unit charge:

$$0.1\,\frac{\text{m}}{\text{sec}} \times 1\,\frac{\text{newton}}{\text{amp meter}} = 0.1\,\frac{\text{newton}}{\text{coulomb}}$$

In the terminology of Sec. 13.2 we would say that motion at 0.1 m/sec across a *B*-field of 1 tesla is equivalent to an electric field of

$$0.1\,\frac{\text{newton}}{\text{coulomb}} = 0.1\,\frac{\text{volt}}{\text{meter}}$$

From one wall of the artery to the other (1 cm across), that *E*-field corresponds to a potential difference of 1 millivolt:

$$0.1\ \frac{\text{volt}}{\text{meter}} \times 0.01\ \text{meter} = 0.001\ \text{volt}$$

Repeat: Moving ions experience a force of 10^{-1} newton/coulomb across the direction of flow. Across the 1-cm diameter, the product of force per unit charge times distance is equal to

$$10^{-1}\ \frac{\text{newton}}{\text{coulomb}} \times 10^{-2}\ \text{meter} = 10^{-3}\ \frac{\text{joule}}{\text{coulomb}} = 10^{-3}\ \text{volt}$$

Riding along across a *B*-field is equivalent to sitting in an *E*-field.

If you repeat the argument with a negative ion, you note that the force on it is in the opposite direction, corresponding to the same electric-field direction and the same potential difference. How much charge (or of which sign) actually moves between the pole faces never enters the argument. No electric current flows along the artery. Current would flow *across* the artery if we wanted to use the potential difference to do work by drawing current between electrodes attached to opposite walls of the artery. Then the potential difference would decrease. But we still want a word for how much there is in the absence of current. That word is e.m.f., **motional e.m.f.** in this case, because it is associated with motion of the fluid across the magnetic field. The letters "e.m.f." actually stand for "electromotive force," but nobody says that, and besides, the word force is really quite confusing, since an e.m.f. is a voltage. Some people use the word *electromotance*. (Recall Chapter 13.)

It may be useful to repeat the derivation of the motional e.m.f. using the symbols v, B, and d for the numbers above. A conductor moving with velocity $\mathbf{v}$ perpendicular to a magnetic field $\mathbf{B}$ experiences an induced electric field

$$E = vB$$

in a direction perpendicular to $\mathbf{v}$ and $\mathbf{B}$. Over a distance d along the direction of the E-field the potential rises by an amount Ed, so if the two electrodes are a distance d apart, the motional e.m.f. is

$$\text{e.m.f.} = Ed = vBd$$

To interpret this formula further, go back to the numbers. With the blood flowing at 10 cm per second, 10 centimeters of blood pass every point in 1 second. This means the blood in 10 cm of artery. That 10 cm of blood (of artery) is 1 cm wide. How much blood flows by in 1 second? An amount 10 cm long and 1 cm wide. That may seem like a strange description (an area?), but we may think of 10 cm × 1 cm = 10 cm^2 of blood passing between the pole faces per second. Why talk about an area of blood? Because the magnetic field lines are perpendicular to that area. The same magnetic field lines that went through one square centimeter go through the next square centimeter one-tenth of a second (= 10^{-1} sec) later. Here we may get some physical intuition by rewriting the units of our *B*-field:

$$1\,\frac{\text{newton}}{\text{amp meter}} = 1\,\frac{\text{volt sec}}{\text{m}^2} = 10^{-4}\,\frac{\text{volt sec}}{\text{cm}^2}$$

Through each cm^2 of area between the pole faces pass 10^{-4} volt seconds of something. That something is *magnetic flux,* which was defined in Sec. 15.4. The flowing fluid intercepts or "cuts" magnetic flux at the rate of

$$\frac{10^{-4}\text{ volt sec}}{10^{-1}\text{ sec}} = 10^{-3}\text{ volt}$$

Another way to see that is to multiply the *B*-field by the rate at which area sweeps past the electrodes:

$$10^{-4}\,\frac{\text{volt sec}}{\text{cm}^2} \times 10\,\frac{\text{cm}^2}{\text{sec}} = 10^{-3}\text{ volt}$$

Of course, there was no magic about changing from meters to centimeters for this part of the story; it is just easier to visualize the flux cutting an area of 1 cm^2.

EXERCISE 1 In our example, a motional e.m.f. of 1 millivolt was induced between electrodes 1 cm apart by motion at 10 cm/sec in a field of 1 tesla.
(a) If the *B*-field is doubled, what happens to the e.m.f.?
(b) If the speed of flow is doubled, what happens to the e.m.f.?

EXERCISE 2 If the average speed of flow were only 5 cm/sec, how large a *B*-field would be needed to raise the e.m.f. to 2 millivolts?

We talked about blood flow. But you realize that the ideas of this section are completely general. Any charge moving across a magnetic field experiences an electric field whose strength is $v_\perp B$, where $v_\perp$ is the component of the velocity perpendicular to the magnetic-field direction. It is as if that electric field is a hidden aspect of the magnetic field that is brought out of hiding by the motion. Later in the book, in the chapter on Relativity, we will pose the question, "motion relative to what?" In whose frame of reference does there have to be motion in order to observe this "hidden" electric field? A magnetic field does not have "its own" frame of reference. It turns out that the answer is, the frame of reference of the observer. This is a profound answer, and it may trigger all kinds of questions in your mind. It is intended to! Some of them won't be answered until you read the Relativity chapter (Chapter 20). But what you already know now is that some electric fields don't originate from charges. Remember, back in Chapter 13, Electrostatics, you learned that electric field lines start on positive charges and end on negative charges. Now we are talking about electric field lines that don't have charges as their source. They don't begin or end at all. They come around and bite their own tails; that is, they are endless loops.

Magnetic forces were introduced in Chapter 15—forces on currents and forces on moving charges. When we talked about sources of magnetic field, we realized that currents and moving charges *are* sources of magnetic field. We now show that this story is only consistent if motion in a magnetic field gives rise to (is accompanied by) an electric field perpendicular to that magnetic field.

16.2 FARADAY INDUCTION

The result of the previous section can be restated briefly as follows: The e.m.f. due to motion of a conductor in a magnetic field is equal to the rate at which the conductor cuts magnetic flux. If you were now asked if it matters whether the conductor moves and the magnet stays fixed, or the magnet moves and the conductor stays fixed, what would you guess?

If your guess is that the effect is the same either way, you would be right. Besides, you would be indicating your belief that what counts is *relative* motion. It was this intuition that was formalized by Albert Einstein in his Principle of Relativity, the cornerstone of the Special Theory of Relativity (Chapter 20). This principle essentially rejects the need for a preferred frame of reference (the conductor's? the magnet's?). It asserts that all inertial frames are on an equal footing. If the laws of electromagnetism are to be invariant under change of reference frame, then it could not make any difference whether the conductor moves or the magnet moves.

With this symmetry in mind, the term motional e.m.f. can be replaced by the more general term **induced e.m.f.**. The formal statement of the law of electromagnetic induction is

Faraday's induction law: The e.m.f. induced in a circuit is equal to the time rate of change of the magnetic flux through the circuit.

There are various ways to change the flux through a circuit. We could

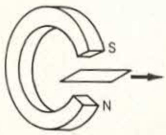
move the whole circuit;

move part of the circuit, that is deform it;

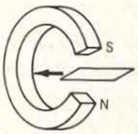
move the source of the magnetic field (the magnet);

change the strength of the magnetic field.

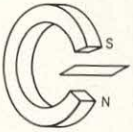
EXERCISE 3 For each sketch of Fig. 16.2 indicate which way the induced current would flow in the circuit. If you have trouble, read about Lenz's Law at the end of this section and try again.



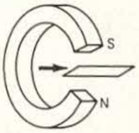
(*a*) Wire loop moves to the right



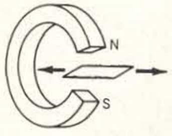
(*b*) Wire loop moves to the left.



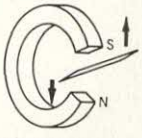
(*c*) Magnet moves to the left



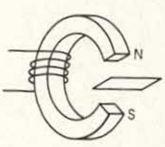
(*d*) Magnet moves to the right



(*e*) Wire loop stretches so as to intercept more flux



(*f*) Wire loop turns counterclockwise about a horizontal axis



(*g*) Current in electromagnet increasing

FIGURE 16.2

Generator

Most electric generators are designed so that coils of wire are rotated in a magnetic field, just as in an electric motor (Fig. 15.2). Suppose that you rotate a single loop of wire about an axis in its plane. The magnetic flux through the loop will be maximum when the field is perpendicular to the plane of the loop. After a 180° rotation, the flux will be maximum going through the other way, so we have to talk about positive flux and negative flux. When the plane of the coil is parallel to the field, the flux through the loop is zero. It is exactly at this zero-flux position that the rate of change of flux is maximum. This is the moment the flux is changing sign. This also happens twice in each rotation, and in opposite directions. Twice during each rotation the induced e.m.f. is maximum, once positive, once negative. Twice during each rotation the flux has a maximum. At these instants the e.m.f. is zero. The voltage in the loop is sinusoidal. This is what is delivered to our homes in the United States via the power lines, alternating 60 times per second. Our electric clocks stay on time because the coils at the generating station are kept rotating at constant frequency.

A vertical-axis wind turbine used for electrical power generation. This turns out to be a particularly efficient geometry. Since wind speed is variable, wind-produced electrical energy is usually stored in batteries.

If current is allowed to flow in the loop in response to the induced e.m.f., there is a magnetic force on the loop opposing its motion. A little right-hand-rule practice will verify this for you. You should satisfy your intuition that you have to do work to generate electric power. That intuition has its formal counterpart in the book-keeping device we called Conservation of Energy back in Chapter 4.

The induced current in turn gives rise to magnetic flux. You can verify that the flux due to the induced current is in such a sense as to tend to oppose the flux change that induced it. That qualitative fact is called **Lenz's Law.** It also is intuitively satisfying. If it were the other way around we would have a vicious circle—more and more flux—and violate Conservation of Energy.

You can see that our ac generator could be run as a motor by feeding it with an alternating current. Its rotation would still induce an e.m.f., appropriately called the "back" e.m.f. because it tends to oppose the current responsible for the force that keeps the loop rotating.

Please don't get the idea that all generators generate alternating current. By cleverly switching the current several times during each rotation a somewhat wobbly direct current (dc) is obtained instead. Conversely, if a direct current is fed into a machine equipped with such a "commutator," it operates as a dc motor. These technological comments are inserted here mainly to make contact with your own experiences with electrical gadgets and toys.

EXERCISE 4 The starting motor of a car is a dc motor. The car battery is a 12-volt battery. When the starter button is first pressed, suppose that the motor draws 100 amps from the battery. Some of the 1200 watts of power provided by the battery go into mechanical work (speeding up the motor, turning the car engine); some go into heating up the wires. As the motor starts to turn faster, the back e.m.f. increases, so the motor draws less current.

(a) When the current is down to 50 amps, at what rate is the battery supplying power?

(b) Argue that the power going into heating the wires is about 25% of what it was initially.

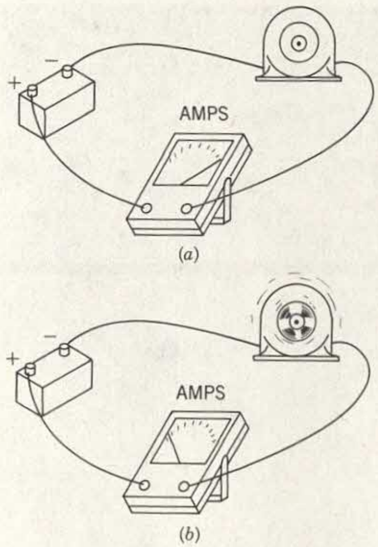


FIGURE 16.3 (a) The motor draws a lot of current at slow speed; (b) and draws very little current at high speed.

16.3 TRANSFORMERS AND ALTERNATING CURRENT

The oscillatory current (ac) that is carried by our power lines has a great advantage over direct current (dc): It is easy to change the voltage, using a **transformer.**

Figure 16.4 is a schematic representation of two coils wound in such a way that all the magnetic flux generated by one has to pass through the other. One way to achieve this is to wrap both coils around the same iron core. Suppose that the coil arbitrarily labeled "primary" is fed an alternating current; then the flux along its axis alternates in phase with the current. When the current (and hence the flux) is increasing, an e.m.f. is induced in the secondary. When the current is decreasing, an e.m.f. of the opposite sign is induced in the secondary. If both coils have the same number of turns, then the flux—and hence the rate of change of flux—is the same in both coils, so the e.m.f. is the same in both. Since the primary and secondary coils are not connected (there is no conducting path between them) such a device is called an isolation transformer. But if the secondary has twice as many turns as the primary, then the flux in the secondary is twice that in the primary, and so is the e.m.f. This is called a step-up transformer. The step-up ratio is just the ratio of the number of turns in the two windings. Your television set has at least one such transformer in it, used to obtain the thousands of volts needed in the electron gun of the cathode-ray tube, with the primary hooked to the 115-volt ac available at the electric outlet.

If, on the other hand, the secondary has fewer turns than the primary, that is a step-down transformer. Most door bells are designed to operate around 16 volts in order to save on insulation, so a step-down transformer in your basement with a turns ratio of 7:1 (primary: secondary) steps down the 115-volt line voltage. The device you might hear humming outside your home on the power pole is also a step-down transformer. For transporting electric power through the power lines to your street, the power company finds it more efficient to use hundreds or even thousands of volts. For example, a 10:1 step-down transformer might convert 1150-volt ac on the pole to the 115-volt ac that enters your home. To appreciate the efficiency, suppose that you need to draw 10 amperes for a 1150-watt home appliance. (115 volts × 10 amps = 1150 watts). To provide the power at 10 times the voltage, only $\frac{1}{10}$ of the current needs to flow in the power lines in the street,

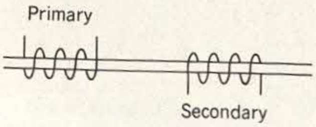
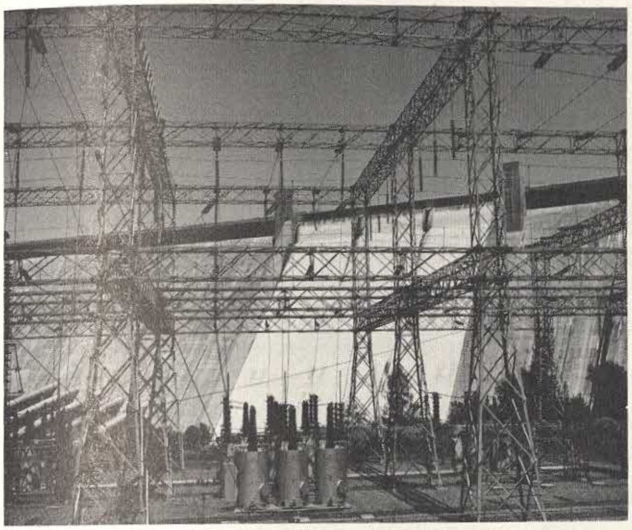


FIGURE 16.4 Transformer.



An electric "switch-yard" with transformers in the foreground. In the background is Shasta Dam, California.

or 1 amp. Remember, the power is the volts times the amps, and there is hardly any loss in the transformer. If the power line has a resistance of 1 ohm, a 1-amp current will give an RI-drop of only 1 volt, wasting 1 watt (= 1 volt × 1 amp) for power in heating the line. Had the feeder line used the 115-volt household voltage, the current in the 1-ohm line wound have been 10 amps, giving an RI-drop of 10 volts and a power loss of 100 watts (= 10 volts × 10 amps) going to waste in heating the line.

As you might guess, the lines used for long-distance transport of electric power use voltages of hundreds of thousands of volts, with great banks of step-up transformers at the generator and step-down transformers at the user end. The choice of 115 volts for the line voltage in our homes is a compromise between efficiency and safety. The Europeans settled for less safety and greater efficiency by choosing 230 volts for their household voltage.[1]

EXERCISE 5 A voltage-doubling transformer has a turns ration of 2:1. There are twice as many turns in the secondary winding as in the primary. The amplitude of the alternating voltage in the secondary is twice that in the primary.

(a) Use Faraday's induction law to argue that the current amplitude in the secondary is half that in the primary.

(b) Use Conservation of Energy to prove the same result. Remember that volts times amps equals watts.

[1] See Sec. 14.5, Electrical Injury. The standard line frequency in Europe is 50 Hz, in contrast to the U.S. standard of 60 Hz. If you take a step-down transformer to Europe in order to use your 115-volt electric razor on their 230-volt lines, don't expect your American electric clock to keep time on their frequency; it will only clock 50 minutes in an hour.

EXERCISE 6 Suppose that the primary and secondary of a transformer are wound in the same sense. Say they are both right-handed windings. Argue that when the primary current starts to flow right-to-left, the induced e.m.f. in the secondary will be in such a sense as to make current flow left-to-right.

EXERCISE 7 **AC Voltages—RMS and Peak-to-Peak**
When we say that our household line voltage is 115 volts ac, what aspect of the alternating e.m.f. does the figure "115 volts" describe? The answer is going to require the rest of this paragraph. To calculate the power taken from the line when a current is drawn, we multiplied the volts times the amps to get the watts. In our example of the 1150-watt appliance, it draws 10 amps from the 115-volt supply. That means it effective resistance is 11.5 ohms (= 115 volts ÷ 10 amps). Repeat:

$$(115 \text{ volts}) \times \frac{(115 \text{ volts})}{(11.5 \text{ ohms})} = \boxed{} \text{ watts}$$

The line voltage, the "115 volts," appears *squared* in the expression for the power consumed by the 11.5-ohm appliance. But the 1150 watts represents an average, not an instantaneous value of the power. This is an average over the entire cycle of the alternating voltage, or over many cycles. The rate at which energy is delivered by the line varies during the cycle, during the $\frac{1}{60}$ of a second in which the voltage reverses twice. To get a particular value of the average (or "mean") power, we needed a particular value of the average (or "mean") *squared* voltage. Now to answer our question: The "115 volts" is the voltage which, when squared and divided by the resistance, gives the average power. It is the square root of the required mean squared voltage. **Root mean square** is abbreviated "RMS". The "115 volts" is the **RMS** voltage in our household electric lines.

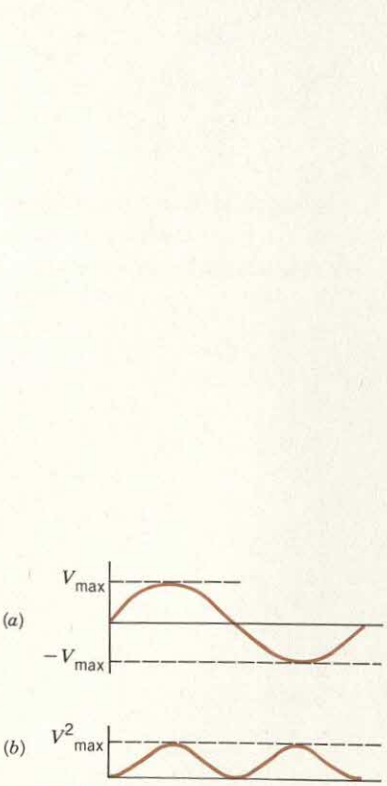


FIGURE 16.5 (a) Voltage plotted versus time; (b) voltage *squared* plotted versus time.

You might wonder how that value is related to the maximum value of the voltage, what you might call the voltage amplitude, V_{max}. In oscillating sinusoidally the voltage goes from zero to V_{max}, back to zero, to $-V_{\text{max}}$, back to zero, and then repeats [Fig. 16.5(a)]. The mean voltage is obviously zero. Figure 16.5(b) plots the square of the voltage. A glance will convince you that the shaded area under the V^2 curve is exactly half the area under the dotted line through $V^2{}_{\text{max}}$; that is, the average value of V^2 is $\frac{1}{2}V^2{}_{\text{max}}$:

$$\langle V^2 \rangle_{\text{average}} = \tfrac{1}{2}V^2{}_{\text{max}}$$

Turning the page upside down is particularly convincing. Taking square roots, we have

$$V_{\text{RMS}} = \sqrt{\tfrac{1}{2}}\, \mathrm{V}_{\text{max}}$$

Now put in $V_{\text{RMS}} = 115$ volts and solve for V_{max}. [Solve it.] (HINT: $\sqrt{\frac{1}{2}} = 0.707$.) Many people talk about the "peak-to-peak" voltage, $2V_{\text{max}}$, which comes out to 325 volts. (Verify.) That is the entire width of the range of the voltage variation during each cycle.

EXERCISE 8 If a 115-volt house voltage is displayed on the vertical input of an oscilloscope, show that the voltage amplitude $V_{\max}$ should be 163 volts.

16.4 OSCILLATOR

An inductance tends to oppose a change in the current it carries. That is Lenz's Law, and we proved it in Sec. 16.2. The current is the source of magnetic flux. Any change in flux is accompanied by an e.m.f. tending to drive current in opposition to the change in the current. The analogy has been drawn to inertia (mass): Just as the kinetic energy of a mass tends to maintain its motion, so the magnetic energy of an inductance tends to maintain the current. If we can supply a "restoring force," then we can make an electrical oscillator analogous to the mechanical oscillator—the mass on a spring. An electrical capacitance supplies just such a restoring force.

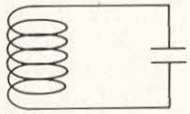


FIGURE 16.6

If the current in an inductance is made to flow into a capacitor and charge it up (Fig. 16.6), the voltage built up on the capacitor will oppose the current, which will decay to zero. At this point the capacitor will start to discharge, building up current in the opposite direction. When the capacitor is discharged, the current has built up to its maximum value and decays gradually while the capacitor charges up with opposite polarity. When the current has dropped to zero, the charge on the capacitor is again maximum, so the current reverses, building up and reaching its maximum just as the capacitor is again discharged and the cycle completed.

The energy is all magnetic when the current is maximum and the capacitor discharged. It is electrostatic when the charge on the capacitor is maximum and the current momentarily zero. This is quite analogous to the mechanical oscillator. There the energy is all kinetic energy when the motion is fastest as the mass passes through its equilibrium position. It becomes all potential energy of the spring when the motion reverses. Energy is dissipated in friction, so mechanical oscillations eventually damp out. In the electric circuit there is always some resistance, so electrical oscillations damp out as electromagnetic energy is dissipated into heat.

The larger the inductance, the longer it takes the current to decay. Similarly, the larger the capacitance, the longer it takes the capacitor to charge up. Increasing the inductance or increasing the capacitance both result in a longer period for the oscillation. Not surprisingly, the period does not depend on the voltage amplitude, but only on the circuit components. The box labeled Oscillator Frequency uses dimensional analysis to guess how the frequency depends on the inductance L and the capacitance C.

An electrical oscillator, like its mechanical analogue, displays the phenomenon of *resonance*. Suppose that an oscillatory voltage is imposed on an LC circuit—a circuit with an inductance and capacitance in series. There is one

OSCILLATOR FREQUENCY BY DIMENSIONAL ANALYSIS

Assume that the frequency ν is given by a formula like

$$\nu = \text{constant} \times L^x C^y$$

The exponents x and y are to be determined by a dimensional argument. Inductance is measured in henrys (= ohm seconds), capacitance in farads (= coulombs/volt = seconds/ohm). Frequency is measured in seconds^{-1}. Keeping units only, we have

$$\text{sec}^{-1} = (\text{ohm sec})^x (\text{sec ohm}^{-1})^y$$

This gives the relation

$$x = y$$

because there are no *ohms* on the left. Equating the exponents of the *seconds* gives

$$x + y = -1$$

determining both exponents to be $-\frac{1}{2}$:

$$\nu = \text{constant} \times L^{-1/2} C^{-1/2}$$

Dimensional analysis cannot give the value of the "constant," of course. A proper derivation gives the formula

$$\nu = \frac{1}{2\pi\sqrt{LC}}$$

frequency for which the response of the circuit is greatest. For given voltage amplitude, the current amplitude is maximum at this resonance frequency. Perhaps you are not surprised that the resonance frequency is almost exactly the natural frequency of the circuit. Right at resonance, the current amplitude is limited only by the resisitance of the circuit. The smaller the resisitance, the sharper the resonance, that is the closer you have to be to the natural frequency to get a large current response to the driving voltage.

Tuning the resonance frequency of a circuit by varying its inductance or capacitance is, of course, what you are doing when you turn the knobs to "tune in" to a radio or television station.

EXERCISE 9 Look inside a radio and try to discover what the tuning knob does. If the knob moves a bank of metal plates relative to another bank of parallel plates, it is varying a capacitance.

16.5 CHANGING ELECTRIC AND MAGNETIC FIELDS—RADIATION

It is comfortable to paraphrase Faraday's induction law by saying that a changing magnetic field induces an electric field. That is quite correct, but you have to avoid misinterpretation. For example, it would be misleading to say that a changing magnetic field is the *cause* of an electric field, as if the one preceded the other. It is more correct to say that a changing magnetic field is indistinguishable from an electric field. But even that can be attacked, because induced electric field lines don't begin and end on electric charges, as do the *E*-lines in electrostatics. The induced *E*-lines don't begin and end at all, but are continuous, just like magnetic field lines.

They are easy to visualize when they are circular, surrounding a cylinder of changing magnetic flux. Faraday's law of induction tells us the e.m.f. around the circle: For a circular *E*-line of radius r, the rate of change of *B*-flux through the circle is the e.m.f.:

$$\text{e.m.f.} = E \times 2\pi r = \text{time rate of change of magnetic flux}$$

Instead of e.m.f.[2] we might use the term *circulation of the electric field.*

Maxwell Induction

Now prepare yourself for a remarkable symmetry. The Faraday Induction Law (Sec. 16.2) states that a region of changing magnetic field is surrounded by *E*-field lines. An analogous law states that a region of changing electric field is surrounded by *B*-field lines! This is called the Maxwell Induction Law. Again, those *B*-lines are easy to visualize when they are circles surrounding a cylinder of changing electric flux. Again, the circulation of the *B*-field is proportional to the rate of change of the *E*-flux:

Maxwell Induction Law

$$B \times 2\pi r = \frac{k_m}{k_e} \times \text{(time rate of change of electric flux)} \qquad \text{(Eq. 2)}$$

The proportionality constant k_m/k_e is explained in Problem 16.9. We will say more about it later. What we want to do now is convince you that Maxwell induction together with Faraday induction implies the existence of electromagnetic waves in free space.

Here is the argument: Maxwell's Induction Law says that if an electric field changes, it is equivalent to (is accompanied by) a magnetic field. That magnetic field, of course, will be changing, so, according to Faraday's Induction Law, it will be equivalent to (be accompanied by) an electric field. That electric field, of course, will be changing. Is that what we mean by radiation? Well, radiation means waves, propagating patterns of field. If a spatial pattern of *E*-field moves, that means the *E*-field is changing in the region through which it moves, and the Maxwell → Faraday → Maxwell → and so forth argument has to work. But if it is to be a true wave, the pattern has to be preserved as it moves, that is, the *E*-field you get from the changing *B*-field you got from the changing *E*-field has to be the same *E*-field you started with. The story has to be self-consistent. The next paragraph will show that this self-consistency implies that electromagnetic radiation in free space has a unique speed!

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[2] Is the induced e.m.f. a voltage? It is certainly measured in volts. The e.m.f. is the work done per unit charge in taking a test charge once around the circle. But it is not a potential difference. Indeed the word *potential* is inappropriate when potential energy cannot be defined. (See Sec. 4.2). But if you were to put a wire on the circle and leave a tiny gap in it, so no current could flow around the circuit, an electrostatic voltmeter connected across the gap would measure the induced e.m.f. If current were allowed to flow, then the voltage measured with an ordinary voltmeter would be the induced e.m.f. reduced by the voltage drop in the resistance of the circuit, the *RI*-drop we met in connection with Ohm's Law in Chapter 14.

Suppose that a block of *B*-field moves through space at speed v. Think of a pattern of field consisting of a front with no field ahead of it and almost uniform field behind it. The usual Faraday's Law argument (see Sec. 16.1, Motional e.m.f.) gives the equivalent *E*-field as

$$E = Bv \qquad \text{(Eq. 3)}$$

If that barefaced oversimplification bothers you, multiply the equation by some length L so that it reads like "e.m.f. = time rate of change of *B*-flux." Anyway, that *E*-field is also such a block, also moving at speed v, and the analogous Maxwell-induction argument gives the required *B*-field as

$$B = \frac{k_m}{k_e} Ev \qquad \text{(Eq. 4)}$$

That proportionality constant, k_m/k_e, appeared in the Maxwell Induction Law when we talked about circular *B*-lines. We won't "derive" it further.[3] All we need to do is note that the B in those two equations has to be the same if this is going to represent a wave, and similarly for the E. If you substitute either equation into the other, the E and B drop out, leaving

$$v^2 = \frac{k_e}{k_m}$$

Put in the values:

$$v = \sqrt{\frac{k_e}{k_m}} = \left(\frac{9 \times 10^9 \text{ newton m}^2/\text{coulomb}^2}{10^{-7} \text{ newton/amp}^2}\right)^{1/2} \qquad \text{(Eq. 5)}$$

$$= (9 \times 10^{16} \text{ m}^2/\text{sec}^2)^{1/2} = 3 \times 10^8 \text{ m/sec}$$

Repeat: The unique speed of electromagnetic waves in free space is 300,000 kilometers per second, or 186,000 miles per second. Strong waves or weak waves, they all travel at the same speed. That is what you saw when the E and B dropped out of the equations for the wave speed v.

In the next chapter this characteristic speed will be called c:[4]

$$c = \sqrt{\frac{k_e}{k_m}} = 3 \times 10^8 \frac{\text{m}}{\text{sec}}$$

This constant of nature is central to the Theory of Relativity. In Chapter 20 it is shown to be a universal speed limit—the highest speed at which anything, not just electromagnetic waves, can travel.

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[3] For derivation see Problem 9.

[4] c for constant? No, the c is for the Latin word *celeritas* meaning speed.

SUMMARY

A device to measure blood flow. E.m.f. generated by motion across a magnetic field.

Induced e.m.f. Various ways to change magnetic flux.

Lenz's Law.

How an electromagnetic generator works.

How a transformer works. Step-up and step-down: What matters is the number of turns in the primary/secondary windings.

RMS and peak-to-peak.

Changing B-field is equivalent to E-field (Faraday induction).

Changing E-field is equivalent to B-field (Maxwell induction).

The mechanism of electromagnetic radiation. The unique speed of electromagnetic waves in free space.

KEY TERMS

Motional e.m.f. Voltage induced by the motion of a charge through a magnetic field.

Faraday's induction law The e.m.f. induced in a circuit is proportional to the time rate of change of the magnetic flux linking the circuit.

Induced e.m.f. Voltage induced in a circuit by a change in the magnetic flux.

Lenz's Law The sense of an induced current is always such as to oppose the flux change that caused it.

Electromotance—e.m.f. Work done per unit charge in taking a "test charge" around a closed path. The circulation of the E-field around a closed path.

Back e.m.f. When an electric motor turns, it acts as a generator opposing the current that drives it. The voltage induced by turning is called the back e.m.f. It is the work done by the motor per unit charge.

AC—alternating current, sinusoidal current In the United States 60-Hz ac is standard. That means the current reverses 120 times each second. In Europe, the standard is 50 Hz. The great advantage of ac over dc (direct current) is that transformers can step up or step down the voltage without the use of rotating machinery.

Transformer Two coils wound on the same core so that the same magnetic flux threads both the primary winding and the secondary winding. The ratio of the voltage amplitudes in the two windings is equal to the turns ratio. A step-up transformer has more turns in the secondary than in the primary.

RMS—root mean square When we say that our house current has "115 volts," we mean that the average of the voltage *squared* is $(115 \text{ volts})^2$. Since the voltage is alternately + and − (forward and back), its average value is zero. But the average *square* voltage tells us how much power can be delivered. A 115-volt (RMS) signal has an amplitude of 163 volts ($= 115 \text{ volts} \times \sqrt{2}$).

Oscillator frequency Natural frequency, resonance frequency. A circuit containing inductance L and capacitance C resonates at frequency $\nu = (1/2\pi)(LC)^{-1/2}$.

Maxwell induction A region of changing electric field is surrounded by a magnetic field.

Displacement current When a capacitor is charging, a current flows into the positive side and out of the negative side. At the same time, the flux of electric field is increasing in the gap. In order to satisfy the desire for *continuity* of current across the gap, we say that the flux change constitutes a displacement current equal to the charging current.

c Speed of electromagnetic waves in free space = 3.00×10^8 m/sec = $(k_e/k_{m)}^{1/2}$.

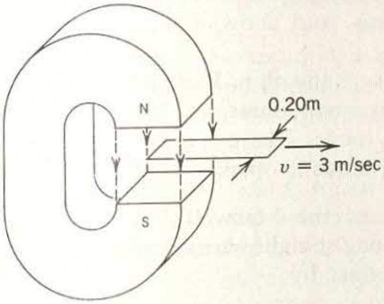
REVIEW QUESTIONS

- In the electromagnetic blood-flow apparatus, suppose that the magnetic field is vertically down. This means that Na^+ ions experience a force to the left of the y velocity, while Cl^- ions experience a force to the right. If the *B*-field is doubled, the motional e.m.f. is multiplied times ________.
- If the speed of flow increases by 1%, the e.m.f. increases by ________%.
- If the electrical resistance of the blood changes, why does this not affect the motional e.m.f.?
- Motion in a magnetic field is one way to induce an e.m.f. Name two others.
- If a conducting loop is rotated in a magnetic field, an alternating e.m.f. may be induced around the loop. Describe a relative orientation of field and loop for which no e.m.f. is induced.
- Lenz's Law says that the field of an induced current is in such a direction as to oppose the flux change that induced that current. The *B*-field inside a clockwise current loop is directed down. If the current decreases, which way is the induced e.m.f.? clockwise/counterclockwise.
- If the primary of a transformer has more turns than the secondary, we call it a step-down transformer. If the turns ratio is 10:1, it steps 115-volt ac down to ________ volts.
- What turns ratio is needed to step 115-volt ac up to 20,000 volts inside a television receiver?
- If the 230-volt line into your house has a circuit breaker that limits the current to 30 amperes, what is the maximum current you can draw from all the 115-volt lines inside? Did your reasoning involve energy (power) considerations?
- In an LC-series circuit oscillating freely, when the current into the capacitor is maximum, the current through the inductor is also maximum / zero, and the voltage across the capacitor is also maximum / zero. At this time, the energy of the oscillator is all electric / all magnetic. One-quarter period later, the energy is all electric / magnetic.
- When the capacitance is increased, the natural frequency of the oscillations is also increased / decreased.

- An electric charge in simple harmonic motion generates electromagnetic waves that go out at the same speed in all directions in free space. Argue that this speed is the speed of light.

PROBLEMS

✶ 1. **Motional E.M.F.** A loop of wire 0.20 m across moves to the right at 3 m/sec as shown, with its left end in a *B*-field of strength 0.5 T (vertically down), its right end essentially in zero magnetic field. Show that the induced e.m.f. in the loop is 0.3 volts.



PROBLEM 1 Motional e.m.f. The dotted *B*-lines are at the edge of the pole faces. The magnetic field is rather uniform over the pole faces and falls off steeply beyond.

✶✶✶ 2. **G.E.K.** An ocean current of 2 m/sec flows between electrodes 10 meters apart in the earth's *B*-field, whose vertical component is 0.8×10^{-4}T.
(a) Calculate the voltage between the electrodes. Oceanographic measurements of water currents are made using such a Geomagnetic Electrokinetograph (G.E.K.)
(b) If the electrodes are attached to a ship drifting with the current, show that the measured voltage will be zero.

✶✶✶ 3. **Motional E.M.F.** What is the voltage appearing between the wing tips of an airplane cruising at 178 m/sec (= 400 miles per hour)? Wingspread = 50 meters. Why is it not possible to measure this voltage with a voltmeter while riding in the airplane?

✶✶ 4. **Plasma Generator—Motional E.M.F.** A stream of hot gases rushes through a pipe 0.5 m in diameter at an average speed of 100 m/sec. Over one section of the pipe a transverse magnetic field of 0.5 tesla is maintained.
(a) Find the force on one positive ion in the partially ionized gas stream.
(b) If you were traveling with such an ion, you would experience an acceleration. Knowing the charge and mass of the ion, you might naively ascribe that acceleration to an electric field of ________ volts/meter.
(c) What is the electric potential difference between the ends of the diameter normal to the *B*-field?
(d) Suppose that a current of 100 amps were drawn from this "electric generator," how many watts of power would it be producing?
(e) As current is drawn, how much work would each coulomb of charge that passes be capable of performing? (HINT: Work per second divided by charge per second = ________)
(f) In 1 second, a length 100 m/sec × 1 sec of gas passes through the *B*-field, having an area of 0.5 m × 10^2 m = 50 m^2. The magnetic flux "cut" by this area is 50 m^2 × 0.5 T = ________.
Check: Is this equal to the potential difference of part (c)? (HINT: Emf = time rate of change of flux.)
(g) If the gas has a density of 0.25 kg/m^3, what is the kinetic energy of 100 meters of gas, i.e., how much kinetic energy does the boiler impart to the gas per second? (HINT: Circular pipe, area = πr^2.)
(h) What fraction of this power does the generator take away? Could one have another generator farther down the pipe?

NOTE: This is not the way electricity is generated in practice, but you will note that it requires no moving parts, no wheels, no bearings. Maybe some day . . .?

✶✶ 5. **Starting Motor** An automobile starting motor might draw 50 amps of current from the 12-volt battery. Suppose that the total circuit resistance is 0.10 ohms.
(a) Calculate the *RI*-drop in the whole circuit.
(b) Calculate the back e.m.f. induced by the rotation of the motor.
(c) The product of the back e.m.f. times the current is the mechanical power output of the motor. Calculate.
(d) Is this greater than the rate at which the resistance of the circuit converts electrical energy into heat? Do the calculation. (HINT: RI^2.)

(e) Argue that the motor would draw less current if it ran faster.

(f) How is it possible for the motor to run faster but to put out less power?

(g) When it does, will the rate of loss RI^2 be greater or less than before?

✶ **6. Induced e.m.f.** A coil has a 7-cm^2 cross section. When carrying a current of 1 amp, the *B*-field inside is 1.0×10^{-3} tesla. Suppose that the current is brought up from zero to 1 amp in one-tenth of a second.

(a) Calculate the e.m.f. induced in a single turn of wire surrounding the coil.

(b) Call the coil the "primary" circuit and the single turn the "secondary." If the primary has 10,000 turns, what is the "back e.m.f." induced in it while the current is increasing to 1 amp?

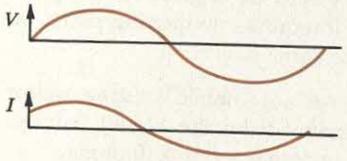
(c) This is a "step-down" transformer. Is the turns ratio the same as the ratio of the voltages induced in primary and secondary? Explain your answer.

(d) Does it matter whether the secondary "turn" is wrapped tightly or looped loosely? Explain.

✶✶ **7. Resonant Circuit**

(a) If an alternating voltage is impressed on a capacitance, the peak current flows one-quarter of a period before the peak voltage is attained. We say that the current "leads" the voltage. Sketch graphs of ac current and voltage using the same time scale. For a circuit with capacitance and resistance in series, the graphs will look like the figure shown. The current leads the voltage by less than a quarter period.

(b) When an ac voltage is impressed on an inductance, the peak current flows one-quarter of a period *after* the peak voltage is attained. We say the current "lags" behind the voltage. Sketch graphs as in part (a).



(c) With an inductance and capacitance is series, the current in the two is in phase, but the voltages are not. Indeed, they are of opposite sign. When an alternating voltage is impressed on a pure resistance, the current is in phase with it. Now suppose that you have all three: resistance, inductance, and capacitance, all in series. Show that, if the capacitance is very small, the current will lead the applied voltage. If the inductance is very large, show that the current will lag the applied voltage. No mathematics is intended here, just a few sentences of explanation.

(d) For a particular choice of inductance and capacitance, the voltage across the one will exactly cancel the voltage across the other. For a given input voltage, this maximizes the current. This is the *resonance* condition! Argue that the current and voltage are in phase at resonance. (HINT: Imagine a large positive voltage across the capacitance, an equally large negative voltage across the inductance, and a small positive voltage across the resistance.)

✶ **8. Dimensional Analysis** The unit of capacitance is the farad. The unit of inductance is the henry.

(a) Recall the definitions, and show that 1 henry × 1 farad = 1 sec^2.

(b) The unit of resistance is the ohm. Remembering that 1 ohm = 1 volt/1 amp, combine ohms and henrys is some way to obtain seconds. (HINT: There is only one way.)

(c) Combine ohms and farads to obtain seconds.

✶✶✶ **9. "Derivation" of the Maxwell Induction Law** At a distance *r* from a long straight wire carrying a current *I*, the magnetic field is given by

$$B = \frac{2k_m I}{r}$$

Another way to say that is that the circulation of the *B*-field around a path surrounding the wire (make it a circle) is

$$\text{circulation} = B \times 2\pi r = 4\pi \times k_m I$$

(a) What is the circulation of the *B*-field around a straight wire carrying 1 amp?

(b) Now suppose that the wire has a gap in it—say, a small parallel-plate capacitor. We might expect that the existence of such a gap would not alter the circulation of the *B*-field around the wire as long as it carries the same current. Of course, charge will pile up on the two sides of the gap, positive charge on one plate, negative charge on the other. Now we want to write the circulation formula not in terms of the current, but in terms of the rate at which the electric field is changing between the capacitor plates. Remember, the current into one plate and out of the other represents the rate at which the charge on the capacitor is changing. The charge on the capacitor is proportional to the flux of electric field between the plates:

$$Q = \frac{1}{4\pi k_e} \times E \times \text{area}$$
$$= \frac{1}{4\pi k_e} \times \text{electric flux}$$

PROBLEM 13.47

hence

$$I = \left(\frac{1}{4\pi k_e}\right) \times \text{(time rate of change of electric flux)}$$

Combine:

$$\text{circulation of } B\text{-field} = \frac{4\pi k_m}{4\pi k_e} \times \begin{matrix}\text{(time rate of change of} \\ \text{electric flux)}\end{matrix}$$

How fast is the electric flux changing in the gap in the 1-amp current? ________ volt m/sec. (HINT: Show that volts × meters is the appropriate unit of electric flux.)

(c) Calculate the proportionality constant k_e/k_m. (HINT: Get the units right. Take the square root of what you get and compare with Equation 5.)

✱✱ 10. **Motional e.m.f. Again** Back to Problem 1, in which an e.m.f. of 0.3 volts is induced in a 0.20-meter-wide loop moving through a 0.5-tesla magnetic field at 3 m/sec. Let the loop have a resistance of 0.15 ohms.

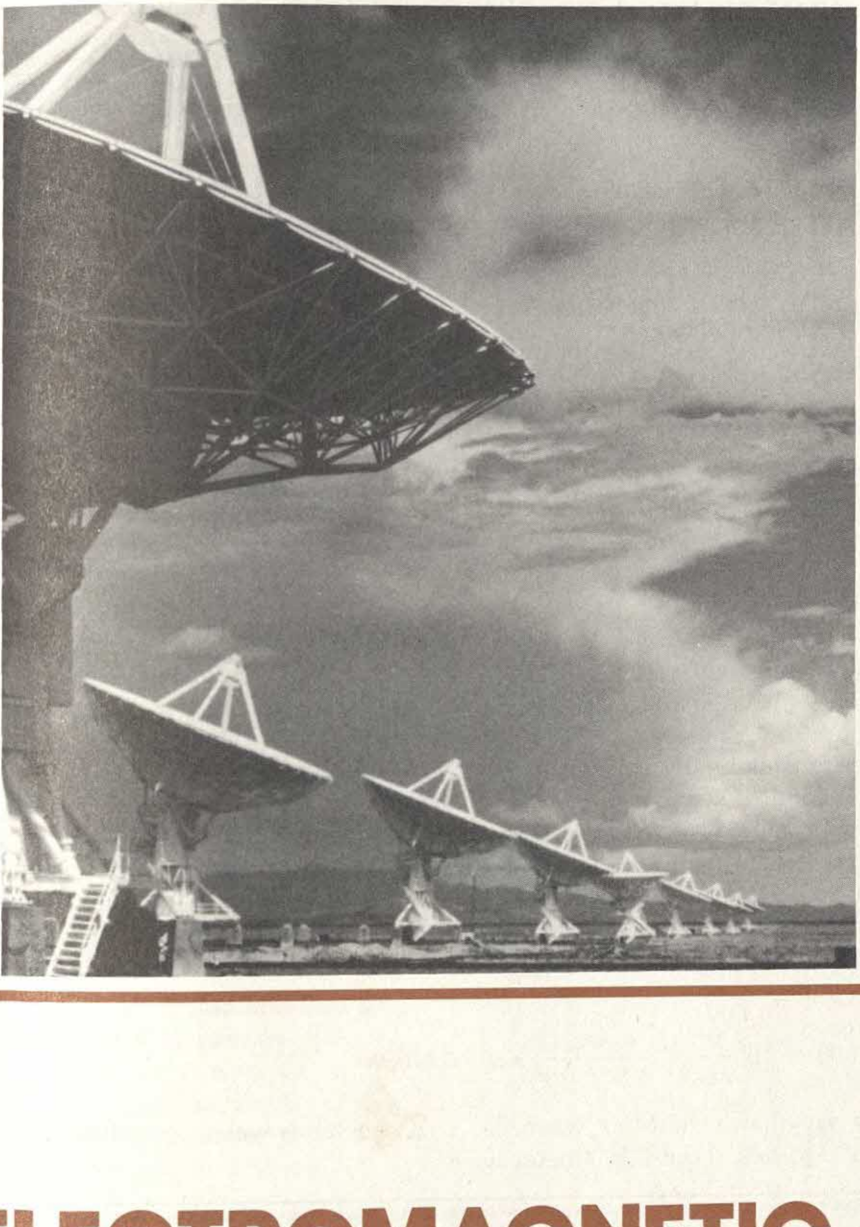
(a) Calculate the induced current in the loop. Clockwise or counterclockwise?

(b) The left-hand end of the wire loop experiences a magnetic force when this current flows across the magnetic field. Calculate magnitude and direction (Chapter 15, Eq. 1).

(c) To keep the loop moving at 3 m/sec, whoever is pulling the loop has to exert an equal and opposite force. Calculate the rate at which this force does work. (HINT: newtons × m/sec = watts.)

(d) The work per unit charge in going around the circuit (the loop) is 0.3 volts. The power dissipated in the resistance of the loop is the work per unit charge times charge per unit time: power = RI^2. Calculate. If the answers to parts (c) and (d) were not the same, would you be worried? In Chapter 4, Energy and Power, we talked about energy bookkeeping. What is happening here is that all the work that goes into pulling the loop at constant speed appears as heat. It has to. There is no place to store energy in this situation.

NOTE: The self-inductance of the loop is so small that the current is essentially constant during the motion, so the magnetic-field energy of the induced current is constant also.



Directional antennas making up the VLA (Very Large Array) radio telescope in Socorro, New Mexico. For accurate location of stellar sources of kilohertz radiation this telescope has an effective diameter of 30 kilometers.

17

ELECTROMAGNETIC RADIATION AND PHOTONS

SOME QUESTIONS THIS CHAPTER WILL ANSWER

Explain why wavelength is inversely proportional to frequency.

Explain what the size of Planck's constant has to do with the energy of a photon.

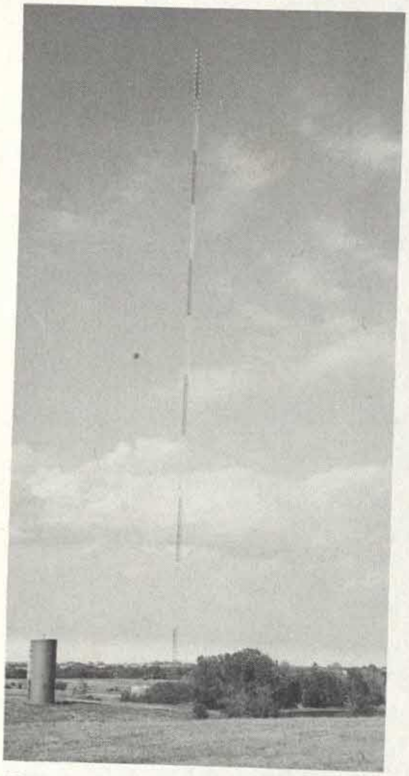
Why is the threshold wavelength for the photoelectric effect different for different metals?

Why does a wave train passed through a fast shutter not have an accurately measurable frequency?

Why does doubling the number of excited atoms in a laser quadruple the intensity of the laser beam?

Argue that the Compton effect offers a direct measure of the momentum of a photon.

What is a Laue pattern?



FM antenna. The middle of the FM band has wavelengths around 3 meters, so an efficient FM antenna might be about 1 meter long. But it has to be high up, because those short waves need line-of-sight transmission. Unlike the waves of the AM band (wavelengths around 300 meters in the middle of the band), they don't diffract readily around large obstacles.

17.1 FREQUENCY AND WAVELENGTH

The top of your **AM** radio dial is marked 1500. That means 1500 kilohertz, or 1.5 megahertz. It means the radio station that broadcasts in that frequency "band" emits electromagnetic waves that repeat their signal patterns around one-and-a-half million times per second. In the vertical antenna at the transmitter the current starts to flow *up* once every two-thirds of a microsecond. It starts to flow *down* one-third of a microsecond after it starts to flow up. The wave travels away from the antenna at 3.00×10^8 meters per second. That, you may recall, is the free-space speed of all electromagnetic waves. In the two-thirds of a microsecond it takes to emit one full wave, the wave travels 200 meters. By the time the current starts to flow *up* the antenna, the part of the wave emitted the last time current started to flow up has gone

$$3.00 \times 10^8 \frac{\text{m}}{\text{sec}} \times \frac{\text{sec}}{1.5 \times 10^6} = 2 \times 10^2 \text{ m}$$

We say that a 1.5-MHz wave has a 200-meter wavelength. Many people prefer to talk about a 200-meter wave.

EXERCISE 1 The bottom of the AM band is 550 kHz. How long are those waves?

EXERCISE 2 A station near the middle of the FM dial transmits at a frequency around 100 MHz. What is the wavelength of those waves?

EXERCISE 3 Commercial television (VHF) uses wavelengths between 1.4 and 5.6 meters. What are the corresponding frequencies?

EXERCISE 4 Light of wavelength 500 nm is green. What is its frequency?

EXERCISE 5 Write down the equation relating the frequency ν, the wavelength λ, and the wave speed c. Remember, the period of the wave is $1/\nu$, and it travels one wavelength in one period.

You may have noticed that the transmitting antennas used to radiate long waves (commercial AM radio) are taller than those used for short waves (police radio, CB). The reason is that for efficient radiation the antenna should not be too much shorter than the wavelength. A quarter wavelength is close to optimal. The same goes for the receiving antenna. That is why you pull out the extensible antennas with which portable TV sets are equipped. You may have noticed that, in general, shorter objects radiate shorter wavelengths. The vibrations of molecules are in the **infrared.** Atoms radiate at frequencies ranging from the infrared up through the **visible,** the **ultraviolet,** and into the **x-ray** region. The electromagnetic radiation from the nucleus is called **gamma radiation;** it has still shorter wavelengths (Fig. 17.1).

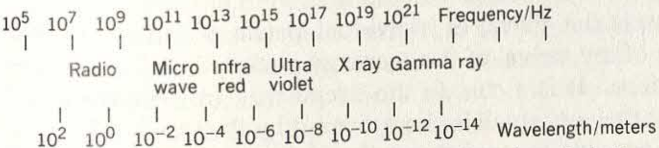


FIGURE 17.1 Scale of frequencies and wavelengths of electromagnetic waves. The visible spectrum occupies less than one-third of a decade between the infrared and the ultraviolet.

Characteristic Spectra

In Chapter 10, Sound and Hearing, we talked about the resonance frequencies of objects of various sizes. That concept came up in connection with standing waves. It was not surprising that the characteristic vibrations of a long string radiated long-wavelength sound waves—low frequencies. The double bass plays lower notes than the violin. The piccolo has a shorter air column than the flute and is used to play higher notes. The characteristic frequencies—we called them natural frequencies—of a string or air column are relatively easy to describe. We talked about a fundamental and higher harmonics, about the shape of the wave. The vibrations of a polyatomic molecule are more complicated. We have to be concerned with many degrees of freedom. Even as simple a structure as an atom can vibrate and radiate at a large number of different frequencies. Rarely is the vibration purely sinusoidal, single-frequency. A collection of molecules may radiate many distinct frequencies at the same time. We say *distinct frequencies*. The spectroscopist says **spectral lines.** The radiation from individual atoms and molecules is not a blur of frequencies. Indeed, each atomic and molecular species has a characteristic spectrum of rather sharp frequencies. Each has its own signature. That is why spectroscopy is the chemist's sharpest tool for identifying chemical elements and compounds.

You might ask why all the atoms of a given species radiate at exactly the same set of distinct frequencies. That question is answered in Chapter 18. As you may have guessed, it has to do with the "quantum" aspects of matter and radiation. In the next section we explore the quantum nature of radiation.

17.2 PHOTONS

A **photomultiplier** is a sensitive detector of visible light. It is really a combination detector and amplifier so that it can respond even in very weak light. The electrical output of such a detector can be read on a meter. At low input levels, the needle quivers. The signal is "noisy." If it is displayed on an oscilloscope, the trace is spiky. In extremely weak light, there is only an occasional spike. Each spike indicates that some light energy has arrived at the detector. But the spikes are not regularly spaced, not rhythmical. They seem to come at random. Sometimes two or three come close together. Sometimes there will be a long period without one.[1]

What we are seeing is the arrival of individual *quanta* of radiation: **photons.** The irregularity of the arrival of these energy packets is not an artifact of the measuring process. It is a clue to the irregularity of their emission. Indeed, the physics of the very small is characterized by the statistical nature of all events. It is impossible to predict exactly when the next photon will be emitted! The best that can be accurately known is the probability distribution of the times of emission. (See Problem 7, Poisson Distribution.)

Size of the Quantum. Photoelectric Effect

Are all the arriving light quanta the same size? Experiments described in this section show that the energy of a photon is determined uniquely by the frequency ν of the radiation. The two are proportional:

$$\text{photon energy} = h\nu$$

High frequencies have energetic photons. The reason we don't notice the "graininess" of light in everyday living is that the proportionality constant h is so small:

Planck's constant $h = 6.6256 \times 10^{-34}$ joule seconds

So the number of photons emitted per second by photon sources like light bulbs and TV antennas is just enormous.

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[1] The movie "Photons" demonstrates this phenomenon beautifully. (Educational Services Inc.; distributed by Modern Learning Aids).

EXERCISE 6 Show that a photon of green light ($\lambda = 500$ nm) has an energy of 2.5 electron volts.

In a photomultiplier tube (see Problem 10 for sketch), light comes in through the glass and is absorbed by a metal "cathode" surrounded by a good vacuum. The energy of a photon can be given to an electron in the metal. If the electron moves off fast enough and in the right direction, it can get out of the metal. Of course, the surface of the metal tries to pull it back in. To get photoelectrons out of the metal, the photon energy has to be big enough to overcome this attraction. The work needed just to pull an electron out of the metal is called the **work function** of the metal. If the energy of the photon is greater than the work function, photoelectrons can be produced. The excess energy can go into kinetic energy of the photoelectrons. The maximum kinetic energy a photoelectron can have is the difference between the photon energy $h\nu$ and the work function ϕ:

Einstein's photoelectric equation: $\text{K.E.}_{\text{max}} = h\nu - \phi$ (Eq. 1)

We return to the photomultiplier in Problem 10, Photomultiplier. Here we concentrate on what we can learn from the photoelectric effect. Suppose that you did not know Planck's constant. Measuring the maximum kinetic energy of the photoelectrons would give you a way of measuring the energy of the photons. Using Eq. 1, all you need is the work function—how hard the metal hangs on to its electrons. There are, to be sure, electrostatic methods of measuring work function. But it comes directly out of a photoelectric experiment, provided that you have light of two or more sharp wavelengths: Measure the maximum kinetic energy of the photoelectrons excited by each spectral color. Plot it versus the frequency ν of the light. According to our Eq. 1, the graph should be a straight line (Fig. 17.2). Its slope is Planck's constant h. If the straight line is extrapolated to zero frequency, the intercept is $-\phi$.

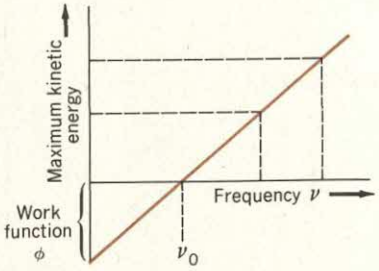


FIGURE 17.2 Photoelectric effect. At the threshold frequency ν_0 the photon energy is just equal to the work function of the photocathode.

Evidently, there is a **threshold frequency** $\nu_{\text{threshold}}$ for the photoelectric effect. Photons with energy less than $h\nu_{\text{threshold}}$ cannot free electrons from the surface of the metal.

work function $\phi = h\nu_{\text{threshold}}$

That threshold frequency is different for different metals. Metals that are chemically very reactive hang on to their electrons weakly and have threshold frequencies in the red or even infrared. Metals with large work functions hang on tightly and have threshold frequencies toward the violet or ultraviolet. Table 17.1 tests your chemical intuition against the measured thresholds. You may want to recall ideas about "electronegativity" and electrode potential from Chapter 13 at this point.

TABLE 17.1 Photoelectric Work Function of Some Metals (Electron Volts)

Cesium	1.8
Potassium	2.2
Sodium	2.3
Calcium	2.7
Zinc	3.1
Lead	4.0
Aluminum	4.1
Silver	4.7
Nickel	5.0

EXERCISE 7 Plot the following photoelectric data in such a way as to find the work function of the metal:

Wavelength of light λ	500	400	nm
Maximum kinetic energy of photoelectrons	0.68	1.30	eV

(HINT: You have the choice of plotting either $\nu = c/\lambda$, or just $1/\lambda$. If you make the latter choice, the slope of the straight line will be hc.)

(a) What is your result for hc, in eV meters?

(b) Use Table 17.1 to identify the metal.

(c) Calculate the threshold wavelength for this metal.

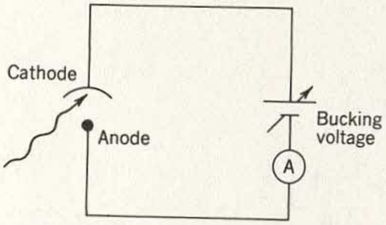


FIGURE 17.3

EXERCISE 8 The sketch (Fig. 17.3) shows the essentials for measuring the photoelectric effect. The cathode is illuminated. The anode is in the dark. When the bucking voltage V is high enough, it cuts off the photocurrent. That is, when $eV >$ maximum kinetic energy, the photoelectrons are driven back into the cathode by the electric field.

(a) Put in the data from Exercise 7. What wavelength of light will have a cutoff of photocurrent when the bucking voltage is 0.3 volts? (HINT: The light is orange.)

(b) If the bucking voltage is reversed, so it becomes an accelerating voltage, predict how the current will depend on V.

EXERCISE 9 What is the shortest wavelength that will excite the photoelectric effect on zinc?

EXERCISE 10 Chapter 12, Wave Optics, made a convincing case for a wave model of light. There is no way a "particle model" of light could explain diffraction. Yet diffraction is a simple consequence of a wave model. With the photoelectric effect, an obvious experiment is to turn up the intensity of the light and see what happens to the photoelectrons. On a wave model, you would predict that their kinetic energy would also increase. That does not happen. The electron kinetic energy depends only on the wavelength (the color) of the light, not on how strong it is. What does happen with increasing light intensity is that the photocurrent grows; in fact, it increases in proportion to the light intensity. Now fill in the blanks:

If the light intensity is doubled, the photocurrent is multiplied times ______, but the bucking voltage needed to cut off the photocurrent is multiplied times ______.

This exercise was intended to convince you that light really does come in quanta. What you notice is that the quantum description is in a sense a particle picture. But it required the idea of wavelength (or frequency), concepts intrinsic to the wave picture. This wave-particle duality is an essential feature of what is called "quantum theory."

Light: Waves or Particles?

In Chapter 12 we studied the diffraction and interference of light. Those two phenomena can only be explained if light travels in the form of waves. Here we learn about the photoelectric effect. That phenomenon requires light quanta—we have called them photons—for its explanation. It tells us that light comes in packages. In order to account for all the properties of light, both the wave and the particle aspects are needed. But our idea of a particle is that it is in a particular place at any particular time. A wave, on the other hand, is continuous. It is distributed rather than localized. What happens in an experiment designed to localize the light in space and time? For example, what does a fast shutter do to a light beam? It chops it, you will say. But what if the light is very weak? Suppose that the light is so weak that, on the average, less than one photon gets through in the time the shutter is open. Can a photon be chopped in half? Surely not! A shutter cannot change the color of the light. The frequency and the wavelength of the light cannot be changed by the chopping. If we think of a photon as a train of waves, what a fast shutter can do is limit the number of waves in the train. But the total energy of the photon is still $h\nu$, quite independent of how many waves there are in the train. The faster the shutter, the smaller the probability that a photon will get through. Again, the quantum description allows only statistical prediction. On any particular opening of the shutter, there is no way to predict whether there will be a photon or not.

EXERCISE 11 A light beam is so weak that it transmits only 10 photons per second. If the shutter is set at 1 second, the average number of photons that get through in one click is 10. With a shutter speed of $\frac{1}{10}$ of a second, the average number is 1. With a speed of 1 millisecond, the average is $\frac{1}{100}$ of a photon. This means that the probability that *any* light gets through when the shutter clicks is 1 chance in 100.

(a) At 1 millisecond, what is the probability that two photons get through? [Answer: Approximately 1 in 10,000.]

(b) Three?

The point is that 1 millisecond is short enough so that it is quite unlikely that more than one photon will make it through: The probability of more than 1 photon in 1 ms $= 10^{-4} + 10^{-6} + 10^{-8} + \ldots = 10^{-4}/0.99$.

(c) Can you prove that? If not, check it on your calculator.

It is as though the photons resist being localized. If you insist on confining a photon within a narrow time interval, it responds by refusing to tell you in advance whether it is going to be there or not. You might wonder whether a diffraction experiment can still be performed with a beam of light so weak that on the average there is less than one photon in the apparatus.

Can one photon interfere with itself? Or does interference require at least two photons?

Experiments have shown that interference effects occur no matter how weak the light, provided you wait long enough. In a double-slit experiment each photon seems to explore both slits. The fringe pattern results because, statistically, more photons end up at the interference maxima, fewer at the minima.

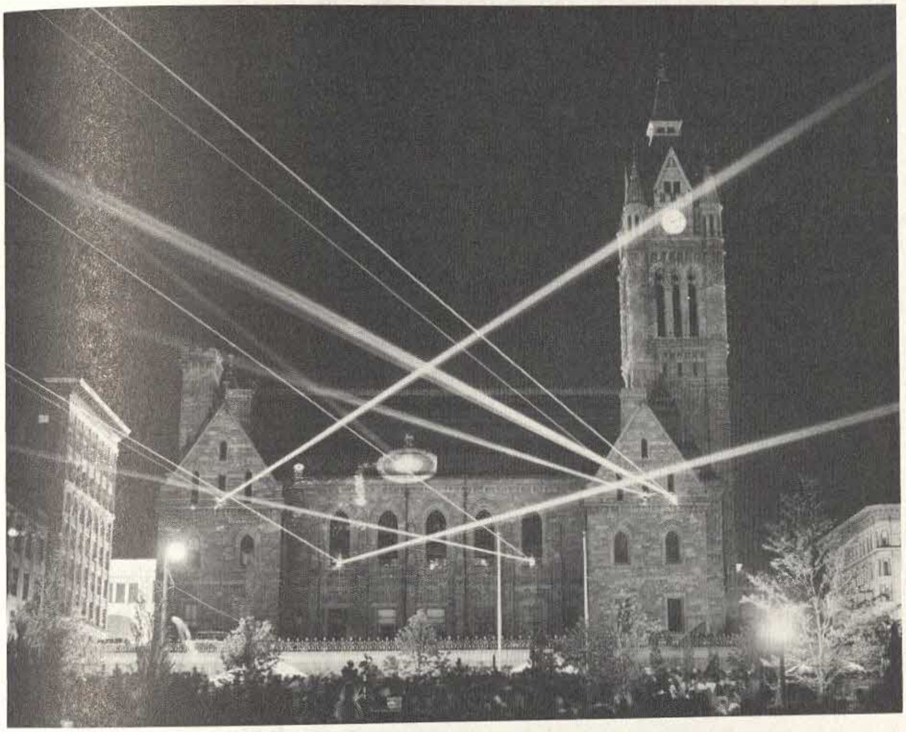
But what if someone really gets inquisitive and wants to know *which* slit of the double-slit apparatus the photon went through? What happens is that any device that could find that out spoils the interference pattern! This result can be stated more generally: Any experiment designed to investigate the particle properties of light spoils the wave properties, and vice versa. That generalization is called the **Complementarity Principle.** It says, in essence, that the wave and particle descriptions can never contradict each other. An experiment to measure frequency or wavelength accurately requires a wavetrain many waves long. Any attempt to localize the light in space or time shortens the wavetrain, spoiling the measurement of its wave properties. (See Problem 16, Bandwidth and Uncertainty.)

17.3 LASER

Remember, the photons come out of a conventional light source quite randomly. The atoms that radiate the light do so independently. Whether one atom is or is not radiating does not affect when a neighboring atom may start to radiate. We say "the photons are not correlated." The technical term is to call them "incoherent." If we think of each atom as emitting a wavetrain with a beginning and an end, the wavetrains coming from different atoms are not in phase.

A **laser** is a light source in which the individual atoms are made to radiate in phase. It may be useful to think of them as holding hands. The electromagnetic field itself provides the coupling among the atoms that keeps them radiating coherently. The word *laser* means *L*ight *A*mplification by *S*timulated *E*mission of *R*adiation. The term **stimulated emission** is the key. The electromagnetic wave incident on an atom excites its oscillation, so that neighboring atoms oscillate in phase. You can guess what is needed for laser action: bathing all the atoms in an electromagnetic wave of very high intensity, very well defined in direction as well as polarization. No attempt is made here to describe how the various kinds of lasers actually work. They all have an accurately plane end window. The light comes out of the end window essentially a plane wave so that the laser beam does not spread out much over short distances.

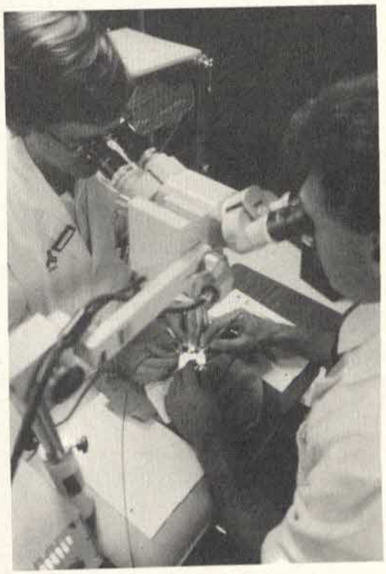
In applications of lasers, the high intensity and the narrowness of the beam are both exploited. Of course, the high intensity is possible *because* the beam remains so narrow. But there is more to it. If 10 oscillators radiate together, so that the resulting electric-field amplitude is 10 times the amplitude of each, the intensity of the wave is 100 times the intensity each one



Laser light show. Strategically placed mirrors reflect the narrow and intense laser beams to make a spectacular display of colored lights. What makes the beams visible is scattering from dust, tiny water droplets, and from density fluctuations in the air (clumps of air molecules). If you don't have occasion to see a laser beam, you may recall having seen a very intense searchlight beam. There you would notice appreciable spreading, almost absent in the laser beam.

would radiate alone. Remember, the energy density of the field goes as the *square* of the field (Eq. 15.4-3). Power goes as voltage *squared.* But doesn't that violate conservation of energy? Shouldn't 10 oscillators have 10 times the power of one oscillator? The answer is yes if you are concerned with an average over all directions. The answer is yes, if the oscillators have random phase. But if they are coherent, then in one direction they all interfere constructively. In that "forward" direction, the intensity goes as the *square* of the number of oscillators. We met this argument back in Chapter 12, Wave Optics. We needed it to explain how the intensity of the principal maxima of a diffraction grating with N lines could be proportional to N^2. The answer there was that the width of the principal maxima was proportional to $1/N$. The total amount of energy still went as N. In the same way, when N atoms in a laser are radiating coherently, the intensity of the emerging light goes as N^2. Again, you will worry about the energy bookkeeping. Note that the beam is confined to a very narrow solid angle, proportional to $1/N$.

You can understand why laser beams are used to guide the surveying teams in building straight tunnels. There, it is the narrowness of the beam that is exploited. Laser surgery exploits the high intensity concentrated in a small area. I had the sight of my left eye preserved thanks to laser surgery on the retina.



Laser surgery.

17.4 THE MOMENTUM OF RADIATION

People talk about the sun "beating down." But you can't actually feel the physical pressure of sunlight. The pressure of light is so small that it requires highly sensitive instruments to detect it. It turns out to be important in astronomy. The large, hot stars are kept from collapsing by radiation pressure. They would be much smaller without the pressure of light coming from the inside to keep the outside layers from falling in.

We can use the ideas of the previous chapter to argue that electromagnetic radiation exerts a force when it is reflected or absorbed. The electric and magnetic field vectors of an electromagnetic wave are always perpendicular to each other, and both are perpendicular to the wave's direction of travel (Fig. 17.4). If the wave meets a free charge, the electric force on the charge will accelerate it. The charge will start to move in the direction of the *E*-field. That direction of motion is perpendicular to the *B*-field, resulting in a magnetic force on the charge along the perpendicular direction—the direction of travel of the wave. The perpendicular component of the force averages to zero because of the sinusoidal nature of the *E* field. But even though the *B* field is also sinusoidal, the Lorentz force is nonvanishing and always in the forward direction. Problem 14 gives the argument that the momentum of the wave is proportional to its energy. Indeed, the ratio is always the same:

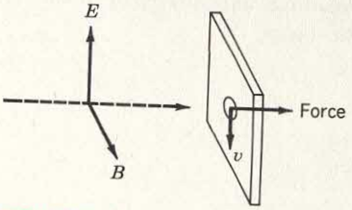


FIGURE 17.4 Momentum of radiation. Electromagnetic wave traveling to the right interacting with an electron. In an *E*-field directed up, electron accelerates downward. In moving down, in perpendicular *B*-field (out), force on electron is to the right.

$$\frac{\text{energy}}{\text{momentum}} = c = \text{speed of electromagnetic waves in free space}$$

Accordingly, if the energy of a photon is $h\nu$, its momentum is $h\nu/c$.

EXERCISE 12 A photon of energy $h\nu$ has wavelength $\lambda = c/\nu$. Write its momentum in terms of λ.

[Answer: Photon momentum $= h/\lambda$

(Eq. 1)

Compton Effect

When a photon interacts with an electron inside a metal and the electron comes out, we call that the photoelectric effect. When a photon interacts with an electron and both electron and photon come out, we call it **Compton scattering.** We might be tempted to say that the photon gives some of its energy to the electron. That would not be a fortunate way to talk. The photon that comes out is a new photon, not part of the old one. Of course, it has less energy. Lower energy means lower frequency, hence longer wavelength than the incident photon.

It is a remarkable feature of the Compton effect that the wavelength difference between the Compton-scattered photon and the incident photon is independent of the wavelength of the incident photon. Problem 19 gives an approximate derivation of that result. The derivation is based simply on the conservation of energy and momentum in the photon-electron collision assuming that the electron is initially free. The discovery of the Compton effect was therefore a compelling piece of evidence that the photon description of radiation is correct. For straight backward scattering, what you might call a head-on collision, the wavelength difference is

$$\lambda_{\text{scattered}} - \lambda_{\text{incident}} = 2\frac{h}{mc} \qquad \text{(Eq. 2)}$$
$$= 4.85 \times 10^{-12}\ \text{m}$$

for scattering by electrons. You may wonder, on seeing the small size of that wavelength change, how people were able to measure it. For visible wavelengths—around 5×10^{-7} m—the wavelength difference would be in the fifth significant figure. The answer is that the Compton effect was discovered with X rays, not with visible light. The wavelength increase predicted by Eq. 2 is a larger fraction of a smaller incident wavelength. For a 1-Å X ray, we are talking about a 5% wavelength change.

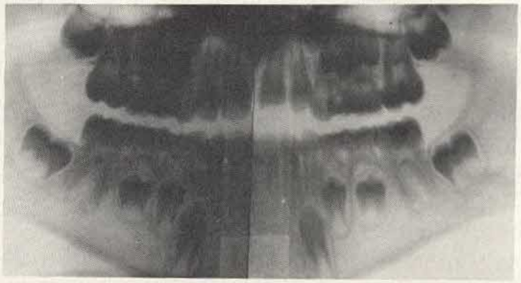
EXERCISE 13 Show that the energy of a 1-Å X ray photon is 12.4 KeV. Your dentist uses X rays with energies 5 or 10 times greater than that, that is, still shorter wavelengths.

EXERCISE 14 What is the wavelength of a 100-keV X ray? If it were Compton scattered straight backward, what would be the wavelength of the scattered photon? The energy? How much energy would the recoiling electron carry off?

17.5 X RAYS

Even though they are invisible, you know a lot about X rays. For instance, you know that they penetrate solid objects much more easily than visible light does. The technical way to say this is that the absorption coefficient of most materials for electromagnetic waves decreases as the wavelength gets shorter. We talk about hard X rays (short wavelength, high photon energy) as being more penetrating than soft X rays (long wavelength, low photon energy). As a rule, materials with heavier atoms have larger absorption coefficients than those with lighter atoms. That is why a dental X ray shows the gum tissue as a gray area, the tooth as a slightly structured white shadow, and the silver filling as a completely unexposed gap.

You also know that exposing your body to X rays is dangerous. This is not just because the X rays get inside you. After all, visible light gets inside you too. Hold your hand up to a strong light and you can see that some red light gets through the soft tissue. But it is completely harmless. Unlike the



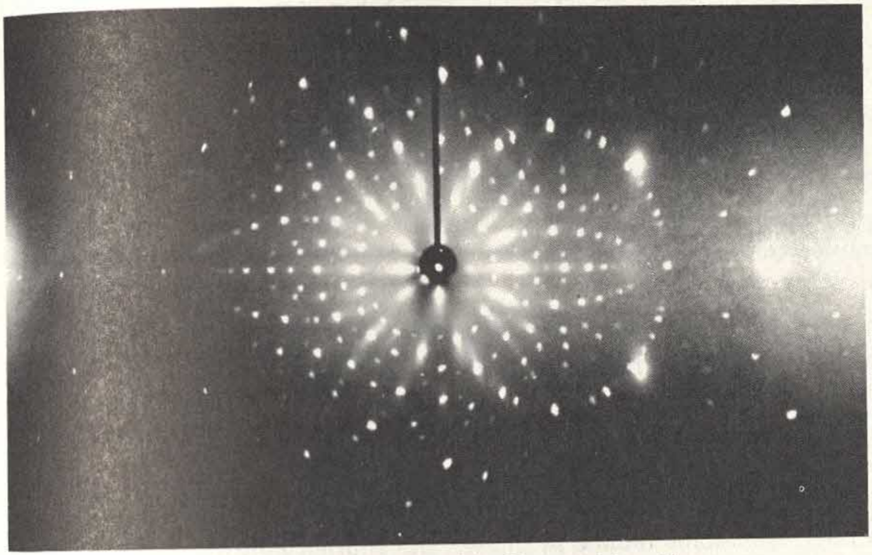
A dental X ray.

An X ray of a human hand.

visible rays, the X rays are an **ionizing radiation.** They are able to tear electrons off the atoms and molecules of the materials they pass through. To intuit why X rays are ionizing and visible rays are not, remember that an X ray photon carries much more momentum than a visible photon.

EXERCISE 15 A photon of red light has an energy around 2 eV. A photon of a dental X ray might have an energy of 80,000 eV. The ratio of wavelengths is 40,000:1, so the ratio of momenta is __________________.

Perhaps you do not know how the wavelengths of X rays are measured. With visible light, the diffraction grating provided wavelength measurements of great accuracy. Gratings can be ruled so that the "slit separation" is just two or three wavelengths of light. That means two or three "orders" of diffraction maxima can be seen on each side of the central maximum, two or three complete spectra of the light being analyzed. An optical grating allows high dispersion, that is, large angles between spectral lines of different wavelength. Those are closely ruled gratings, of course. If such gratings were to be used for X rays, with wavelengths thousands of times smaller, there would be thousands of orders, and very little dispersion. Good wavelength measurements require gratings with even closer spacing between the "slits." Nature has provided such gratings: crystals. Instead of manmade rulings, there are rows of atoms, spaced with a regularity no ruling engine could achieve. Of course, the rows are in three dimensions, not just one. That makes the diffraction patterns somewhat complicated. Instead of a pattern of lines, we have a pattern of spots, called **Laue pattern.** Problem 21



Laue pattern from sodium chloride crystal (ordinary table salt).

shows how the positions of Laue spots can be used to interpret the spacings between planes of atoms in the crystal in terms of X-ray wavelength. Of course, once the wavelength of the X rays is known, the **X-ray diffraction** technique can be turned around and used to investigate the structure of crystals.

SUMMARY

Some wavelengths of electromagnetic waves are

Typical AM radio station	300 m
Typical FM radio station	3 m
Typical TV station	2 m
Microwave oven	12 cm
Infrared	greater than 700 nm
Visible	400 to 700 nm
Ultraviolet	less than 400 nm
Typical dental x-ray	0.01 nm

Free-space speed is universal = 30 cm/nsec.

Small antennas are best for radiating short wavelengths.

Atoms and molecules radiate electromagnetic waves of well-defined frequencies. These spectral lines are highly specific identifiers of their source.

The spiky fluctuations of weak light are evidence of its discrete nature: Light comes in packages (photons). Each wavelength has a unique photon energy: $E = h\nu = hc/\lambda$.

According to Einstein's photoelectric equation $h\nu = \phi + \text{K.E.}_{\text{max}}$, the energy of the photon goes partly into work to overcome the surface barrier, partly into kinetic energy of the photoelectron.

Photoelectric measurements with several wavelengths give a straight-line graph whose slope gives Planck's constant and whose intercept gives the work function.

The chemically more reactive metals tend to have lower work functions.

A photon cannot be chopped in half. But a shorter wavetrain has a less-well-defined frequency.

The wave description and the photon description of electromagnetic radiation are complementary. The measurement of a particle property spoils the accuracy of a later measurement of a wave property, and vice versa.

In a laser, the atoms radiate in phase. The emitted light beam can be very narrow and very intense.

The momentum carried by an electromagnetic wave is proportional to its energy.

In Compton scattering, the wavelength increase is independent of the incident wavelength. Accordingly, the Compton effect is "big" for x rays.

X rays are an ionizing radiation.

X-ray diffraction is used to study the structure of crystals.

KEY TERMS

c Speed of electromagnetic waves in free space = 2.9979×10^8 m/sec.

ELF Extremely Low Frequency radiation. A few hundred hertz and below. Useful if penetration into seawater is desired, as in communication with submerged submarines.

Radio wave An electromagnetic wave whose principal frequencies are in the range of a few kilohertz to a few hundred megahertz.

Microwave Electromagnetic wave whose principal frequencies are in the range 10^9 to 10^{12} Hz.

Infrared I.R., also called heat radiation. Electromagnetic waves whose principal frequencies are in the range 10^{12} to 0.5×10^{15} Hz. Since the microwave and IR regions overlap, the distinction between them is based on source: Microwaves are produced by manmade oscillators, infrared by molecules.

Visible light Frequency range 0.4×10^{15} to 0.7×10^{15} Hz. The transition to the infrared on the long-wavelength end and to the ultraviolet on the short-wavelength end of the visible spectrum is different for different individuals.

Ultraviolet Frequency range 10^{15} to 10^{16} Hz. This is the part of the sun's radiation responsible for the "gorgeous suntan."

X ray Frequency range 10^{16} Hz and above.

Gamma ray Electromagnetic radiation originating in the atomic nucleus. Frequencies usually above 10^{20} Hz.

AM Amplitude modulation. A common way of coding information to be transmitted via radio waves. The high-frequency sinusoidal "carrier" wave is multiplied by the low-frequency (usually audio) signal to be sent.

FM Frequency modulation. Another way of coding. Instead of modulating the amplitude of the carrier wave, the information (audio?) signal modulates the frequency of the carrier.

Spectral line A radiation of sharply defined frequency.

Line spectrum Radiation consisting of several sharply defined frequencies. Individual atoms radiate line spectra.

Band spectrum Radiation with closely spaced groups of spectral lines giving the appearance of a continuous range of frequencies. The rotational and vibrational motion of molecules gives rise to band spectra.

Photomultiplier Light detector in which a photoelectron triggers an avalanche of secondary electrons, resulting in a pulse of current.

Photon A quantum of electromagnetic energy. The energy of a photon is $h\nu$, where ν is the frequency and h is Planck's constant.

Planck's constant $h = 6.6256 \times 10^{-34}$ joule sec
$= 4.1356 \times 10^{-15}$ eV sec
$= 1.2397 \times 10^{-6}$ eV meter/c

Photoelectric effect A photon is absorbed by a solid, resulting in the emission of an electron from its surface.

Einstein's photoelectric equation The maximum kinetic energy of photoelectrons from a given surface is $h\nu - \phi$, where ϕ is the work function of the surface.

Work function Energy need to pull an electron out of the surface of the solid. Related to threshold wavelength for photoelectric effect, $\lambda_{\text{threshold}} = hc/\phi$.

Threshold frequency The lowest frequency of light for which a particular material exhibits the photoelectric effect. Related to work function $\phi = h\nu_{\text{threshold}}$.

Wavetrain Sequence of repeated wave patterns. The phase of the oscillations advances continuously and uniformly from the beginning of the wavetrain to the end, "without stuttering." Two independently emitted photons are not, in general, in phase, so we speak of each as constituting a separate wavetrain.

Laser *L*ight *A*mplification by *S*timulated *E*mission of *R*adiation. The term is used for a light source using this principle. The photons emitted by a laser are emitted in phase, that is, coherently. As a result of having long wavetrains, laser light has extremely sharp frequency definition, as well as accurate directionality.

Stimulated emission An electromagnetic wave can excite an atom to radiate in phase with it.

Radiation pressure A surface absorbing electromagnetic waves experiences a force in the direction of travel of the waves. A perfectly reflecting surface experiences twice as much force as an absorbing surface.

Photon momentum $h\nu/c$.

Compton effect When a photon interacts with a free electron and imparts momentum to it, another photon comes off. Energy and momentum are conserved in the interaction. As a consequence, the wavelength shift between the incident and scattered photon is independent of the wavelength of the incident photon.

Ionizing radiation Photons in the far ultraviolet have enough energy to cause ionization when they are absorbed. So do x-rays and gamma rays. The term is also used for beams of *particles* (electrons, neutrons, alpha particles, etc.) of sufficient energy to cause radiation damage.

Laue pattern The diffraction pattern formed on a photographic emulsion by an x-ray beam scattered from a crystal.

X-ray diffraction The study of crystal structure using X rays. By extension, the same technique applied to liquids, polymers, and other large molecules.

Bragg's Law $n\lambda = 2d \sin \alpha$, where d is the spacing between crystal planes, α is the scattering angle, and n is an integer.

Complementarity principle The wave description and the particle description of a physical situation never contradict each other. An experiment designed to measure a wave parameter destroys knowledge of the complementary particle parameter, and vice versa.

Uncertainty principle A measurement of the energy (e.g., of a photon) with precision $\pm\Delta E$ introduces an uncertainty Δt into a subsequent measurement of the time it is emitted given by $\Delta t = h/\Delta E$, where h is Planck's constant.

REVIEW QUESTIONS

- Some books quote the speed of light as 1 foot per nanosecond. How far off is that? ________%
- If ν is the frequency of a periodic wave, then $1/\nu$ is called its ________________.
- In one period, a wave travels a distance equal to its ______________________.
- The product $\lambda\nu$ of the wavelength λ times the frequency ν is called _______. For electromagnetic waves in free space this is a universal constant c = _______.
- What is the period of a 1.5 MHz wave? Its wavelength?
- How far does it travel in one microsecond? In two-thirds of a microsecond?
- The energy of a photon is proportional to the frequency / wavelength / of the radiation.
- Planck's constant is: a universal constant / the same for all metals but different for nonmetals / different for different metals.
- In the term *work function*, what work is meant?
- Which has the larger work function, tungsten or potassium? Which of those two might you use for making a light meter for use in photography?
- If a metal has its photoelectric threshold in the ultraviolet, it does / does not show a photoelectric effect for green light.

- According to Einstein's photoelectric equation, plotting the maximum kinetic energy of photoelectrons versus ______________ yields a straight line. The intercept of that straight line with the "energy" axis yields the ______________.
- Describe an experiment for measuring the maximum kinetic energy of photoelectrons.
- Chopping a light beam with a fast shutter: can / cannot change the energy of the individual photons. It makes the energy: less well / better / defined.
- Interference effects: require at least two photons / show up regardless of how weak the light.
- The momentum transported by a light beam per unit time is proportional to: the wavelength / the energy transported per unit time.
- What is the relationship between the momentum of a photon and the wavelength of the radiation?
- In Compton scattering, momentum is transmitted by the incident photon to the recoil electron so that the scattered photon has: more / less energy than the incident.
- The wavelength shift $\lambda_{\text{scattered}} - \lambda_{\text{incident}}$ in Compton scattering:
  - is greater for high-energy photons.
  - is smaller for high-energy photons.
  - is independent of the energy of the incident photons.
- Dental X rays: pass right through / are strongly attenuated by a silver filling.
- X-ray diffraction is a technique for the study of: cavities in teeth / fractured bones / crystal structure.

☐ QUESTIONS FOR FURTHER THOUGHT

α. **TV Towers** End-fed television antennas radiate most efficiently if they are about one-quarter of a wavelength long. The commercial TV stations broadcast on wavelengths between 1 and 6 meters. Why are the transmitting towers so tall—often over a thousand feet? (See photo on page 482.)

You may need binoculars to convince yourself that the antenna is just a short vertical wire at the top. If you leaf back to the Wave Optics chapter (Chapter 12) to refresh your memory about diffraction of waves, you can convince yourself that waves a few meters in wavelength are transmitted essentially along the "line of sight." Hills and tall buildings are many wavelengths tall and wide, so their "shadows" are quite sharp. Waves diffract around an obstacle only when the dimensions of the obstacle are comparable to, or less than, one wavelength. By reversing that same logic, you will conclude that AM radio stations can be received at much greater distances, distances so great that the curvature of the earth gets between the transmitting and receiving antennas. Use the sagitta formula, $y = \frac{1}{2}x^2/R$ (Sec. 5–2), to calculate how far from a 300-meter-high TV tower a line-of-sight signal can be received at sea level.

Your rabbit-ear receiving antenna should also be approximately $\frac{1}{4}$ wavelength long. Argue that this is the distance between a node and the adjacent antinode of a standing wave. See Sec. 10.4 in the Sound and Hearing chapter to review resonance and standing waves.

β. **Light or Sound?** In vibrating, a molecule close to other molecules radiates elastic waves together with electromagnetic waves. These infrared waves travel together, and it is not even useful to try to distinguish them. The IR frequencies are above the range of hearing and below the range of seeing. Yet people talk about infrared light rather than about infrared sound. Can you guess why?

γ. **Narrow Slit** Bearing in mind how the wavelength of visible light is usually measured, explain why the word *spectral line* is used for an electromagnetic radiation of sharply defined frequency.

NOTE: When the frequency is not sharp, the word *band* is used.

δ. **Signal-to-Noise Ratio** At low light levels, the output of a light detector like a photocell is noisy. At high levels of illumination, the electrical output appears very stable. Explain.

ε. **Ultraviolet Light** Visible radiation passes through transparent matter without causing any appreciable ionization. Soft X rays ionize heavily along their path. How about ultraviolet, the frequency range between the visible and the X-ray region? Is UV ionizing radiation?

NOTE: Some swimming pools use UV to kill germs in their filtering systems. People using sun lamps are instructed to wear dark glasses.

ζ. **Photographic Process** The light-sensitive emulsion consists of tiny grains (crystallites) of silver bromide (AgBr) suspended in gelatin. Impurity atoms have trapped some electrons so that some Ag$^+$ ions are reduced to Ag atoms. When exposed, the absorption of each photon on the grain frees an electron to move within the emulsion. That electron reduces another silver ion, which joins existing silver atoms to form a silver speck on the grain. The collection of silver specks is called the latent image. Later, in the darkroom, the developer dissolves the silver bromide, and silver ions migrate toward the speck and stick to it. Each exposed grain becomes a grain of silver. The development process essentially acts as an amplifier, putting millions of silver atoms where the latent image had only a few.

(a) What is the purpose of the gelatin?

(b) After developing, the unexposed parts of the film are quite transparent. In other words, the film becomes a negative. Why?

(c) Why can the developing solution not be used over and over?

η. **Reflected or Absorbed** Explain why the pressure of light exerted on a perfectly reflecting mirror is twice as great as the pressure exerted by the same light beam on a black object.

θ. **Color Television and X Rays** Color TV tubes may use electron beams with energies as high as 20,000 electron volts.

(a) What is the shortest X-ray wavelength that could be emitted from the front of the tube?

(b) Why are you advised not to sit too close to your color TV?

ι. **Can You Trick the Double Slit?** In talking about photons in connection with a double-slit interference pattern, the question "Which slit did the photon go through?" is somehow forbidden. The Complementarity Principle says that it is a question we are not allowed to ask. What if we try to trick the photons by putting a photon detector into one of the two slits? Argue that the trick is self-defeating. If the detector is 100% efficient, that is, detects all the photons that go through that slit, that is just like closing the slit; so the interference pattern will disappear. If the detector is not 100% efficient, some of the photons will get past the detector without registering. We should get a blurry interference pattern. Of course, for those photons responsible for the pattern we have no information about which slit they chose.

ϰ. **Photons and Double-Slit Interference** You have read that any attempt to find out which of the slits the photon chooses results in destruction of the interference pattern. One way to think of that is as an aspect of the Complementarity Principle: Localizing the photon destroys knowledge of its wavelength, blurring the pattern.

Don't make the mistake of thinking that interference involves more than one photon. An early experiment to dispel the notion that it takes two photons to "interfere" used light of such low intensity that the probability that more than one photon be traveling through the apparatus at any given time was very small. Suppose that you had a picowatt (10^{-12}-watt) source of 500-nm (green) photons. Then the average time between photon emissions is _______ seconds. If the apparatus is 1 meter long, the time a photon spends traveling through it is _______ seconds. At any given instant, the probability that there be a photon in the apparatus is the ratio of the travel time to the average time between photons, or 1 part in _______. You should have got a number less than 1%. Argue that the probability that there be two photons at any particular instant is less than 1 part in 10,000.

PROBLEMS

✶ 1. **The Range of the Visible** Visible light has a range of wavelengths from 400 nm (violet) to 700 nm (red).
(a) What is the corresponding range of frequencies?
(b) Suppose that we laid out the palette of spectral colors like a piano keyboard, that is, on a logarithmic scale (see Chapter 10). What fraction of an octave does the visible frequency spectrum span? Remember, an octave represents a doubling (or halving) of frequency.
[Answer: 0.8]

✶ 2. **Project Seafarer** Radio waves do not penetrate far into good conductors of electricity. As a rule, the shorter the wavelength, the shorter the penetration depth. Project Seafarer uses signals at extremely low frequencies (E.L.F.) to communicate with submerged submarines. A typical frequency might be 76 Hz.
(a) What is the corresponding wavelength?
(b) Why do you think the Navy chose wilderness areas to bury the antennas for this communications system?
(c) Pure water is actually not a good conductor of electricity. Why is seawater a good conductor?

✶✶ 3. **Microwave Cooking** Waves with wavelengths of several centimeters are efficiently reflected by metallic (very good) conductors but penetrate a few centimeters into ionic (less good) conductors. In a microwave oven the waves are focused so that they are most intense inside the wet food. Why can the food be cooked wrapped in dry paper without cooking the paper?

✶ 4. **Black-Body Radiation** The radiation from a glowing coal is a blur of frequencies. In a solid or liquid, there are so many degrees of freedom, so many possible motions that the sharp characteristic frequencies (spectral lines) of the individual atoms and molecules are broadened into wide frequency bands. A hot solid radiates a continuous spectrum of frequencies that is characteristic of the temperature of the solid, not of the material of which it is made. This spectrum is called the *black-body radiation spectrum*. The word *black* implies that the solid body is able to absorb radiation of all frequencies, not just of a discrete set of frequencies. You may have observed that the coal glows red when you cook hot dogs, and glows orange and then yellow as the temperature is raised in a glass-making furnace. The order of the spectral colors goes right along the frequency scale. The hotter the enclosure, the bluer the glow. That means there is a peak in the spectrum. The higher the temperature, the higher the peak frequency. The *Wien displacement law* states that the two are proportional: Stated in terms of the wavelength of the peak ($\lambda = c/\nu$), it says

$$\frac{c}{\lambda_{\text{peak}}} = (1.034 \times 10^{11}\ \text{Hz/Kelvin})\ T$$

or

$$\frac{hc}{\lambda_{\text{peak}}} = 4.97\ kT$$

where k is Boltzmann's constant 1.38×10^{-23} joules/Kelvin.

(a) Show that assigning a surface temperature just over 6000 K to the sun puts the peak of the spectrum at the observed wavelength around 500 nm (green).

(b) The 1978 Nobel Prize in physics was awarded to Penzias and Wilson for the discovery of the "three-degree background radiation" coming from all directions in space. At what frequency would that peak? Can you guess what kind of a detector they used?

★★★ **5. Black-Body Spectrum (a Numerical Calculation)** The distribution of radiant energy in frequency is given by

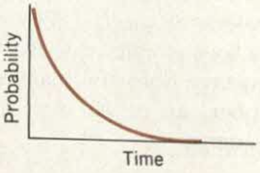
$$\frac{8\pi h\nu^3/c^3}{(e^{h\nu/kT} - 1)}\ \Delta\nu$$

where h is Planck's constant, k is Boltzmann's constant, c is the speed of light, and $\Delta\nu$ is a narrow band of frequencies centered at frequency ν. How much more energy (%) is there in a 1-Hz band near the peak of the spectrum (see Problem 4) than in a 1-Hz band at twice the frequency, which would be in the ultraviolet?

[Answer: 18 times as much, that is 1700% more.]

★★ **6. Random Events** If photons are arriving at a detector at such a rate that, on the average, out of 100 consecutive microsecond intervals only *one* has a photon arriving during it, argue that the probability that *no* photon will arrive in any particular microsecond interval is 99%.

★★ **7. Poisson Distribution** (Mathematical!) With purely random events, how are the times between events distributed? If you know that the events occur 1 second apart on the average, then the probability of occurrence per unit time is one per second. In other words, the probability that an event will occur in any particular microsecond is one in a million. The probability that *no* event will occur in a particular microsecond is $1 - 10^{-6} = 0.999999$. That is close to exact, because the chance that two or more events will occur in the same microsecond is so small. The probability of no event in two microseconds is $(1 - 10^{-6})^2 = 0.999998$. The probability of no event in three microseconds is $(1 - 10^{-6})^3 =$ ______________. The probability of no event in a million microseconds is $(1 - 10^{-6})^{1{,}000{,}000} =$ ______________. If your calculator is not up to that kind of arithmetic, approach it gradually. Work with tenths of a second, for example: $(1 - 10^{-1})^{10} =$ ______________. Then try hundredths: $(1 - 10^{-2})^{100} =$ ______________. And so forth. This is called a limiting process. With patience, you can get as close to the "answer" as you like. The answer has a symbolic name. It is called $1/e$. Now you can show that the probability that no event will occur in two seconds is $1/e^2 =$ ______________. In general, the probability of having an "empty" time interval t is $e^{-t/\text{second}}$. The (remarkable?) conclusion is that the most likely time interval between events is zero. The figure is a graph of this decaying exponential.



PROBLEM 7 Probability distribution of intervals between random events.

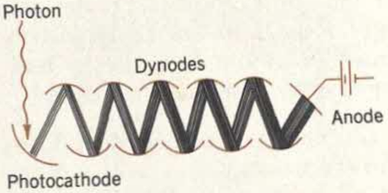
NOTE: You may be worried that something was put over on you. Is the probability of no event in a time interval t the same as the probability distribution of time intervals between events? No, but the two are proportional. The probability of no event for a million microseconds followed by an event in the next (one-million-and-one-th) microsecond is $e^{-1} \times 10^{-6}$. Also, whether an event did or did not occur just before the start of the one-second interval is irrelevant for a truly random process.

✶✶ **8. Photons from a Light Bulb**
(a) Try to estimate how many photons per second are emitted by a 100-watt bulb. Suppose that about 10% of the 100 watts gets out as visible light. The light looks pretty yellow, so an average wavelength might be 550 nm.
(b) Assume that the pupil of the eye has an open area of 20 mm^2. How many photons per second enter when the eye is 10 meters from a 100-watt bulb? (HINT: The area of a sphere is $4\pi r^2$.)
(c) How far away from a 100-watt bulb would an eye receive only 100 photons per second? At such low light levels you would start to notice the graininess of light, that is, you would see the bulb twinkle.

NOTE: Faint stars twinkle because few photons per second get to your eye. Bright stars twinkle too. That is due to temperature and density fluctuations in the atmosphere.
(d) We are 1.5×10^{11} m from the sun and receive 1.4×10^3 watts/m^2 of radiant energy from it. How far away from a star of comparable luminosity would our eye receive only 100 photons per second? What assumptions did you make?

✶ **9. Signal-to-Noise Ratio** In Sec. 10–9 you learned that the random thermal motion of the air molecules limits the useful sensitivity of the ear. The same limitation applies to the eye. A molecule of the retina has an average kinetic energy about two orders of magnitude below the energy of a photon. Planck's constant is small but not too small for the eye to be a photon detector. Calculate the kinetic energy of a gas molecule $\frac{3}{2}kT$, where $k = 1.38 \times 10^{-23}$ joules/Kelvin and T is the absolute temperature, for a body temperature of 37°C (= 310 K), and express in eV.

✶ **10. Photomultiplier** In the photomultiplier tube an incoming photon strikes the *photocathode* and excites a photoelectron. This is accelerated by an electric field to strike an electrode called a dynode, where it excites several secondary electrons, which go on to the next dynode, and so on. Suppose that there are 10 dynodes, as in the figure, and each electron excites 4 secondaries on the average. Then each photon results in 4^{10} = over a million electrons out of the last dynode to hit the *anode* and contribute to the pulse of measuring current.
(a) If the light signal is very weak—say 100 photons per second—what is the average measuring current?
(b) Suppose that the accelerating voltage in the last stage of amplification is 90 volts. How much energy is there in each "spike" or pulse in the output?
[Answer: 9×10^7 eV.]
(c) What is the average power into the measuring circuit?
________ watts



PROBLEM 10
Schematic of the photomultiplier. This one has 10 stages of amplification.

✶ **11. Solar Battery** A photodiode is a photon detector that converts the energy of incident photons into electrical signals. In the dark, the sandwich of semiconducting layers having varying impurity content (the "*p-n* junction") is just a capacitor. In the light, a photon absorbed close to the junction can give its energy to an electron, which can cross the junction. This results in a brief burst of current. A silicon photodiode, for example, can absorb photons of energy greater than 1.1 electron volts. The voltage generated across the junction is somewhat less than 1 volt. To get an idea of the current and power such a device can supply, suppose that direct sunlight (energy flux approximately 1 kW/m^2) falls on 1 cm^2 of diode surface. Suppose that 5% of the incident photons each excite an electron across the junction at a potential of .75 volts.
(a) The "average" solar photon has an energy of 2.5 eV. What is the maximum current that can be drawn from this solar cell?
(b) How much power does it supply?

✶✶ **12. Coherent Sources** Three vertical radio antennas standing side by side are connected so that they radiate in phase. If turned on individually, at a distance of 1 km each one radiates an electric field of amplitude 1 mV/m. With all three connected, the electric field at 1 km along the perpendicular (central maximum) direction is 3 mV/m. Remember that the intensity is proportional to the square of the electric field.
(a) By what percentage is the intensity reduced if one of the three is turned off?
[Answer: 55%]
(b) By what percentage is the intensity further reduced if one of the remaining two is turned off?

✶✶ **13. Laser Surgery** Suppose that a 10-watt laser is 2% efficient so that the beam carries 0.2 joules of light energy per second. Assume that the tissue to be burned has the density (1 gm/cm^3) and specific heat (4.2 joules per gram per degree) of water. If the beam has a cross section of 10^{-3}mm^2 and penetrates 1 mm into the tissue, by how many degrees will a 5-millisecond pulse heat the tissue?

✶✶ **14. Momentum and Energy Transported by Electromagnetic Waves** An electromagnetic wave with an electric field amplitude E_o has magnetic field amplitude $B_o = E_o/c$. An electron in a field E_o would experience a force eE_o. If it moves in the direction of that force at speed v, the work done by that force per unit time is eE_ov. The magnetic force eB_ov is in the direction perpendicular to both B and v, that is, along the direction of travel of the wave. So the momentum transferred from the wave to the electron is directed along the direction of travel of the wave. The ratio of the power eE_ov supplied by the wave to the force eB_ov it exerts is $E_o/B_o = c$. Since power is energy transferred per unit time and force is momentum transferred per unit time, we have that the ratio of the energy of the wave to its momentum is a universal constant:

$$\frac{\text{energy}}{\text{momentum}} = c$$
$$= \text{speed of electromagnetic waves in free space}$$

(a) If all the light from a 100-watt bulb (i.e., 10 watts) is concentrated in a parallel beam, how much force will it exert on an absorbing surface?
(b) Argue that the force on a perfectly reflecting surface will be twice that.

✶✶ **15. Solar Sailcraft** A solar-powered space vehicle is to have a large reflecting sail pushed by the pressure of the sunlight. Near the earth, the flux of radiation from the sun is 1.4 kW/m^2.
(a) Calculate the force per square kilometer of sail facing the sun.
(b) Suppose that the sail material (aluminum-coated mylar?) has a mass of 1 gram per square meter of sail. What is the maximum acceleration the vehicle could achieve, neglecting the mass of passengers and freight? Did you have to take into account the gravitational attraction of the sun ($= 6 \times 10^{-3}$ newton per kilogram)?

✶✶ **16. Bandwidth and Uncertainty** This problem has to do with how accurately frequency and wavelength can be measured for a wavetrain. The greater the number of waves in the train, the better the possible accuracy. Without worrying about the precise definition of "uncertainty," we can say that with a wavetrain 100 waves long a frequency measurement good to one part in 100 is possible. This is because the ends of the wavetrain are not well defined. It is hard to tell whether there are 99 waves or 100 or 101. Take a 500-nm photon (red) emitted in 10 nsec.
(a) Show that its frequency is 5×10^{14} Hz. Accordingly, in 10^{-8} seconds, a wavetrain 5×10^6 cycles long passes any point.
(b) A measurement of frequency would involve counting waves and dividing by the time it took them to pass. Neglecting any error in timing 10 nanoseconds, the uncertainty in the frequency measurement is 1 part in 5×10^6, or ________ Hz.
(c) A measurement of wavelength would involve measuring the length of the wavetrain and dividing by the number of waves in the train. Again neglecting any error in the length measurement, the wavelength can be measured to 1 part in 5 million. For a wavelength of 5×10^{-7} m, this gives a wavelength uncertainty of ________ m. We now repeat the argument, letting N be the number of waves in the wavetrain, ν the frequency, and λ the wavelength. The uncertainty in N we call ΔN, and so forth. We set $\Delta N = 1$. If the wavetrain takes a time t to pass, then its frequency is $\nu = N/t$, hence $\Delta\nu = 1/t$. More carefully stated, this is called the Bandwidth Theorem. The space occupied by the wavetrain is how far it travels in the time t, or $L = ct$. Since the wavelength is defined as $\lambda = L/N$, show that $\Delta\lambda = \lambda^2/L$.
(d) The photon energy is $h\nu$, and it can also be known to an accuracy of one part in N. Show that the uncertainty in the photon energy is h/t. This is one form of the Heisenberg Uncertainty Principle.

✶ **17. Momentum Uncertainty.** (Continuation of Previous Problem) The momentum of a photon is $h\nu/c$. The uncertainty in the frequency is $\Delta\nu = 1/t$, where t is the time interval within which the photon is constrained, that is, the uncertainty in the time of its emission. In this time t it travels a distance $L = ct$. Show that the uncertainty in the photon's momentum is h/L. This is another form of the Heisenberg Uncertainty Principle.

✶✶ **18. Compton Effect** A 1-Angstrom x ray ($\lambda = 10^{-10}$ m $= 0.1$ nm) has a head-on interaction with a stationary electron. "Head-on" implies that the scattered x ray will go back the other way.
(a) Show that the incident photon energy $h\nu$ is 0.0124 MeV.
(b) Since the momentum of a photon is the energy divided by the speed of light, we can write the momentum of our incident photon as 0.0124 MeV/c. (Using eV/c as a momentum unit is conventional in high-energy physics.) The forward momentum of the incident photon goes into forward

momentum p of the recoiling electron plus backward momentum $-h\nu^*/c$ of the scattered photon:

$$\frac{h\nu}{c} = p - \frac{h\nu^*}{c}$$

The energy difference $h\nu - h\nu^*$ all goes into kinetic energy of the electron:

$$h\nu - h\nu^* = \frac{p^2}{2m}$$

Eliminate $h\nu^*$ between these two equations to arrive at an equation for the recoil momentum of the electron. Show that the solution of this quadratic equation is $p = 0.024$ MeV/c. Why do you not have to worry about the other root of the quadratic?

(c) Find the wavelength $hc/h\nu^*$ of the scattered photon. You can use either the energy or the momentum conservation equation.

(d) Show that the wavelength change is 0.0485 Å.

✶✶ 19. **Compton Effect: Derivation** We want to show that the wavelength difference $\Delta\lambda$ between the incident and scattered photons is independent of the incident wavelength:

$$\Delta\lambda = \lambda^* - \lambda = hc\left(\frac{1}{h\nu^*} - \frac{1}{h\nu}\right) = hc\left(\frac{h\nu - h\nu^*}{h\nu^* \cdot h\nu}\right)$$

The numerator in that last parenthesis is just the kinetic energy K of the electron. To get the denominator, write the equations of energy and momentum conservation (see Problem 17) in the form

$$h\nu - h\nu^* = K, \qquad h\nu + h\nu^* = pc$$

Then square them and subtract. Show that the denominator comes out $\frac{1}{4}(p^2c^2 - K^2)$. Dropping that K^2 gives $\Delta\lambda = 2\,h/mc$. Show that.

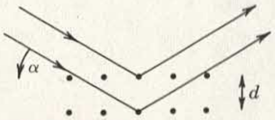
NOTE: In a proper relativistic derivation, kinetic energy is not $p^2/2m$ but rather $\sqrt{p^2c^2 + (mc^2)^2} - mc^2$. That makes the derivation exact. Relativistic mechanics is taken up in Chapter 20.

✶ 20. **Production of X Rays** The source of the X rays in an X-ray tube is a metal target bombarded by a beam of fast electrons. As these electrons are brought to a stop in the target, they radiate a continuous spectrum of frequencies. This radiation is called *Bremsstrahlung,* literally "braking radiation." High-energy electrons can also penetrate deep within the atomic core and excite high characteristic frequencies, lines of the X-ray spectrum of the target metal. What is the shortest x-ray wavelength that can be excited by a 100-keV electron beam?

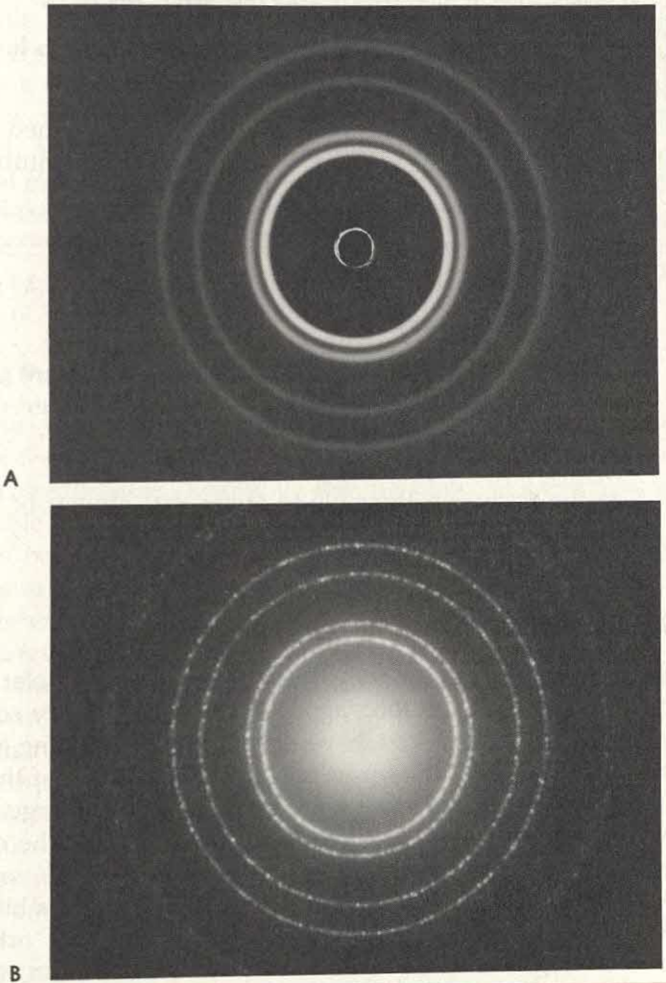
Patient in a CAT (computerized axial tomography) scan machine. This is an x-ray machine that uses a computer to produce an essentially three-dimensional picture of an internal organ.

NOTE: That is a beam of electrons that has been accelerated through a potential difference of 10^5 volts in the electron gun.

✶✶ 21. **Bragg's Law** In diffraction from a three-dimensional crystal, the intensity maxima occur at those angles at which adjacent planes of atoms contribute constructively interfering rays. If successive parallel planes are a distance d apart, then rays making an angle α with those planes interfere constructively, provided that the path difference $2d \sin\alpha$ is an integer number of wavelengths of the X rays (see the figure). This is called **Bragg's Law:** $n\lambda = 2d \sin\alpha$. In the cubic crystal NaCl (table salt), the planes parallel to the cleavage directions are 5.63 Å apart. At what angle α is the Laue spot corresponding to $n = 1$ for X rays of wavelength 1 Å? Show that this spot is actually at an angle 2α from the direction of incidence. If you have a magnifying glass, look at a grain of salt to see how beautifully perpendicular the cleavage planes are.



PROBLEM 21 Show that the path difference between waves reflected from the upper plane and waves reflected from the lower plane is $2d \sin\alpha$.



Diffraction pattern obtained *(a)* with X rays and *(b)* with electrons incident on a polycrystalline sample of aluminum.

18

THE QUANTUM NATURE OF MATTER

SOME QUESTIONS THIS CHAPTER WILL ANSWER

Explain how atomic energy levels give rise to sharp lines in the emission spectra of elements.

If the wavefunction of a particle has a well-defined wavelength, how does the De Broglie relation determine the momentum?

For a particle confined in one dimension, how much higher in energy is the first excited state than the ground state?

For a hydrogen atom, how much higher in energy is the first excited state than the ground state?

Explain how the Uncertainty Principle takes account of the perturbation the observer introduces into the system being observed.

How is electron diffraction observed?

Use the Exclusion Principle to argue that lithium ($Z = 3$) cannot have all its electrons in an $n = 1$ state.

18.1 SOME PARADOXES

The idea that an atom is something like a miniature solar system was probably planted with you long ago. Was it in elementary school? All through your science courses you have seen pictures showing electrons in orbits around the nucleus, drawn in a way to remind you of the orbits of planets around the sun. There was a massive positively charged nucleus strongly attracting those light negatively charged electrons. The electrons managed to avoid falling in by whirling around at great speed. If you took a chemistry course, the pictures became less fanciful but perhaps a bit vague. Instead of talking about orbits, the chemists used words like **orbital** and **electron cloud.** If they turned mathematical, they used the term **wavefunction.**

It certainly is an appealing model of the atom: Electrons revolving in stable orbits around a nucleus to which they are attracted by the electrostatic force. Such a model seems to explain how the atom can radiate light of a single frequency: According to the electromagnetic theory of Chapter 16, an accelerated charge radiates electromagnetic waves. If a charge moves in a circle, it should radiate electromagnetic waves whose frequency is precisely the frequency of the circular motion. But that argument contains some paradoxes.

Conservation-of-Energy Paradox

If an atom radiates away electromagnetic energy, that energy has to come from somewhere. Throughout this course, accurate energy bookkeeping has always been possible. Where does the atom get the energy it radiates away? Of course, it could get energy from the electrostatic field of the positive nucleus and the negative electrons. The revolving electron could move closer to the nucleus as it radiates, lowering its electrical potential energy. The electrostatic force is attractive, so energy bookkeeping would be satisfied this way. Such a spiraling in with a continuously shrinking orbit radius is what the laws of mechanics and electromagnetism predict. It is exactly what happens with artificial satellites when they lose energy through air friction. They spiral closer and closer to the earth. As the orbit gets smaller, the frequency increases. That is because the inverse-square force increases as the separation decreases. But there goes our model with its stable orbit! Stable orbits were needed to explain the sharp spectral lines. We confront an apparent paradox.

Identical-Atoms Paradox

Here is another one. How much alike are two atoms of the same element? The answer is that they are completely identical. They are so identical that we call them indistinguishable. Logically, the operation of interchanging two indistinguishable atoms is not an operation at all. There is no way to tell whether it has been performed or not! A quick digression: You have probably heard about isotopes. Those are atoms having the same chemical properties but different nuclear mass. So when we say atoms of the same element are indistinguishable, we should really say atoms of the same isotope. There is no way to distinguish one gold atom from another. They don't wear shirts with numbers.

As you know, there is a countable number of elements. Gold atoms are quite different from platinum atoms. Gold has atomic number 79. That means the gold nucleus has 79 protons and the gold atom has 79 electrons. Platinum has atomic number 78; the platinum atom has 78 electrons. There is no such thing as a slightly more goldlike atom of platinum or a slightly more platinumlike atom of gold. There are no in-between atomic species.

Stability-of-Matter Paradox

If the electrons could spiral in toward the nucleus, wouldn't the atom eventually collapse? There is nothing in the laws of mechanics or the laws of the electromagnetic field that explains why the positive and negative charges don't just neutralize each other—with a bang—and wipe each other out. You probably learned in a chemistry course what holds a material like NaCl (table salt) together. The positive sodium ions, Na^+, attract the negative chloride ions, Cl^-, and they form a regular crystal lattice. The electrostatic forces give a quantitative account of the chemical binding of the ions. You may have been taught that all chemical forces are essentially electrostatic in nature. But you probably did not learn why the Na^+ does not fall into the Cl^-. You may have been told that they touch. What is "touching" all about? What is responsible for the short-range repulsive force between atoms? What keeps matter from collapsing? The ideas of "classical physics"—mechanics and electromagnetic theory—cannot explain the stability of matter.

Quantum Mechanics

The paradoxes are resolved by the **quantum theory** of matter, also called quantum mechanics. The laws of the electromagnetic interaction, as we studied them in Chapter 13, are correct and useful in describing "large-scale" physics. Together with the laws of mechanics ($F = ma$), they could explain macroscopic phenomena. But on the atomic scale those laws seem to be violated. The microscopic world does not conform. Quantum theory seems to work all the time. It explains things down to the very smallest-size objects we know. Also, proceeding to larger and larger objects, the predictions of quantum theory agree more and more closely with those of the classical laws. Classical physics may be considered the large-scale approximation to quantum-mechanical physics.

18.2 STATIONARY STATES AND STANDING WAVES

If you have leafed through this chapter, you noticed how few Problems it has at the end. This is because quantum mechanics is more mathematical than this book wants to be. This chapter does not attempt to show you quantum-mechanical calculations. It remains rather descriptive and mostly qualitative.

Here is the general idea: In classical physics, we talk about particles, each having a well-defined position and momentum. Those are idealizations, to be sure. In quantum mechanics, the *particle* is to be represented by an abstract something called a *wavefunction*. The wavefunction carries the information for finding the *probability* of the particle's having certain values of the position and momentum variables. You can see that quantum mechanics is intrinsically statistical.

Those wavefunctions are the solutions of **wave equations.** A wave equation is a partial differential equation with the space coordinates (x, y, z) and a time coordinate (t) as independent variables. The dependent variable, the wavefunction, is usually called Ψ (pronounced psi).

If a wavefunction $\Psi(x, y, z, t)$ is a sinusoidal wave, that is, a wave with a sharply defined wavelength λ, then the particle described by that wavefunction has a sharply defined momentum p given by the **De Broglie relation:**

$$p = \frac{h}{\lambda} \qquad \text{(Eq. 1)}$$

EXERCISE 1 Does the same relation hold between the momentum and the wavelength of a photon? Yes / No. (See Chapter 17.)

But suppose that we are interested in things that don't change in time. Physicists use the term *stationary states.* Then the time variable t drops out. The wavefunctions for stationary states describe *standing waves.* Even without leafing back to Chapter 10, Sound and Hearing, you are likely to remember a special feature of standing waves. Here was the one subject in physics where *integers* appeared! We counted nodes; we counted antinodes. A stretched string can vibrate with one antinode; that is called its fundamental mode. It can vibrate with two antinodes, an octave higher. But there is no standing wave with one-and-a-half antinodes. The statement itself is nonsense. Counting yields discrete numbers of nodes and antinodes. That is why standing waves are always characterized by integers. Now you know how integers get into quantum mechanics: They label the possible stationary states. Instead of having a continuous spectrum of allowed energies, an atom has a discrete spectrum. We use the term *energy levels.* The integers characterizing the stationary states of a system are called its **quantum numbers.**

That is all pretty abstract. An example may help.

One-Dimensional Example

Of course, there is no such thing as a one-dimensional atom. But just to be able to talk about something simple, suppose that an electron were to be confined to move between rigid walls a distance L apart. From Chapter 10 you recall the argument about the possible standing waves. The fundamental mode has the longest wavelength, λ_1.

Fundamental:

$$L = \tfrac{1}{2}\lambda_1$$

For the next-longest wavelength λ_2 we have

Second harmonic:

$$L = \tfrac{2}{2}\lambda_1$$

For the third harmonic we get

Third harmonic:

$$L = \tfrac{3}{2}\lambda_3$$

Figure 18.1 reminds you how that goes. Now bring in the De Broglie relation (Eq. 1) that gives the allowed values of the momentum. For the fundamental mode, the momentum is

$$p_1 = \frac{h}{\lambda_1} = \frac{h}{2L}$$

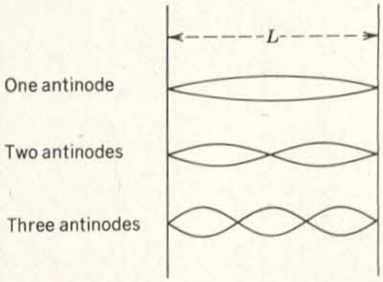


FIGURE 18.1 Standing waves between rigid walls: (a) one antinode; (b) two antinodes; (c) three antinodes.

For the higher modes we get

$$p_2 = \frac{h}{\lambda_2} = \frac{h}{L}$$

$$p_3 = \frac{h}{\lambda_3} = 3\left(\frac{h}{2L}\right)$$

$$\vdots$$

$$p_n = n\left(\frac{h}{2L}\right)$$

The allowed values of the momentum are integer multiples of the lowest value. Now for the allowed values of the energy. The electron's energy $\frac{1}{2}\,mv^2$ can be written in terms of its momentum p as $p^2/2m$. This gives an energy spectrum:

$$E_1 = \frac{p_1^2}{2m} = \frac{h^2}{8mL^2}$$

$$E_2 = \frac{p_2^2}{2m} = \frac{4h^2}{8mL^2}$$

$$\vdots$$

$$E_n = n^2\,\frac{h^2}{8mL^2}$$

in which the allowed energies are 4, 9, 16, 25, and so on, times the energy of the lowest state. That lowest state is conventionally called the **ground state.** The higher states are called excited states: first excited state, second excited state, and so forth.

This one-dimensional example is fictional. Its use here is to develop a vocabulary. Its energy spectrum is quite unlike the spectra of real atoms. In real atoms, the energy difference between the ground state and the first excited state is larger than the energy differences between higher excited states. The higher the quantum numbers, the smaller the energy differences between successive excited states. Finally, there is an energy at which the electron is completely "free." The work to free an electron from the ground state is called the **ionization energy** of the atom.

Hydrogen Atom

Hydrogen has only one electron. Not only is the hydrogen atom simple, but its energy spectrum is also the prototype for the more complicated many-electron atoms. Here is the model: The nucleus is a proton, with a single positive charge e. It is 1836 times more massive than the electron (mass m), which has a single negative charge $-e$. So the motion of the proton is quite negligible. The force pulling the electron in toward the nucleus when it is a distance r away is

$$\text{electrostatic force} = \frac{k_e e^2}{r^2}$$

The ground-state energy (see Problem 2) is

$$\frac{-2\pi^2 m\,(k_e e^2)^2}{h^2} = -1\ \textbf{Rydberg} \qquad \text{(Eq. 2)}$$

Note that this is negative. The energy scale has been chosen so that zero energy means "free electron" with negligible speed. All bound states are negative-energy states. The absolute value of the ground-state energy of hydrogen is called 1 Rydberg and abbreviated 1 Ry.

EXERCISE 2 Substitute the known values for the constants m, $k_e e^2$, and h, and verify that the ground-state energy is -13.6 electron volts. Here are the values:

$m_{\text{electron}} = 0.911 \times 10^{-30}$ kg
$k_e e^2 = 14.4 \times 10^{-10}$ eV m
$h = 4.135 \times 10^{-15}$ eV sec

HINT: 1 eV $= 1.602 \times 10^{-19}$ joules

The formula for the energies of the excited states is

$$\begin{array}{l}\text{energy spectrum of}\\ \text{the hydrogen atom}\end{array}\quad E_n = -\frac{1}{n^2}\,\text{Ry} \qquad \text{(Eq. 3)}$$

The spectrum of allowed energies is shown in Fig. 18.2. Look how the energies of the higher states get closer together: The first excited state has energy $E_2 = -\frac{1}{4}$ Ry; that is $\frac{3}{4}$ Ry above the ground state. The second excited state has energy $E_3 = -\frac{1}{9}$ Ry. That is only $(\frac{1}{4} - \frac{1}{9})$ Ry $= \frac{5}{36}$ Ry above E_2.

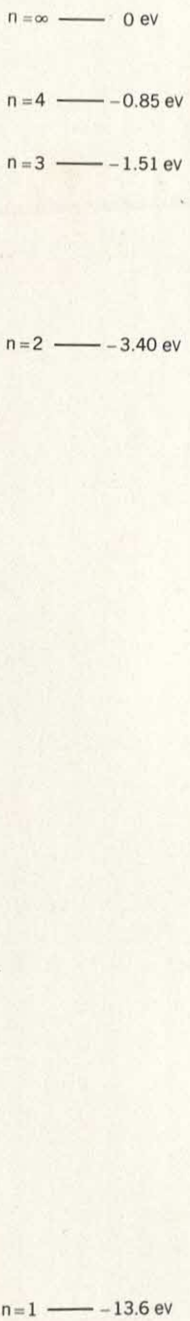


FIGURE 18.2 Allowed energies of the hydrogen atom. The limit $n \rightarrow \infty$ corresponds to ionization.

$$E_n = -\frac{1}{n^2}\,\text{Ry} = -\frac{1}{n^2}\,(13.6\ \text{eV})$$

EXERCISE 3 What is the energy difference $E_4 - E_3$? ______ Ry

EXERCISE 4 Out of the infinite number of bound states in that 13.6-eV energy range, all but nine states have energy levels in the upper 1% of the range. True / False.

Notice that you have not been shown the calculation that gave that energy spectrum. What went into the calculation was the inverse-square force, also known as the **Coulomb field,** and a standing-wave condition. It goes without saying that everything about this atom is spherically symmetrical. What came out of the calculation—the solution of a differential equation—is a formula for the allowed energies with only one quantum number, n. It is called the *principal quantum number,* and it takes all integer values from 1 to infinity. When n is infinite, we say the atom is ionized: The electron's energy is zero. Of course, there are positive energy states too, but they are not quantized. A free electron is not in a stationary state. Its wavefunction is not a standing wave.

Transitions between States

A quantum system can go from one stationary state to another by interacting with something outside itself. We shall be talking about an atom interacting with the electromagnetic field. If the interaction results in an atomic transition from a higher state to a lower state, that is radiation. If the transition is from a lower state to a higher state, that is absorption. In either case, the total energy is conserved: The energy difference between the quantum states is the energy of the photon radiated or absorbed.

Now you know how the energy levels of atoms are studied. For *emission spectroscopy,* a gas of atoms is somehow excited to higher states. One way is to use a gas discharge. The familiar neon tube is an example. The gas is put in a high electric field. Electrons accelerated by the field can collide with atoms and give up enough energy to excite the atoms to higher energy levels. When an atom makes spontaneous transition to a state having a lower energy level, it radiates a photon whose energy is equal to the difference between the two energy levels. For *absorption spectroscopy,* radiation with a continuous band of frequencies is passed through the gas to be studied. The spectral lines corresponding to the energy-level differences of its atoms will be missing from the radiation coming through.

EXERCISE 5 For the hydrogen atom, a transition from the second excited state (energy E_3) to the first excited state (energy E_2) results in the radiation of a photon of energy $\frac{5}{36}$ Ry (= 1.89 eV).

(a) Repeat the argument to show that. See Eq. 3.
(b) Show that the wavelength of the photon is 656.1 nm (red). See also Problem 3, **Balmer Series.**

In Chapter 16, we started to understand electromagnetic radiation in terms of changing electric and magnetic fields. In Chapter 17, we learned that the energy of electromagnetic waves is quantized. Now we have made the connection between the quantum states of the radiating atoms and the quanta of the electromagnetic field, the photons.

18.3 UNCERTAINTY PRINCIPLE

As long as the electron remains in one of the stationary states, there is no radiation. What is the electron doing while it is in a stationary state? It does not really do us any good to ask the question. There is no conceivable way of finding out. To find out, we would have to interact in some way with the electron. In interacting, we would perturb the stationary state. We are presented with a fundamental barrier to finding out the answers to certain questions!

This limitation is called the Heisenberg **Uncertainty Principle.** As you might guess, it has given rise to a good bit of philosophical speculation. For example: Is the limitation on our knowledge a temporary one, or is it permanent? Is it just that at the present stage of our science this is the best we can do, or has Nature imposed the quantum-mechanical uncertainty on us forever? Most physicists believe that the quantum theory is indeed fundamental, and that we are stuck with the limitations it imposes. No matter how closely we may observe, some phenomena will always be unpredictable.

The degree of uncertainty that has been imposed on us has to do with the size of Planck's constant h. Suppose that we want to know where the electron is. Speaking somewhat fancifully, we want to look at it, or take a snapshot of it. That means illuminating it, shining light at it. If we use "light" of long wavelength, we get a fuzzy picture. It is hard to localize an object to much better than one wavelength, just because of diffraction. (See Sec. 12.3.) Good localization requires scattering a short-wavelength radiation. But here comes the catch: The shorter the wavelength of the photon, the larger its momentum (Sec. 17.4). In being scattered, the photon transfers momentum to the object doing the scattering. In being "seen," the electron recoils. The more closely it has been localized, the more momentum has been transferred to it during the scattering. By finding out about the electron's position, we have had to sacrifice knowledge of its momentum. As stated here, the uncertainty in position Δx is approximately one photon wavelength λ. The momentum uncertainty Δp is approximately the photon momentum h/λ. (Eq. 17.2) Evidently, the two uncertainties are inversely proportional:

$$\Delta p \approx \frac{h}{\Delta x} \qquad \text{(Eq. 4)}$$

The experimenter who does not want to spoil the electron's momentum by bouncing a high-momentum photon off it is condemned to use a long-wavelength photon and gets a fuzzy picture. If he wants a sharp picture, he has to use a short-wavelength photon, but that transfers a lot of momentum to the electron.

Equation (4) may be considered the quantitative statement of the Heisenberg Uncertainty Principle. In the form presented here, it is not very quantitative, though. The wiggly ≈ sign means "approximately." That is just because we have chosen not to give a precise definition of the word "uncertainty." The statisticians might talk about standard deviation or some other error estimate, some precise way of measuring the statistical spread of the measurements. They might come up with a formula different from our Eq. (4) by a factor 2 or π or perhaps 2π, depending on the definition they chose for Δx and Δp.

It might be useful to repeat Eq. (4) a few more times. If we write it

$$\Delta x \approx \frac{h}{\Delta p}$$

it says that an experiment that measures the x-component of the momentum of our object (the electron) within a precision Δp spoils its position coordinate x so that it is uncertain by $\pm h/\Delta p$. It does not say anything about how bad the experiment might be. It does say that regardless of how good the experiment, subsequent measurements of the object's position will show that minimum statistical scatter.

Writing Eq. (4) in the form

$$\Delta x \, \Delta p \approx h$$

makes it look more symmetrical. This emphasizes that position and momentum are complementary variables. Improving the measurement of the one implies sacrificing accurate knowledge of the other. It does not matter which measurement is done first. The second measurement is spoiled by the first. The observation has perturbed the object being observed.

But isn't that a universal problem? Doesn't every experiment in some way spoil the thing being experimented on? Doesn't the act of putting a thermometer in the test tube change the temperature of the contents? Don't political pollsters subtly influence the opinions they are supposedly probing, just by the act of asking a question? Of course! Complementarity is everywhere. There are uncertainty principles in all branches of learning. What is special about Heisenberg's is the appearance of Planck's constant h. It puts a universal lower limit on how little a measurement disturbs what it measures. No matter how carefully the measurement is carried out, the minimum scale of graininess imposed by Planck's constant sets a minimum for the uncertainty of complementary measured quantities.

18.4 DIFFRACTION OF PARTICLES

If the electron is a wave, then it should exhibit diffraction and interference effects. Since you know about electron microscopes, that is not news to you.

What can a 40-keV electron microscope do better than a 10-keV electron microscope? You will say, the electrons are flying twice as fast.

EXERCISE 6 Why not 4 times as fast?

Twice the momentum means half the De Broglie wavelength (Eq. 1). A shorter wavelength means better resolution. The ability of a wave beam to resolve fine details is limited to a scale of detail not finer than about one wavelength. This is true whether it is a beam of light or a beam of electrons or any other wave. The image of two objects less than one wavelength apart blurs enough so that it is hard to tell whether there are two objects or only one. The 40-keV electron beam has twice the resolving power of the 10-keV beam.

EXERCISE 7 Calculate the De Broglie wavelength of a 10-keV electron. [Answer: 0.012 nm] Compare with the wavelength of a 10-keV photon. Note that the electron has 10 times the resolving power of the X ray.

Even the 10-keV electron microscope is 50,000 times better than an optical microscope at resolving details.

EXERCISE 8 Where does that factor 50,000 come from? (HINT: Calculate the wavelength of green light, which is roughly in the middle of the visible spectrum.)

Every household owns electron beams these days. There is at least one in your TV tube. But where is the evidence that those electrons are really diffracting? You will say, quite correctly, that the blur in those electron-microscope pictures is evidence. Really convincing evidence for diffraction is an interference pattern. Historically, it was the observation of a diffraction pattern formed by an electron beam that made the case for the wave nature of matter. Electron diffraction patterns looking just like x-ray diffraction patterns were obtained by reflecting an electron beam from the surface of a crystal. The patterns satisfied Bragg's Law (Chapter 17, Problem 21), with λ the De Broglie wavelength of the electrons. The photograph shown on page

505 compares an electron diffraction pattern with an X-ray diffraction pattern from the same crystal. You had a right to expect the pictures would look similar! The point is, the formalism of quantum mechanics is not just a successful prescription for explaining the properties of atoms. The electron waves are really waves.

Quantum mechanics is not restricted to electrons, of course. Diffraction effects have been observed with heavier particles as well. Neutron diffraction is a technique for studying the vibrations of crystals. There are proton microscopes. Also, diffraction of various ion beams has been used in the study of surfaces. Why not diffraction of baseballs? It is just that the momentum—even for a slow pitch—is so large. That makes the wavelength very short, according to the De Broglie relation. For such a short wavelength, the separation between interference maxima and minima is too small to be observed.

EXERCISE 9 A baseball has a mass of 0.2 kg. A slow pitch might have a speed of 10 m/sec, giving a momentum of 2 kg m/sec. Calculate the De Broglie wavelength. How closely ruled would a diffraction grating have to be to space the maxima 10^{-4} radians apart?

HINT: $m\lambda = d \sin \theta$

Philosophical Digression

We have been talking about matter waves. What is waving? When we watch water waves on a pond or in the bathtub, the surface of the water is really moving. Those waves are a concrete thing. They can be seen and felt. But when an electromagnetic wave travels through free space, what is actually happening? What moves? When an electron wavefunction changes in time, what is it that is moving? The answer for the electromagnetic wave is that E-fields and B-fields are changing. The answer for the electron wave is that the psi-function is changing. If you do not find those answers deeply satisfying, console yourself. The history of science represents a steady progression toward more and more abstract explanations. Each generation of students finds itself confronted by more sophisticated abstractions. How well does our culture prepare us to deal with such abstract ideas? What each of us finds satisfying is, after all, conditioned by our culture. Science is part of that culture.

18.5 THE EXCLUSION PRINCIPLE AND THE PERIODIC TABLE

In your chemistry courses you may have learned that the stationary-state wavefunctions of an electron are characterized by four quantum numbers. In Sec. 18.2 you learned that only one quantum number determines the energy levels of the hydrogen atom. In other words, different stationary states can have the same energy level. Such states are called **degenerate.** In the hydrogen atom, the electron moves in a $1/r^2$ field. Some of the degeneracy of the pure **Coulomb field** is lifted in the many-electron atoms. Less symmetry usually means less degeneracy. It turns out that the principal quantum number n can still be given meaning in the more complicated atoms. The force acting on any one electron is not so very different from a nice spherically symmetric inverse-square force.

Here is a table of the degeneracies of the first few energy levels of the Coulomb field:

Principal quantum number n	**Number of states having the same energy**
1	2
2	8
3	18
4	32

That is a mathematical result. The proof is rather involved. But the result is intensely interesting for chemistry when combined with another principle: The **Pauli Exclusion Principle.**

Exclusion Principle	No two electrons can occupy the same quantum state.

In one fell swoop the existence of the chemical elements is explained! Here is the story of the Periodic Table.

Periodic Table of the Elements

Hydrogen has only one electron. In the ground state, it is in the $n = 1$ energy level.

A nucleus with two protons in it can bind two electrons to form an atom. In the ground state they can both be in the $n = 1$ level. We know that because the table gives the degeneracy of the $n = 1$ level as 2. Do not be surprised that the electrons in the ground state of helium are more tightly bound than the electron in hydrogen. After all, there is twice as much charge in the nucleus attracting them. That makes for more compact electron wavefunctions, even though the two electrons do repel each other. Anyway, helium is chemically very different from hydrogen.

How about a nucleus with three protons, atomic number $Z = 3$? It can hold on to three electrons to make an electrically neutral atom. According to the degeneracy table and the Exclusion Principle, there is room for only two electrons in the $n = 1$ state. So the third electron has to go into an $n = 2$ state. The $n = 2$ wavefunctions are much more spread out than the $n = 1$ wavefunctions. You already know that an $n = 2$ electron has higher energy; that is, it is less tightly bound. What you expect from this argument you probably know from chemistry: Lithium is chemically very reactive.

How much room is there in the states labeled $n = 2$, the $n = 2$ "shell"? The degeneracy table says it can hold 8 electrons. With 2 in the $n = 1$ shell and 8 in the $n = 2$ shell, an atom with 10 electrons would have the two inner shells filled. Atomic number $Z = 10$ is neon. As you might expect, neon is chemically rather inert. Having a filled shell, it is chemically similar to helium.

The element with atomic number $Z = 11$ is sodium. With just one electron in the $n = 3$ shell, it would be expected to show chemical properties similar to lithium. It does!

WARNING: Don't take sodium metal out of its bottle on a humid day.

This section is starting to look like a chemistry course. The point was just to show how the Exclusion Principle explains (1) the diversity of the chemical elements and (2) the periodic chemical similarities between elements as electron shells are filled and new ones begun. Look in the back* at the **Periodic Table of the Elements.** The column on the extreme right contains the atoms with filled shells: helium, neon, argon, krypton, xenon, and radon. They are called the noble gases. On the extreme left is a column of elements with just one electron in the outer shell: After hydrogen, there is lithium, sodium, potassium, rubidium, and cesium. All are highly reactive. Their hydroxides are corrosive bases. In column II are elements with two electrons in the outer shell: beryllium, magnesium, calcium, strontium, and barium. They are called alkali earth metals. Again, they show chemical similarities.

You might be able to guess why the $n = 3$ shell does not fill up all the way before the $n = 4$ shell starts to fill. That has to do with the lifting of the "Coulomb" degeneracy by the mutual repulsion of the electrons. But enough chemistry for now. For the purposes of this chapter, you need to know only enough chemistry to know that quantum mechanics works.

You should know that quantum mechanics gives a prescription for correctly calculating the energy levels of the atoms as well as all their other properties. Quantum mechanics works for molecules; it works for crystals. Slowly people are learning how to make it work to explain the properties of disordered systems like liquids, and of the highly organized systems that make up living things.

SUMMARY

The planetary model of the atom contains some contradictions:

1. It predicts electron orbits that spiral inwards toward the nucleus, radiating at increasing frequency.
2. It predicts a continuous spectrum of energies. It cannot explain why the atoms of a given species are identical.
3. It cannot explain the stability of atoms, molecules, or solids.

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*Inside back cover

The quantum theory does not ascribe sharply defined position and momentum to a particle. It assigns it a wavefunction, which permits finding the probability distribution of position and momentum.

If the wavefunction has a sharp wavelength, the particle has a sharp momentum given by the De Broglie relation $p = h/\lambda$.

Stationary-state wavefunctions are standing waves. They are characterized by a set of quantum numbers, usually integers.

The energy levels of the hydrogen atom depend only on the principal quantum number n, which takes on the values 1 to infinity: $E_n = -(1/n^2)$ Ry. The higher levels get closer, converging asymptotically to zero energy. That corresponds to a free electron, that is, an ionized atom.

The transition from a higher to a lower energy level is accomplished by the emission of a photon in such a way that energy is conserved.

A wavefunction that is compact cannot have a well-defined wavelength. Conversely, the sharper the wavelength, the more spread-out the wavefunction. This mathematical theorem results in the Uncertainty Principle: Sharp position implies fuzzy momentum, and vice versa. $\Delta x \approx h/\Delta p$.

The wave nature of electrons was confirmed by the observation of diffraction patterns formed by electron beams, with wavelengths given by the De Broglie relation.

Diffraction patterns can be observed for electrons, protons, and neutrons because their De Broglie wavelengths are comparable to atomic dimensions. The De Broglie wavelengths of macroscopic objects like baseballs are too small to observe even at slow speeds.

The Exclusion Principle allows only one electron in each quantum state. It explains the diversity of the chemical elements.

Atomic number Z = the number of protons in the nucleus = the number of electrons in the atom. As Z increases, the filling of successive electron shells described by the principal quantum number n results in periodic chemical similarities of the elements.

The $n = 2$ energy level is eightfold degenerate. That means that eight different wavefunctions have the same energy. We say the degeneracy is lifted if some interaction causes those wavefunctions to have slightly different energies. Adding more electrons lifts the Coulomb degeneracy.

KEY TERMS

Orbital Stationary-state wavefunction. Chemists' terminology distinguishes atomic orbitals and molecular orbitals.

Electron cloud The statistical distribution of electrons in an atom given by the total atomic wavefunction.

Wavefunction Psi function, also called probability amplitude. A function of the space and time coordinates of one or more particles that contains the maximum information about the quantum system.

Isotopes Two or more atomic species having the same atomic number but different mass.

Atomic number Z The number of protons in the nucleus = the number of electrons in the neutral atom.

Quantum mechanics The complete formalism of theoretical physics valid on the atomic scale of sizes and beyond (i.e., still smaller). Developed by De Broglie, Schrödinger, Heisenberg, Born, Jordan, Pauli, Dirac, and others during the 1920s. Also called quantum theory or wave mechanics.

Bohr theory A treatment of the hydrogen atom invented a decade before quantum mechanics to explain the atomic spectrum. The assumption was that the electron moves in orbits according to classical mechanics, but only those orbits obeying the "quantum condition" $(mv)(2\pi r)$ = integer × h are allowed and stable.

Correspondence Principle For large values of the quantum numbers the results of quantum theory must agree with the results of classical theory.

Classical theory Electromagnetic theory together with Newtonian mechanics ($F = ma$).

De Broglie relation The De Broglie wavelength of a particle is inversely proportional to its momentum according to $\lambda = h/p$, where h is Planck's constant. (Pronounced: D'Broilya)

Stationary state A standing-wave wavefunction. A quantum state that does not depend on the time.

Wave equation Usually refers to the Schrödinger equation. In general, a partial differential equation whose solution is a wavefunction.

Quantum number One of the labels of a stationary-state wavefunction. For a particle moving in three-dimensional space, the wavefunctions are characterized by four quantum numbers. In atomic physics the following names are conventional:

n = principal quantum number
l = orbital quantum number
m = magnetic quantum number
s = spin quantum number

Ground state The stationary state of lowest energy.

Ionization energy The work needed to raise an electron from the ground state to the zero-energy (or "free") state.

Coulomb field A force field that is spherically symmetric and falls of as $1/r^2$.

Rydberg Abbreviated 1 Ry. The ionization energy of hydrogen $= 2\pi^2 m\,(k_e e^2)^2/h^2 =$ 13.60 eV. Sometimes the definition is changed by dividing by hc, so the Rydberg is a reciprocal length instead of an energy: $13.60\ \text{eV}/hc = 1.097 \times 10^7\ \text{m}^{-1}$. In this form it is more useful to spectroscopists, who deal in reciprocal wavelengths, called wavenumbers, rather than conventional energy units.

Balmer series The spectral lines of atomic hydrogen corresponding to transitions to the $n = 2$ level from higher energy levels. The first four lines of the Balmer series are in the visible. The wavelengths λ_n are given by the formula

$$\frac{1}{\lambda_n} = 1.097 \times 10^7\ \text{m}^{-1}\left(\frac{1}{2^2} - \frac{1}{n^2}\right), \qquad n = 3, 4, \ldots$$

Emission spectrum The line spectrum of radiation emitted by a gas excited by an electric arc or spark or simply by being heated to high temperature.

Absorption spectrum The wavelengths missing when radiation with a continuum of wavelengths is passed through a gas.

Heisenberg Uncertainty Principle The statistical spread of a wavefunction in space, say in the x-direction, and the statistical spread of the x-component of the momentum are inversely proportional: $\Delta x \cdot \Delta p \approx h$. Given the De Broglie relation, this is a mathematical consequence of the Fourier integral theorem relating the spectral composition of a wavepacket to its spatial extent. Alternative statement: A measurement of momentum spoils the sharpness of localization in space, and vice versa.

Davisson–Germer experiment The first demonstration (1926) of electron diffraction, which established the wave nature of the electron and the validity of the De Broglie relation.

Pauli Exclusion Principle Each quantum state can contain at most one electron.

Degeneracy When two or more quantum states have the same energy, they are said to be degenerate. The number of states having the same energy is called the degree of degeneracy. Thus the $n = 1$ level of hydrogen is doubly degenerate. The $n = 2$ level has eightfold degeneracy. A perturbation that breaks (spoils) a symmetry is said to lift (spoil) a degeneracy.

Periodic Table of the Elements A table of all the chemical elements arranged in order of atomic number Z in such a way that chemically similar elements appear in the same column.

Complementarity Principle A wave picture and a particle picture are complementary descriptions of any physical system. Which picture is needed depends on the measurement to be performed. For every particle variable (like position) there is a complementary wave variable (like wavelength or momentum). A precise specification of one of a complementary pair of variables spoils the precision of the other.

Wave-particle duality A picturesque way of saying complementarity.

REVIEW QUESTIONS

- According to classical electromagnetic theory, if a charge goes around in circles taking time T for each revolution, what frequency of electromagnetic waves should it radiate?
- According to a classical planetary model of an atom, explain why one would expect the frequency to keep increasing.
- Is such a model in conflict with the observation of sharp spectral lines? Explain.
- Chlorine has atomic number 17. This means that all chlorine nuclei contain ________ protons. Some have atomic weight 35; some have atomic weight 37. We say that there are two naturally occurring *isotopes* of chlorine, which are chemically similar / very different.
- Is there any way a chlorine atom can show its age? What is the point of the question?

- A wavefunction with a well-defined wavelength describes a particle with a well-defined ____________________.
- If the momentum of a particle is doubled, its wavelength is multiplied times ________.
- The product of the momentum of a particle times its wavelength is a universal constant called ____________________.
- A standing wave cannot have less than one antinode. In quantum mechanics, that fundamental mode would be called the ground state / first excited state.
- Two stationary states having different quantum numbers may have different energy levels or they may have the same energy level. This latter case is called: degeneracy / decadence / moral turpitude
- The higher states of the hydrogen atom are more / less degenerate than the lower ones.
- As the principal quantum number gets larger and larger, the electron wavefunction spreads out more and more. When the electron is free ($n \to \infty$) we say that the atom is: quantized / ionized / split
- The ionization energy is the work needed to take an atom from its ground state to the ____________________ state.
- The ionization energy of hydrogen is 1 Ry. The nth energy level has energy $-(1/n^2)$ Ry. The energy needed to free an electron from the $n = 2$ level is _____ Ry.
- The photon emitted when hydrogen makes the transition from the $n = 2$ level to the ground state ($n = 1$) has an energy of ________ Ry. (HINT: This is an ultraviolet photon.)
- The photon emitted in the transition form the $n = 3$ to the $n = 2$ level has longer / shorter wavelength than the photon corresponding to the transition from $n = 4$ to $n = 2$.
- When a particle has a well-defined De Broglie wavelength, we say that it is in a state of sharp momentum. Explain. In this case we say that its wavefunction is widely extended or its position is quite fuzzy. The spatial extent of the wavefunction is:

  proportional / inversely proportional

  to the statistical spread of the momentum. How does Planck's constant enter the relationship?
- A clear electron diffraction pattern permits the measurement of wavelength, hence of momentum. The more sharp spots there are in a Laue pattern, the better the measurement of wavelength. Argue that the more Laue spots you see, the less well defined is the position of the electron.
- A wave beam of width w spreads into an angle of minimum size θ given by $\sin\theta = \lambda/w$ (Chapter 12). Assume that the beam is many wavelengths wide. How much more / less does a 10^4-eV electron beam spread than a 10^2-eV beam?

- How much more does a 99-eV beam spread than a 100-eV beam?
- If the beam were a million wavelengths wide, how could diffraction be observed? Be quantitative.
- The $n = 1$ shell can hold only two electrons. Explain how the chemical properties of helium follow from its having a filled electron shell.
- Account for the chemical similarities between lithium ($Z = 3$) and sodium ($Z = 11$).

PROBLEMS

★★ **1. The Size of Atoms from Dimensional Analysis** The universal constants $k_e e^2$, h, and m (= mass of the electron) can be combined in such a way as to yield a quantity having dimensions of length. How close does this come to the diameter (10^{-10} m) of the hydrogen atom?

★★ **2. Bohr Theory** In 1913, a decade before the invention of quantum mechanics, Niels Bohr explained the spectrum of the hydrogen atom using a planetary model with a "quantum condition" for stationary orbits. With hindsight, this "quantization of angular momentum"

$$mvr = (\text{integer}) \times \frac{h}{2\pi}$$

is seen to be a standing-wave condition: The number of De Broglie wavelengths around the circumference $2\pi r$ of a circular orbit has to be an integer: $2\pi r = n\lambda = nh/mv$. In a circular orbit, the electron's acceleration is v^2/r (Sec. 5.2). The centripetal force responsible for this acceleration is

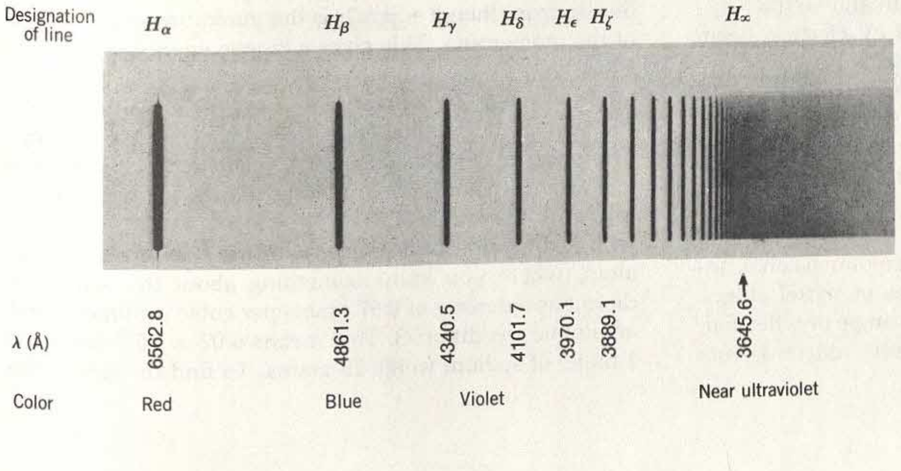
$$\frac{k_e e^2}{r^2} = m\frac{v^2}{r}$$

or

$$mv^2 r = k_e e^2$$

(a) Combine this with the quantum condition to show that the speeds in the various allowed orbits are $\frac{1}{2}$, $\frac{1}{3}$, $\frac{1}{4}$, and so on, of the speed $2\pi k_e e^2/h$ in the first Bohr orbit ($n = 1$).
(b) Show that the allowed orbit radii are 4, 9, 16, . . . times the radius of the first Bohr orbit.
(c) Given the formula $-\frac{1}{2}k_e e^2/r$ for the total energy of an electron in a circular orbit, show that the allowed energies are $\frac{1}{4}$, $\frac{1}{9}$, $\frac{1}{16}$, . . . of the lowest allowed energy ($n = 1$).
(d) Show that the energy of the first Bohr orbit is $-2\pi^2 m(k_e e^2)^2/h^2$, in exact agreement with quantum mechanics.

★★ **3. Balmer Series** The spectral range over which the eye is sensitive to light varies from individual to indi-



Balmer series. In the spectrum of atomic hydrogen, the first few Balmer lines (red, blue-green, violet) are in the visible. All of the other lines of the hydrogen spectrum are either infrared or ultraviolet.

vidual. In the spectrum of hydrogen, only transitions to the $n = 2$ level result in visible radiation.

(a) Given the energy-level formula $E_n = -(1/n^2)$ Ry with 1 Ry = 13.602 eV, calculate the wavelengths of the first four lines of the Balmer series, corresponding to transitions to $n = 2$ from $n = 3, 4, 5$, and 6, called Balmer alpha, beta, gamma, and delta, respectively.

[Answer: $\lambda_{\text{Balmer }\alpha} = 656.2$ nm is red; Balmer beta is blue-green; Balmer gamma is violet]

(b) Is Balmer delta violet or ultraviolet? You will find disagreement on the answer. Some people can see it.

(c) Prove the formula $\lambda_n = 3646 \text{ Å } n^2/(n^2 - 4)$.

✶✶ **4. Singly Ionized Helium** The energy levels of He^+ can be calculated in the same way as those of hydrogen. The only difference is that the electrostatic force on the electron is $2k_e e^2/r^2$, that is, double that in hydrogen, because the nucleus contains two protons.

(a) Show that the ground-state energy of He^+ should be 4 times that of hydrogen (Eq. 2).

(b) Write the formula for the excited levels analogous to Eq. 3.

(c) Are any of the spectral lines corresponding to transitions to the $n = 2$ level in the visible? Show your calculation.

(d) Predict the energy levels of doubly ionized lithium, Li^{2+}

✶✶ **5. Isotopes** Chlorine has two naturally occurring isotopes, ^{35}Cl and ^{37}Cl. The atomic weight of natural chlorine is 35.457. What fraction of natural chlorine is ^{35}Cl?

NOTE: If you want great accuracy, you can use accurate values of the two atomic masses: 34.969 and 36.966. There are no "formulas" in the chapter for this problem. Just figure it out. (HINT: What is a weighted average? If you don't know, Chapter 2, Problem 23.)

✶ **6. Electron Diffraction** Davisson and Germer discovered electron diffraction using a 54-eV electron beam reflected from the surface of a nickel crystal.

(a) Consider the nickel atoms as forming a "grating" with spacing $s = 2.15$ Å. Use the "grating equation" $n\lambda = s \sin \phi$ to calculate the angle ϕ for an $n = 1$ maximum, assuming normal incidence. (HINT: First calculate the De Broglie wavelength λ.)

(b) Since the beam penetrates several atomic layers into the crystal, the problem is actually three-dimensional. Intense reflections are from entire planes of nickel atoms, with the angle of incidence equal to the angle of reflection. Constructive interference between beams reflected from successive planes occurs when the angle α made by the beam with the planes satisfies Bragg's Law, $n\lambda = 2d \sin \alpha$. Here the distance d is the separation between planes of atoms. For nickel this is 0.925 Å. Verify that the angle α for an $n = 1$ reflection comes out 65°.

NOTE: The calculation leading from the 2.15 Å to the 0.925 Å has been left out. It is a little tricky. If you insist on knowing how to do it, your instructor can show you.

✶ **7. There Are No Electrons in the Nucleus** From atomic scattering experiments the diameter of the atomic nucleus is known to vary from 3×10^{-15} m for the proton to 18×10^{-15} m for the uranium nucleus. Use an argument based on the Uncertainty Principle with $\Delta x = 2 \times 10^{-14}$ m to show that an electron confined to that small a space would have a momentum around 6×10^7 eV/c.

NOTE: The use of eV/c, where c is the free-space speed of electromagnetic waves, is conventional as a unit of momentum in high-energy physics. At such high speeds, the formula $p^2/2m$ for the kinetic energy is no longer valid. The kinetic energy of such a fast electron would be around 6×10^7 eV. But the electrostatic potential energy $-k_e e^2/r$ with $r = 10^{-14}$ m is only 1.4×10^5 eV. (Show.) This means that electrostatic forces could not possibly confine an electron in so small a space. The Uncertainty Principle thus proves that there are no electrons in the nucleus. Yet electrons are known to come out of the nucleus in beta decay (radioactivity). The discovery of the neutron (1931) resolved this paradox. The neutron can beta-decay into a proton and an electron.

✶✶ **8. Size of Atoms from the Uncertainty Principle** According to the Uncertainty Principle, it takes energy to confine an electron in a small space. In an atom, that is electrostatic energy. If the diameter $2r$ of an atom is taken to be the uncertainty Δx in a position coordinate of the electron, then $p \approx \pm h/2r$ is the minimum x-component of the momentum. This gives a kinetic energy of $p^2/2m = h^2/8mr^2$.

(a) To estimate $2r$, set that kinetic energy equal to the absolute value of the total energy of confinement at distance r from the nucleus, which is $\frac{1}{2}ke^2/r$ for hydrogen. The value you arrive at is only about 5 times too big for hydrogen.

(b) Put in the numbers.

(c) But compare, for example, with the size of the sodium atom, where you know something about the solid. Sodium has a density of 0.97 grams per cubic centimeter and an atomic weight of 23. That means 6.02×10^{23} atoms (= 1 mole) of sodium weigh 23 grams. To find the size of the atom, make some reasonable assumption about how the atoms are stacked in the crystal.

★★ **9. Energy-Time Uncertainty** A perturbation can cause a quantum jump, that is, a transition from one stationary state to another. In principle, a stationary state has a well-defined value of the energy. However, the probability of a transition to another state implies an uncertainty in that energy. The greater the transition probability, the fuzzier the energy of the state: The energy uncertainty ΔE is inversely proportional to the mean lifetime Δt of the state according to $\Delta E\ \Delta t \approx h$ = Planck's constant. This is another form of the Heisenberg Uncertainty Principle. The spectroscopist conventionally calls ΔE the *width* of the state, because it is measured in terms of the width of a spectral line.

(a) The lifetime of a typical excited atomic state is 10^{-8} sec. What is its width, in eV?

(b) What is the uncertainty in the frequency of the photon emitted in the transition? ________ Hz.

(c) A visible photon has an energy around 2.5 eV. What is the fractional uncertainty in its energy or frequency, expressed in parts per billion?

(d) Compare with Problem 17.16, Bandwidth and Uncertainty. Show that it makes sense to talk about the photon as a wavetrain 6-million waves long.

★★ **10. Filled Electron Shells** Carbon ($Z = 6$) has four $n = 2$ electrons. Silicon ($Z = 14$) has four $n = 3$ electrons.

(a) When carbon combines chemically with other elements, why do the two $n = 1$ electrons not participate in the chemical bond?

(b) Why are the $n = 1$ and $n = 2$ electrons not involved in the chemistry of silicon?

(c) Why are carbon and silicon chemically similar?

★★★ **11. Strength of Materials by Dimensional Analysis** In Chapter 3 we asked why we are not twice as tall as we are. That has to do with the strength of bone. It turned out that the strength (the yield stress) of all the tough materials is about the same, around 10^8 newtons per square meter. Now we are in a position to look for the fundamental reasons why there are no materials, say, a hundred times stronger. Put in only those constants of nature common to all materials:

Coulomb's Law constant	$k_e e^2$
Mass of the electron	m
Planck's constant	h

Use dimensional analysis to combine these three constants in some way $(k_e e^2)^x (m)^y (h)^z$ to make a quantity having the dimensions of a yield stress, N/m^2. Remember, a ballpark figure is all you expect from dimensional analysis.

★★ **12. Fine-Structure Constant from Dimensional Analysis** The universal constants $k_e e^2$, h, and c can be combined in such a way as to yield a dimensionless number. With the $k_e e^2$ in the numerator, this is a rather small number. Multiplied by 2π, it is called the *fine-structure constant* and has the value $\frac{1}{137}$.

(a) Show this.

(b) Look at Problem 2, Bohr Theory, and verify that the speed of the electron in the first Bohr orbit of hydrogen is $\frac{1}{137}c$.

★★ **13. Electron in a Box** The energy levels of a particle in a cubical box of dimensions $L \times L \times L$ are given by the formula $(n_1^2 + n_2^2 + n_3^2)\, h^2/8mL^2$.

(a) How does this differ from our one-dimensional example? Conjecture why there are three quantum numbers instead of one.

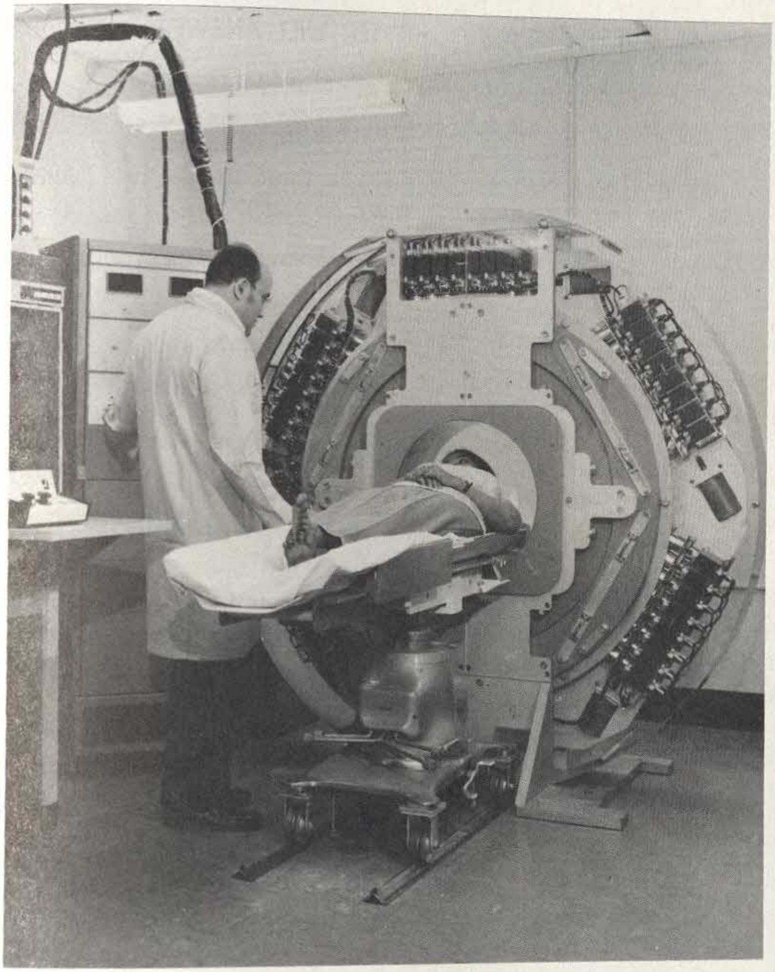
(b) Show that the energy of a quantum state having a particular set of quantum numbers (n_1, n_2, n_3) is proportional to the minus two-thirds power of the volume of the box.

NOTE: This result of the "free-electron theory of metals" is closely obeyed by real metals.

(c) Show that the level with $n_1 = 1$, $n_2 = 1$, $n_3 = 2$ (the first excited state) is triply (threefold) degenerate. That means there are two other stationary states having the same energy.

(d) What is the degeneracy of the level with $n_1 = 1$, $n_2 = 2$, and $n_3 = 2$?

(e) Show that the level with $n_1 = 1$, $n_2 = 2$, $n_3 = 3$ is sixfold degenerate.



Positron Emission Tomography (PET). By computer analysis of gamma rays coming from the annihilation of positrons in human tissue, this instrument constructs pictures that show features of internal organs such as the blood flow.

19

RADIOACTIVITY AND THE ATOMIC NUCLEUS

SOME QUESTIONS THIS CHAPTER WILL ANSWER

How much ionizing radiation is good for you?

What is meant by nuclear binding energy?

If radium has a half-life of 1620 years and the earth is billions of years old, why does any radium remain on earth?

Why can't a nuclear power reactor using uranium fuel explode like an atomic bomb?

How does a breeder reactor work?

How many half-lives does it take for 99.9% of a radioactive sample to decay?

If a radiation dose of 400 rem is 50% lethal, how is it possible to have radiation therapy with a dose of 5000 rem?

In a Central-European mountain village there is a warm spring. Its water, rich in minerals, has been famous since prehistoric times for its miraculous cures. The patient bathes in the special water in order to treat any of a variety of diseases. Such mineral-water baths, generally called "spas," exist in many countries. Some have been considered sacred for thousands of years. More recently, promoters have been advertising the radium content of their mineral waters. Medically, the therapeutic effects of the baths is not understood. Is it the radiation that heals?

We don't generally think of therapy involving radium as prehistoric medicine. Using the invisible rays emitted by radioactive substances to treat cancer is modern radiological practice. Yet we learn that the same rays that can arrest cancer can cause cancer. What is the nature of this radiation and how much of it is good for you?

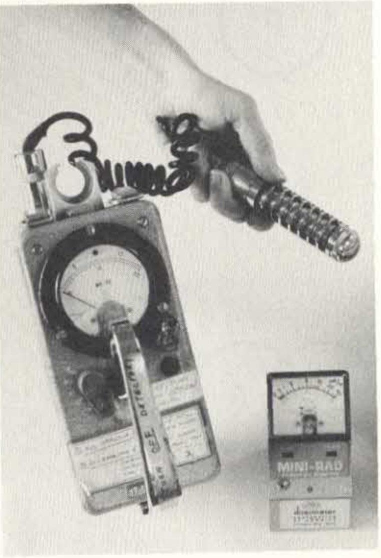
In this chapter you will learn about the origin of the various kinds of **radioactivity.** One of these is **nuclear fission,** which is the basis of the nuclear reactor and the "atomic" bomb. You will learn about other applications of radioactivity. You will learn something about radiation protection and how we measure the amount of radiation the body receives in a given exposure. But you will not learn how much is "good for you." Because even though we are constantly being bombarded by rays coming from the ground we walk on, the water we drink, and the air we breathe, we still don't know if any of it is beneficial to health.

This natural background radiation has always existed on earth. It is responsible for some of the mutations; that is, it has provided many of the genetic changes for biological evolution. The nuclear bomb explosions of World War II were the first large-scale exposure of a human population to high doses of radioactivity. This was followed in subsequent years by smaller doses as a result of radioactive fallout from atmospheric testing of nuclear weapons. Fallout has been strongly implicated as a cause of leukemia and chromosome breaking, and most above-ground nuclear tests have now been stopped by international agreement. Nevertheless, radioactive pollution represents a major health hazard. Among the potential polluters are the increasing number of nuclear power reactors, whose fuel and whose wastes are both radioactive. Their monitoring and control pose sensitive public-policy issues for the coming years.

The invention of the nuclear reactor in the 1940s, with its ability to make artificial radioisotopes, has given the medical profession quite an arsenal of radioactive "sources" for use in both diagnosis and therapy. Today there is a specialty called *nuclear medicine,* but radioactivity is a common tool in every branch of biological and physical science. In the next section we single out two familiar radioactive species, uranium and radium, in order to talk rather generally about the disintegration of the atomic nucleus.

19.1 UNSTABLE NUCLEI: URANIUM

The basic fuel of the nuclear reactor is uranium. Look at the Periodic Table (inside back cover). Uranium, atomic number 92, is the heaviest naturally occurring element. This is the element that led to the discovery of radioactivity. In 1896 Becquerel noticed that a uranium salt had fogged a photographic emulsion wrapped in black paper. The uranium had emitted invisible radiation that could penetrate objects opaque to visible light. The photographic emulsion acted as a radiation detector.[1] It reminded Becquerel of X rays, which Röntgen had discovered the year before. From study of the radiation and careful chemical analysis, it is now known what happens when the uranium gives off its rays: It changes into another element. The nucleus transforms.



A Geiger counter shown held in hand. The electrical connection to the box (i) connects the counter's electrodes to the high DC voltage, and (ii) allows the electrical pulses generated by the detection of ionizing radiation to be fed to a loudspeaker and to a small computer memory. On the model shown, the dial reads counting rate. The small instrument on the right is a dosimeter (see Section 19.6).

If you wanted to go prospecting for uranium deposits, you would use a Geiger counter with a small loudspeaker, a radiation detector you can hear. As your search gets "hot," its rate of clicking goes up. When your counter gets near a sample of uranium, what you hear is a jumbly rat-tat-tat, not rhythmical but obviously random. Sometimes there are long gaps between counts; sometimes they come in rapid-fire succession. Each click represents the disintegration of one uranium nucleus.

Why do these nuclei come apart? To answer that question, we should really ask first, "What holds a nucleus together?" The nuclear "glue" is called the *strong interaction* or simply the **nuclear force,** a force tending to bind all **nucleons** (protons and neutrons) together. It is a most remarkable force. What is remarkable about it is how strong it is when the nucleons are close together and how weak it is when they are far apart. Also, that it keeps the nucleons from coming *too* close. Take uranium. The atomic number $Z = 92$ tells you that there are 92 protons in each uranium nucleus.

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[1] Film badges are still used to monitor radiation exposure. They are commonly worn by people who work around radioactivity and other sources of "ionizing radiation."

Proton, *alias* hydrogen ion. Repels other protons electrostatically because of its positive charge. But when they get close enough, the *nuclear force* takes over and they attract both protons and neutrons.

Neutron. Has no charge. Attracts both protons and neutrons. By itself unstable, but needed to make larger nuclei stable.

Deuteron. A stable nucleus, the nucleus of heavy hydrogen. It takes 2.2 million electron volts (MeV) to pull it apart.

Alpha particle, also known as a helium nucleus. So stable, it takes 28 MeV to pull it apart into four nucleons.

FIGURE 19.1

Those protons repel each other because each one has a positive electric charge. The closer they get, the stronger that electrostatic repulsion. In the **isotope** uranium-238 (written $^{238}_{92}U$) there are also $238 - 92 = 146$ neutrons. These 238 nucleons are crowded into a space less than 10^{-14} meters (= 10 fm) in diameter. The electrostatic repulsion between any 2 protons that close together is gigantic, and the mutual repulsion of all 92 tends to blow the nucleus apart. The neutrons are uncharged, so they experience only the nuclear force and help bind the whole thing together, but this is effective only when they all get close enough—around 10^{-14} m—from each other. It takes an enormous force to push a collection of protons and neutrons that close together, overcoming the repulsive electrostatic force (Fig. 19.1). But once they are close enough to stick together, once the strong interaction has dominated the electrostatic repulsion, it takes an even greater force to pull them apart again. They pop right into their closest-crowding configuration. Note that the hard-core repulsion part of the nuclear force at very small distances is essential to keep the nucleus from collapsing.

This qualitative description of the nuclear force has said nothing about how it depends on the relative numbers of neutrons and protons. But just knowing about its short range allows you to understand nuclear stability: There has to be a delicate balance between the short-range nuclear attractive forces and the long-range electrostatic repulsive forces. Too large a nucleus is unstable because nucleons on opposite sides of the nucleus are far apart and attract each other only weakly. Indeed many of the nuclei near the heavy end of the Periodic Table tend to be unstable, that is, radioactive. An unstable nucleus can make a quantum transition to a configuration of nucleons having a lower energy. It can do this in a number of ways:

(α) It can throw off one or more nucleons, thus becoming smaller. Often what is emitted is a package of four nucleons called an **alpha particle.** This package consists of two protons plus two neutrons bound together to form the nucleus of ordinary helium, $^{4}_{2}He$. Because the radioactively emitted alpha particle has considerable energy, it is also called an alpha ray, and the radioactive process is called alpha decay.

(β) A neutron inside the nucleus can change into a proton, or vice versa. When the neutral particle transforms into the positive particle, a negative particle has to be emitted to conserve charge. It is an electron. Such an electron coming from a nucleus is called a **beta particle** or a beta ray, and the process is called beta decay. For the change of proton into neutron the emitted particle has to be positive. It is called a **positron** and is in every way like an electron except that it has the opposite sign of charge. Note that the mass of a beta-minus (electron) or **beta-plus** (positron) is tiny compared to the mass of a nucleon—about one two-thousandth. The transformation of a proton into a neutron inside the nucleus can also be accomplished by the capture of an electron from outside.

(γ) The nucleus can shift to a lower-energy configuration without emitting anything having mass. In this case the extra energy goes off as a photon. Such a photon is called a **gamma ray,** and the radioactive process is called gamma decay. Often alpha decay and beta decay are quickly followed by gamma decay. In fact, what went through Becquerel's black paper was gamma rays. The similarity to X rays was no coincidence. A gamma-ray photon and an X-ray photon are completely indistinguishable. They differ only in origin. The gamma ray comes from inside the nucleus, the X ray is produced outside.

Please don't think that those 238 nucleons in the uranium nucleus live happily together in that small space for a well-defined time period and then split. Each disintegration is completely unpredictable. The only kind of prediction one can make is a statistical one, a prediction about a large number of nuclei: Approximately one ^{238}U nucleus out of every 6.4 billion will decay in a year. Half of them live almost 5 billion years. But there is no way to predict which uranium-238 nucleus will live another 5 billion years and which one will disintegrate in the next second. Radioactive decay is a random process, and there is no way to distinguish one ^{238}U nucleus from another.

Radium—The Curie

Radium, a radioactive element once widely used in medicine, is also an alpha emitter. Its nucleus is less stable than ^{238}U, or perhaps we should say "shorter-lived." The radium nuclei decay faster. If we had a large number of radium nuclei, half of them would disintegrate—give off an alpha particle—in 1620 years. That may sound like a long time; but compare it with the uranium half-life of 5 billion years (4.47×10^9 years, according to recent measurements)! So the alpha radiation from a large number of radium nuclei is more intense than that from the same number of uranium nuclei. Their **activity** (= number of disintegrations per second) is almost 3 million times as great. (The ratio of the half-lives is $4.47 \times 10^9 \div 1620 = 2.76 \times 10^6$.) That high activity is one of the features that made radium so useful in medicine.

One gram of radium emits about 3.7×10^{10} alphas per second. Because of radium's historical importance, that number has become the international unit of activity:[2]

$$3.7 \times 10^{10} \text{ disintegrations per second} = 1 \text{ curie} = 1 \text{ Ci}$$

Thirty-seven billion events per second is, of course, an enormous rate. Hospitals are not likely to have that much radium in one place; it costs too much. Their sources will have activities in the millicurie or microcurie range:[3]

$$1 \text{ mCi} = 3.7 \times 10^7 \text{ disintegrations per second} = 3.7 \times 10^7 \text{ Bq}$$
$$1\ \mu\text{Ci} = 3.7 \times 10^4 \text{ disintegrations per second} = 3.7 \times 10^4 \text{ Bq}$$

Even so, such sources will be stored in thick-walled lead containers to protect hospital personnel against accidental exposure to the radiation (more about radiation shielding in Sec. 19.6). We now turn our attention to a family relationship between radium and uranium.

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[2] Pierre and Marie Curie discovered radium in 1898.

[3] 1 Bq = 1 disintegration per second. The unit of activity is named in honor of the French physicist Henri Becquerel, the discoverer of radioactivity.

19.2 RADIOACTIVE SERIES

You may wonder how it is that there is any measurable radium left in the world if every 1620 years half of all the radium disappears. Perhaps 1620 years sounds long enough so that we shouldn't have to ask that question. But if you think about the age of the earth—several billion years—you do wonder: After 1620 years, only half of the radium that was there originally was left. After another 1620 years, only half of that remained, or one-quarter of the original amount. After 3 times 1620 years, only one-eighth of the original amount of radium remained;

after 4 times 1620 years → $\frac{1}{16}$;
after 5 times 1620 years → $\frac{1}{32}$;
after 6 times 1620 years → $\frac{1}{64}$;
. . .
after 10 times 1620 years → $\boxed{\quad\quad}$[4]

That is still only 16,200 years, and yet less than one-tenth of one percent of the original radium remains undecayed. Even on a time scale of millions of years, hardly any of the "original" radium would be left. And the earth has been around for billions.

EXERCISE 1 What fraction of the radium would be left after 32,400 years (= 20 times 1620 years)?

The fact that we still find radium in the rocks of the earth suggests that Nature must have a mechanism for creating new radium. The clue to the mechanism is that radium is found in rocks containing uranium. Radioactivity is a process in which one element is transformed into another: Uranium-238 goes through a series of radioactive transformations and eventually ends up as radium-226. The intermediate steps of such a radioactive series or chain can be alpha (α) decays or beta (β) decays, with gamma (γ) decays interspersed.

The radioactive chain that converts uranium to radium starts with that slow alpha decay in which the $^{238}_{92}\text{U}$ nucleus loses 4 nucleons, leaving 234 nucleons, of which 90 are protons. Atomic number $Z = 90$ means that this "daughter" nucleus is an isotope of thorium, written $^{234}_{90}\text{Th}$. (See the decay chart, Fig. 19.2). Thorium-234 is radioactive and emits a fast electron, a beta ray. A beta emission does not change the nucleon number but converts a

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[4] $\frac{1}{1024}$

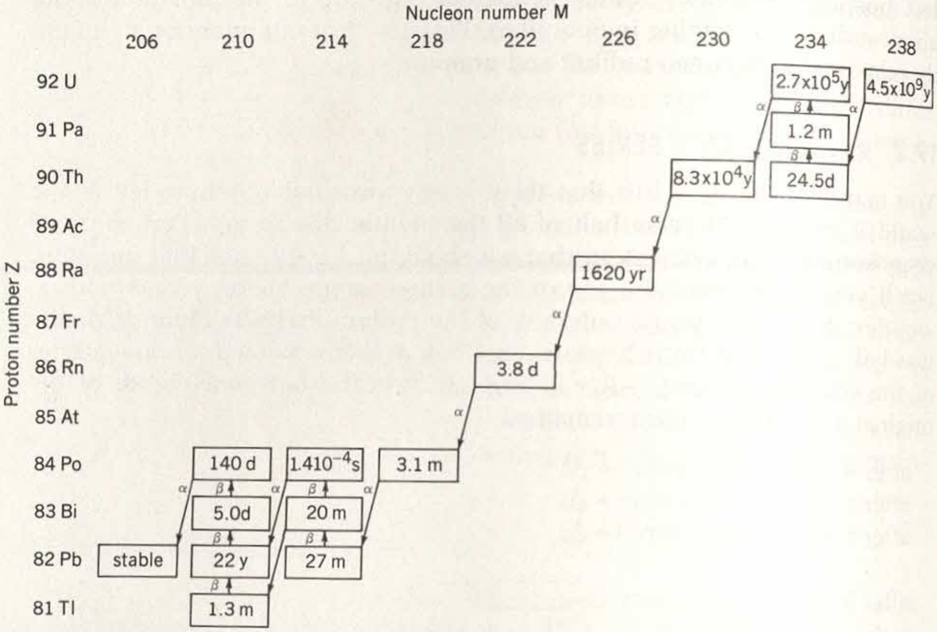

FIGURE 19.2 Uranium-radium radioactive series. The decay scheme starts with uranium-238 in the upper right-hand corner and ends with lead-206 in the lower left. Note that alpha decays decrease the proton number Z (= atomic number) by two and the nucleon number M (= mass) by four. Beta decays increase the proton number by one and leave the nucleon number unchanged. The number in each box is the half-life of the nuclide, in years (y), days (d), hours (h), minutes (m), or seconds (sec).

neutron into a proton inside the nucleus. In this chain, two beta decays follow in quick succession, leaving a nucleus still having 234 nucleons, of which 92 are protons. Atomic number $Z = 92$ means that it is an isotope of uranium again, $^{234}_{92}$U. After three radioactive transformations, we have gone from uranium back to uranium. We have two isotopes of the same element, ^{238}U the mother and ^{234}U the great-granddaughter. Being isotopes of the same element they occupy the same place in the Periodic Table and have the same proton number Z. Consequently, they have the same chemical properties. They differ only in the number of neutrons in the nucleus, and hence in mass or nucleon number.

Our chain continues with two more alpha decays: ^{234}U has a daughter (230 nucleons with $Z = 90$) called ^{230}Th, and a granddaughter (226 nucleons, $Z = 88$), which is ^{226}Ra, the common isotope of radium. That is the one whose radioactive half-life is 1620 years.

The radioactive chain goes on until a stable or nonradioactive nucleus is reached. In this case that is ^{206}Pb, an isotope of lead.[5]

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[5] We can analyze how much lead-206 is present in uranium ores and determine the age of those rocks by assuming that there was no lead and radium when they solidified, that is, roughly when the earth was formed. Such **radioactive dating** assigns the earth an age of 3.9 billion years.

Nuclear Systematics

Please don't try to memorize that decay scheme. By now, you know that physicists are not primarily interested in just knowing facts. We want to use facts to arrive at generalizations, and eventually at predictions. Why haven't we written down the laws that tell you, for example, what combinations of nucleons will cohere and make nuclei? What nuclei are radioactive? What is their decay mode? The reason is that nuclear theory is (a) difficult and (b) quantitatively still pretty rough. We can make some general statements about what nuclei are likely to exist and what decays are possible.

Is there a nucleus consisting of two protons? It has not been found. That doesn't surprise you. The short-range nuclear force, even though it is very strong, is not strong enough to overcome the electrostatic repulsion. But we also don't find a nucleus consisting of two neutrons! We do find one made up of one proton and one neutron. That's the deuteron, the nucleus of heavy hydrogen ($^2_1$H), called deuterium, and it is stable. We find a nucleus consisting of two protons and a neutron, and it is stable ($^3_2$He). We find one consisting of one proton and two neutrons ($^3_1$H), but it is radioactive. This isotope is called tritium, and it likes to beta-decay: One neutron changes into a proton, and an electron is emitted. But it takes its time: Half of all ^{3}H nuclei live over 12.26 years. The nucleus with two protons and two neutrons, on the other hand, is fantastically stable. That is the alpha particle, $^4_2$He, the nucleus of the more prevalent helium isotope.

You were promised generalizations; what you got was more facts. But now you might guess a rule: Having roughly the same number of neutrons and protons leads to stability. Too many neutrons, and you get beta decay. This rough equality of neutron and proton number turns out to be the stability criterion for the light nuclei. But as you saw, lead-206 is stable, and it has 124 neutrons to 82 protons. Figure 19.3 is a chart of neutron number versus proton number for the stable nuclei. Note how it curves, favoring more neutrons for larger nuclei.

If a nucleus has too many neutrons to be stable, are they thrown out? That can happen, but it is not usual. Usually, the neutrons are transformed into protons; they beta decay. If a nucleus has too many protons for stability, are they emitted? No, that seems to happen very rarely. An extra proton can change into a neutron, either by emitting a positron or by electron capture. Alternatively, it can escape in company with three other nucleons, bound up in an alpha particle.

These are approximate rules and not very quantitative. The one quantitative rule that is always obeyed is that only those radioactive processes occur which lower the energy of the configuration of nucleons. The chemist would say, only exothermic processes occur spontaneously.

EXERCISE 2

(a) When a tritium nucleus ($^3_1$H, also called a triton) beta-decays, what is its decay product?

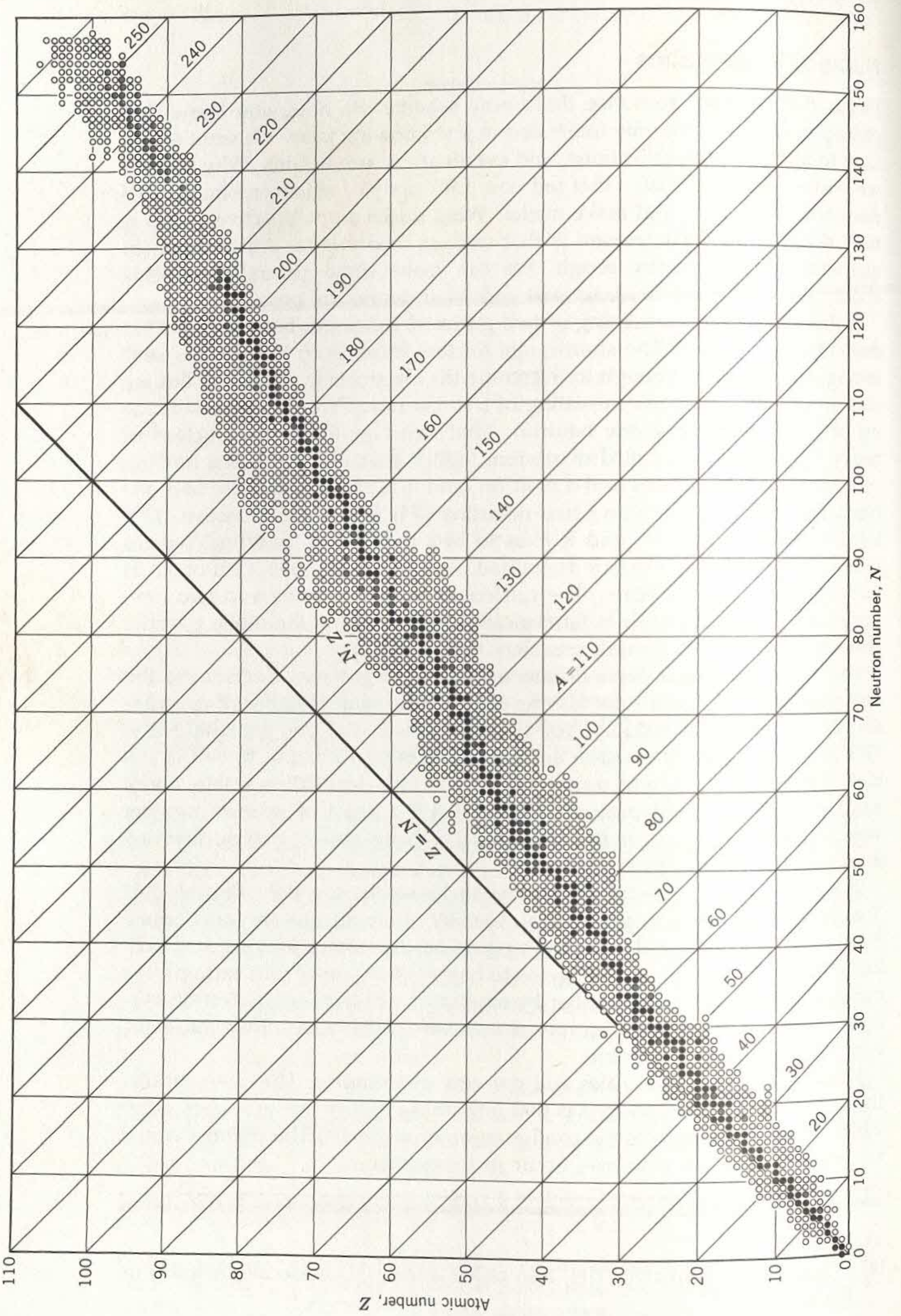

FIGURE 19.3 Stable nuclei. On the lower left, near the beginning of the Periodic Table, the stable nuclei cluster near the $N = Z$ line. In heavier nuclei, relatively more neutrons are needed for stability.

(b) When a helium-6 nucleus beta decays, what is its decay product? (HINT: Refer to the Periodic Table, inside back cover.)

Branching

You may ask, how does a nucleus decide which decay mode to choose? Should it emit an alpha or a beta? Sometimes both are possible, that is, both lead to nucleon configurations of lower energy. Then there is no way to predict reliably which way a nucleus will go. All one can tell is the odds. Take the nucleus bismuth-214, which we meet in the radioactive chain going from uranium-238 to lead-206. (See Fig 19.2). ${}^{214}_{83}\mathrm{Bi}$ can alpha-decay to thallium-210, or it can beta-decay to polonium-214. The alpha decay is 50 times more probable than the beta decay. This means that 98 times out of a 100 that nucleus will choose the alpha-decay route. We say there is a *branching ratio* of 50:1 in favor of the alpha decay. If that all sounds to you as though Nature is playing dice, remember, indeterminacy lies at the heart of the quantum theory.

In the next section we discuss a branching in which there are more than two different ways a nucleus can come apart.

EXERCISE 3 **Radioactive Series**

Approximately 0.7% of the uranium found in nature is ${}^{235}_{92}\mathrm{U}$, a radioactive isotope decaying by alpha emission with a half-life of 7.1×10^8 years.

(a) What is its daughter nucleus?

(b) The thorium isotope ${}^{231}_{90}\mathrm{Th}$ decays by beta emission. Its daughter nucleus decays by alpha emission. What is its granddaughter?

[HINT: The element $Z = 89$ is called actinium (Ac).]

(c) ${}^{227}_{89}\mathrm{Ac}$ beta-decays with a 21.6-year half-life. What is its daughter nucleus?

(d) ${}^{227}\mathrm{Th}$ alpha-decays with an 18.5-day half-life. What is its daughter nucleus?

NOTE: You get very good at this kind of hopping around the Periodic Table if you play "Disintegration," a board game in which the random deal of a deck of cards, labeled "alpha decay," "beta decay," "neutron absorption," and so on, replaces the randomness of Nature.[6] A player "loses" when he lands on a stable isotope.

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[6] Available from Relativistics, Inc., (P. O. Box 1064, Greensburg, Pa. 15601) or make your own rules and play on a standard Chart of the Nuclides, available from Educational Relations, General Electric Co., Schenectady, N.Y.

19.3 NUCLEAR FISSION

While 99.3% of the uranium in the rocks of the earth is ^{238}U, that long-lived alpha emitter (half-life 4.5×10^9 years), the other 0.7% is ^{235}U. This isotope is also alpha-radioactive, with a half-life of 7×10^8 years. The nucleus with one additional neutron, ^{236}U, is not found in nature. Yet in a way, ^{236}U is the central character of our story. It is also unstable, that is, radioactive, and has several different ways of coming apart. It can disintegrate into two lighter nuclei, emitting a few neutrons in the process and a great deal of energy. This is called nuclear fission. ^{236}U does this in a fraction of a second. That is why you won't find any ^{236}U around.

What makes fission so interesting is the possibility of a chain reaction, a self-sustained "burning": ^{236}U is formed when the ^{235}U nucleus absorbs a slow neutron. Because so much energy is released in the fission process, the neutrons come flying off at high speeds. After they are slowed down, other ^{235}U nuclei can absorb them and undergo fission, releasing further neutrons, and so on. Such a self-sustained nuclear reaction heats up the steam that runs the turbines in our "atomic power" stations. In a nuclear reactor, the chain reaction is carefully controlled. An uncontrolled chain reaction is an "atomic bomb."

Political Digression on Nuclear Bombs

Atomic power is in the news enough these days so that you are probably curious about the danger of nuclear explosions, whether accidental or deliberate. You can't get a self-sustained fission reaction in a small piece of ^{235}U, because too many of the neutrons will escape through the surface. It's like trying to set a thin piece of twine on fire. You can't, because it cools off too fast. But as you go to larger pieces of ^{235}U, you reach a critical size at which neutrons are produced as fast as they escape. A larger piece will explode spontaneously. To make a bomb, you need to bring together two or more small pieces to make one large piece, and—bang. There is always a stray neutron around to set things going.

If it's so simple to make an atomic bomb, why hasn't some band of terrorists long ago stolen some reactor fuel and held a city or two for ransom? The reason is that nuclear reactors don't need pure ^{235}U. In nature, the rare isotope ^{235}U always occurs mixed with 138 times as much ^{238}U, which is not readily fissile (does not undergo fission when bombarded with slow neutrons). In fact, ^{238}U is neutron-hungry, so it tends to quench the chain reaction. Getting rid of the ^{238}U is not easy. Since ^{235}U and ^{238}U are isotopes of the same element, no chemical process can separate them. Isotopic enrichment is an elaborate and expensive process carried out in huge factories, using a technique called gaseous diffusion. To make reactor fuel, which is about 3% ^{235}U, only a four- to fivefold enrichment is needed. To make a bomb, you need much purer ^{235}U. The conventional ^{235}U reactor has its dangers all right. It might overheat; it might leak; it has radioactive waste products. But the reactor cannot be made into bombs by terrorists. (See Problem 10, Spent Fuel Rods.)

Plutonium

Uranium-235 is not the only nucleus capable of fission when it absorbs a neutron. It happens to be the only one found in reasonable quantities in nature. Others can be made in nuclear reactors, that is, by bombarding heavy nuclei with neutrons. Of greatest military and industrial importance is the manufacture of plutonium ($^{239}_{94}\mathrm{Pu}$), a **trans-uranic** element, not found in nature at all:

$$^{238}_{92}\mathrm{U} + {}^{1}_{0}n \rightarrow {}^{239}_{92}\mathrm{U} \xrightarrow[23\ \mathrm{min}]{\beta^-} {}^{239}_{93}\mathrm{Np} \xrightarrow[2.3\ \mathrm{days}]{\beta^-} {}^{239}_{94}\mathrm{Pu}$$

The arrow $\xrightarrow{\beta^-}$ indicates a beta decay; the number underneath the arrow is its half-life. Plutonium, whose name evokes thoughts of the underworld,[7] is the material of choice for making nuclear fission weapons. It also makes a cheap fuel for nuclear power generation. Why cheap?

A plutonium reactor can be a *breeder*. This means that the same reactor in which the fission of plutonium is used as an energy source also makes more plutonium out of natural uranium. The fission of plutonium is accompanied by the release of two or three fast neutrons. Slowing these neutrons—cooling them off—means transmitting their kinetic energy through heat exchangers that power an engine, which can generate electricity. Most of the slowed-down neutrons are needed to keep the chain reaction going, that is, to cause more plutonium nuclei to fission. But some of the neutrons may be used to convert $^{238}\mathrm{U}$ into more plutonium. Hence the name "breeder reactor." In the long run, it may even produce more fuel than it uses.

Another Political Digression

If breeder reactors essentially make their own fuel, why do we talk about an energy crisis? Why aren't we quickly building breeders to satisfy the electric power needs of the future? The numbers are impressive: A ton of reactor fuel gives off as much energy in fission—nuclear burning—as 2 million tons of coal in oxidation—chemical burning. What are we waiting for?

This is a hot political question, and passions run high on both sides. A list of some of the reasons given for delaying large-scale construction of plutonium reactors for power generation follows.

1. Plutonium-239 is radioactive, with a 24,000-year half-life for alpha decay. So it has almost 200-thousand times the activity of uranium, atom for atom. How can we be sure none will get into our food or water? Aside from the possibility of leaks, there is the worry about health hazards associated with radioactive dust in the air.

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[7] In Roman mythology, Pluto was king of Hades.

2. The fission products of plutonium are extremely radioactive. How does one dispose of them safely? Burial in deep mines is one answer, but there are obvious problems in getting them there and storing them safely. Actually, this is an unsolved problem for uranium reactors as well.
3. It is easy to make a nuclear bomb out of plutonium. Unlike ^{235}U, plutonium can be chemically separated from ^{238}U. This raises the problem of guarding a plutonium reactor against malicious diversion of even small amounts of nuclear fuel. Enough plutonium for a bomb can probably be carried in a brief case.[8]
4. The technology of breeder reactors is recent and has had little time for testing. Yet it is much more novel, elaborate, and sophisticated than that of the uranium reactors used in power plants and nuclear ships. Estimates vary enormously both as to the probability of large-scale accidents and the capital cost of "sufficiently safe" construction. Nonetheless, there are dozens of breeders in operation throughout the world.

EXERCISE 4 Nuclear Fission

A typical fission reaction using ^{235}U is

$$^{235}_{92}\mathrm{U} + {}^{1}_{0}n \rightarrow {}^{236}_{92}\mathrm{U} \rightarrow {}^{144}_{56}\mathrm{Ba} + {}^{89}_{36}\mathrm{Kr} + 3\,{}^{1}_{0}n$$

in which three neutrons are emitted, leaving radioactive barium and krypton as *fission fragments*. Since more neutrons are given off than are absorbed, this reaction could be self-sustaining. Verify that the number of protons and neutrons is the same on the right as on the left.

Complete the following reaction,

$$^{235}_{92}\mathrm{U} + {}^{1}_{0}n \rightarrow {}^{236}_{92}\mathrm{U} \rightarrow {}^{140}_{54}\mathrm{Xe} + {}^{94}_{38}\mathrm{Sr} + \underline{\qquad\qquad\qquad}$$

in which the fission fragments are xenon and strontium. How many neutrons are emitted? Could this become a chain reaction?

EXERCISE 5 Breeder Reactors

The name *breeder reactor* and the fact that it makes its own fuel tend to give the impression that we get something for nothing, possibly violating some conservation law.

(a) What raw material is used up in the manufacture of plutonium?
(b) Why is this so much cheaper than ^{235}U?
(c) The fission of one ^{239}Pu nucleus results in the release of about 200 MeV of energy, which shows up as kinetic energy of the fast neutrons and the fission fragments. How much energy is released in the fission of 1 mole (239 grams) of plutonium? (HINT: 1 mole = 6.02×10^{23} atoms.)

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[8] Or better, in two brief cases. Safety first! Actually, critical size is still a military secret—highly "classified" information.

(d) How about one ton of plutonium? (HINT: 1 ton = 10^3 kg = 10^6 grams.) Express this energy in kilowatt-hours. (HINT: 1 eV = 1.6×10^{-19} joule. 1 watt hour = 3600 joules.)

(e) In some cities, electric power costs around 5¢/kWh, delivered to the home. What fraction of this would be fuel cost if the fuel were uranium at $40 per pound?

NOTE: With coal-fired power plants, over half the cost goes for fuel. As you see, virtually the entire cost of nuclear power would be capital cost. Then you have to ask not for the cost per kilowatt-hour, but rather for the cost per kilowatt. Current capital costs of generating plants are around $600 per kW of electric power for coal-fired, $1200 for nuclear plants.

19.4 APPLICATIONS

In a nuclear power reactor, as in a nuclear bomb, what we are after is the heat given off in nuclear fission. The neutrons and the fission fragments come off at high speeds. As they are slowed down in being scattered by surrounding nuclei, these neighbors become hot. If you are designing a bomb, you want the chain reaction to proceed so fast that most of the fissions will occur before the high pressure has time to blow the thing apart. You want to attain temperatures of millions of degrees. If you are designing a power reactor, you want to control the rate of the reaction so the temperatures are high enough to run a steam turbine efficiently but not so high as to melt the pipes. The intense flux of neutrons throughout the reactor core presents problems of safety for maintenance personnel and of radiation damage to the metal making up the structure.

The high-flux neutron beam is precisely what is exploited in other applications of nuclear fission. Dozens of different radioisotopes can be made by putting nonradioactive materials in a beam of neutrons. A list of just a few of the uses for such *induced radioactivity* follows. Of course, the neutron beam is itself a research tool. For example, in the analytic technique called *neutron activation*, the secondary radiation following neutron bombardment has characteristic spectra for different elements. This can be used to identify even trace quantities of certain elements in a target sample. (See Problem 13.) A medical use of neutron activation is given in Problem 12.)

Diagnosis

A gamma emitter can be injected into an internal organ as a way of obtaining a photograph of that organ. The different stopping power of different kinds of tissue for short-wavelength photons allows tumors to be detected in this way. Many hospitals now replace the film with banks of gamma counters hooked into sophisticated computers. One such technique, called computerized axial tomography (CAT or CT) has become very prominent.

Tracers

Tritium (^3_1H) is chemically identical to ordinary hydrogen (^1_1H), but beta-decays with a 12-year half-life. Tritiating chemical compounds containing hydrogen—replacing some hydrogen atoms by tritium—is a way of tagging or "labeling" those hydrogen compounds and following them through various chemical and biological changes. This is just one example of a *radioactive tracer*. Tracer techniques have proved useful in agricultural research, in reactionrate chemistry, in physiology, even in just finding gas leaks.

Usually, the tracer is monitored with a radiation counter. Or it may be desirable to localize the tracer with a photograph, using *autoradiography*. When tissue containing radioactive tracer is placed in close contact with a photographic film, the precise structure in which the tracer has concentrated may be identified. Even the distribution of tracer with that structure can be resolved if the range of the ionizing radiation is sufficiently short.

Therapy

The radiologist now has a wide choice of alpha, beta, and gamma sources for radiation therapy. They differ in half-life, in the energy of the radiations, and in the rate at which they are eliminated from the body. Also, certain elements tend to concentrate in specific regions of the body. For example, iodine goes preferentially to the thyroid gland. Consequently, ^{131}I is particularly useful for irradiating the thyroid while limiting the radiation dose to the rest of the body.

The idea in radiation therapy is to destroy pathological growth. The photons (gamma rays) or particles (alphas and betas) emitted by radioactive nuclei have high-enough energies to create ion pairs in their path through matter, that is, to break chemical bonds. A trail of ions in a living cell is likely to damage the cell. One way is by breaking chromosomes. In the treatment of cancer with radiation the idea is to destroy the malignant cells.

Radioactive Energy Source

Most fission fragments have very short half-lives so that the core of a nuclear reactor has high fluxes of beta and gamma radiation. In some of the radioactive chains by which they decay, nuclei occur with half-lives of a few months or years. An example is strontium-90, produced as a result of ^{235}U fission:

$$^{235}_{92}U + ^{1}_{0}n \text{ (slow)} \rightarrow ^{144}_{56}Ba + ^{90}_{36}Kr + 2\,^{1}_{0}n \text{ (fast)}$$

The beta decay of barium-144 is so fast that its half-life, probably milliseconds, has not been measured. The beta decay of krypton-90 has a 33-second half-life, and is quickly followed by another:

$$^{90}_{36}\text{Kr} \xrightarrow[33\text{ sec}]{\beta^-} {}^{90}_{37}\text{Rb} \xrightarrow[174\text{ sec}]{\beta^-} {}^{90}_{38}\text{Sr}$$

The chain does not end there, but ^{90}Sr has a 28-year half-life for beta decay. It can be chemically separated from other material removed from the reactor and makes a lightweight energy source now used in artificial satellites. Exercise 6 gives an idea of the power avaialble from just a few ounces of such a fission product.

You may have read about strontium-90 in quite a different context. Its high specific activity made strontium-90 a particularly dangerous component of the radioactive fallout from nuclear test explosions. Settling over pasturage and thus getting into the milk, strontium-90 was ingested by people who drank that milk. Since strontium is chemically similar to calcium, it tends to be incorporated into the bone. Children, of course, form bone rapidly, so those who were children in the early 1960s, before atmospheric testing was banned by international agreement, may still carry some of that radioactivity around.

EXERCISE 6 **Radioactivity as an Energy Source**

This is just to give an idea of the power available, say, from 1 mole of strontium-90, just 90 grams. ^{90}Sr emits a 0.5-MeV beta ray, with a half-life of 28 years. Compare this with radium, which has a half-life of 1620 years, whose activity we know: 1 gram of ^{226}Ra has about 3.7×10^{10} disintegrations per second; so 1 mole has 226 times that many. One mole of ^{90}Sr will have an activity 1620/28 times that, or ____________________ disintegrations/sec. Multiply times 0.5×10^6 eV to get the energy per second (remember that 1 eV $= 1.6 \times 10^{-19}$ joules). This gives you the available power = ________ watts. (1 watt = 1 joule/sec.)

NOTE: The daugher nucleus $^{90}_{39}$Y is also a short-lived beta emitter. If we included its radiation, that would effectively double our estimate of the power available. Incidentally, that daughter also emits gamma rays, so you don't want to sit too close. Gammas have a long range in air.

19.5 HALF-LIFE

If a hospital owns a radium source, the knowledge that half of that valuable radium will have decayed in 1620 years probably doesn't bother the hospital administrator too much. But if he buys some phosphorous-32, which has a **half-life** of 14.3 days, he knows he has a wasting asset. In two weeks, half of what he has bought will have decayed away. In a month, three-quarters of his investment is gone. Of course, the rapid decay is the key to some of the uses for ^{32}P and other short-lived radioisotopes. For example, they can be injected into the patient for radiation therapy, and the entire dose of radioactivity is delivered in a short time period. Even if the chemical substance is slow to be eliminated by the body—and phosphorous might well remain chemically bound—its activity will have decreased 75% after two half-lives. Repeat: After one half-life, half of the activity remains; after two half-lives, only half of that half is left, 25% of the original activity. One month after an injection of ^{32}P, the patient has received 75% of all the radiation that sample of ^{32}P will ever emit—three times as much as it is destined to emit forever after. After three months—six half-lives down—only 1½% of the activity is left.

The kind of arithmetic we have just been doing is made graphic by plotting a **decay curve.** If we plot the fraction of nuclei remaining undecayed versus time, we get the concave curve of Fig. 19.4. The rate at which nuclei disintegrate at any point in time is proportional to the number of nuclei you have left at that time. This law reflects the independence of the individual nuclei: The disintegration of one nucleus in no way affects the disintegration rate of its neighbors. So the time for half the population to decay is independent of the size of the population. That is the time we have called the *half-life*.

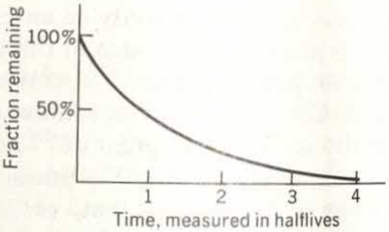


FIGURE 19.4 Decay curve. Note that moving the time origin to the right gives a curve having the same shape, just scaled down.

If you look closely at the decay curve of Fig. 19.4, you notice that the shape of the curve does not depend on when you start counting time. For instance, if you were to "start the clock" at the time called "1 half-life" instead of at zero, the ordinate you used to call 50% would be called 100%, and all other values would just be scaled down by one-half. When time advances one half-life, the fraction remaining is cut by a half.

Convince yourself that after 10 half-lives only one nucleus in 1024 remains: less than 0.1%. Argument:

1 half-life → 1 in 2

2 half-lives → 1 in 4

3 half-lives → 1 in 8

. . .

10 half-lives → 1 in ☐

That means that a radioactive sample is 99.9% "gone" after 10 half-lives. We say that the fraction remaining decays *asymptotically* to zero.

Show that it takes 20 half-lives for 99.9999% of a sample to decay, that is, for only one nucleus in a million to remain. Here is the algebraic formalism: The symbol τ_{half} represents the half-life:

Fraction remaining after 1 half-life is

$$f(\tau_{\text{half}}) = \tfrac{1}{2}$$

Fraction remaining after 2 half-lives is

$$f(2\tau_{\text{half}}) = (\tfrac{1}{2})^2 = \tfrac{1}{4}$$

Fraction remaining after 3 half-lives is

$$f(3\tau_{\text{half}}) = (\tfrac{1}{2})^3 = \tfrac{1}{8}$$

. . .

Fraction remaining after p half-lives is

$$f(p\tau_{\text{half}}) = (\tfrac{1}{2})^p$$

Try substituting $p = 10$ and verify what we said above about 10 half-lives.

So the decay curve (Fig. 19.4) is a graph of the mathematical function *f(t)*, the fraction of radioactive nuclei remaining after a time $t = p$ half-lives:

$$f(t) = (\tfrac{1}{2})^{t/\tau_{\text{half}}} \qquad \text{(Eq. 1)}$$

About 4 half-lives out, it becomes hard to draw the curve, just because of the scale we chose in order to fit our graph on the book page. The thickness of the line drawn through the points is comparable to the value of f we are trying to plot.

The choice of the half-life as the time constant was just a convenience, of course. If the time for half the nuclei to decay is a constant, then the time for, say, three-quarters of the nuclei to decay is also a constant, namely 2 half-lives. (OK?) But any other fraction would do as well for defining a time constant.

Now you ought to worry about what happens if the time $p\tau_{\text{half}}$ is not an integer number of half-lives. For example, what fraction remains after half a half-life?

$$f(\tfrac{1}{2}\tau_{\text{half}}) = ?$$

You can try to read it from the graph, but that may sound like cheating. How was the smooth curve in the graph drawn, anyway?

Let's call the answer x. After half a half-life, a fraction x remains of the original number of nuclei. But another half a half-life later, what remains is a fraction x of that fraction. Repeat: After 2 half-lives, a fraction x^2 remains. But 2 half half-lives is 1 half-life, after which the fraction remaining is $\tfrac{1}{2}$:

$$x^2 = \tfrac{1}{2}$$

So x is the square root of $\tfrac{1}{2}$, or 0.707. Look at the graph again. How well was the curve drawn? And the formula? If we set $p = \tfrac{1}{2}$ in the formula

$$f(p\tau_{\text{half}}) = (\tfrac{1}{2})^p$$

we get

$$f(\tfrac{1}{2}\tau_{\text{half}}) = (\tfrac{1}{2})^{1/2}$$

Taking a number to the one-half power means taking its square root. That may not be news to you, but it does show you how to fill in the rest of the curve with nothing fancier than a square-root calculator. For instance, to find the fraction remaining after one-quarter of a half-life, you have to take the square root of the fraction remaining after one-half a half-life:

$$f(\tfrac{1}{4}\tau_{\text{half}}) = (\tfrac{1}{2})^{1/4} = ((\tfrac{1}{2})^{1/2})^{1/2} = \sqrt{0.707} = 0.841$$

Repeat: After one-quarter of a half-life, 84.1% of the nuclei remain.

More Math: Logarithms

If somebody asked you how long (how many half-lives) it takes until only 10% remain, that is, how long it takes for 90% to decay, you could look at the curve and say, "just a little over three half-lives." To do better, you could start on the square-root-taking process and get an answer by successive approximations. Tiresome? You would see the advantage of having printed tables or a button on your calculator to solve this sort of problem to four or five significant figures. Chances are you are familiar with some tables you could use: tables of logarithms.

The question being asked is: To what power do you raise $\frac{1}{2}$ to get $\frac{1}{10}$?

$$(\tfrac{1}{2})^p = \tfrac{1}{10}$$

The power p is the number of half-lives until only one-tenth of the original nuclei remain. How do you solve an equation like that for p? If you take the reciprocal of both sides,

$$2^p = 10$$

the question becomes, to what power do you raise 2 to get 10? That is a question for a table of logarithms to base 2; you probably don't have one of those. Your log table used base 10. It can answer the question, "To what power do you have to raise 10 to get 2?" That will get you the reciprocal of what you are after:

$$10^{1/p} = 2$$

The answer is 0.301. If your calculator does not have a "log" function, you might try an old log table or slide rule to check this; the number 0.301 is called the common logarithm of 2. Repeat:

$$10^{0.301} = 2$$

Turn that around to read

$$2^{1/0.301} = 10$$

The reciprocal of 0.301 is 3.32. That answers our question:

$$(\tfrac{1}{2})^{3.32} = \tfrac{1}{10}$$

It takes 3.32 half-lives until only one-tenth of the nuclei remain.

Let's go over that again quickly. You want to solve the equation

$$\left(\tfrac{1}{2}\right)^p = \tfrac{1}{10}$$

Take the logarithm of both sides:

$$-p \log 2 = -\log 10$$

If your logs are to base 10, that is, common logarithms, the log of 10 is 1, the log of 2 is 0.301:

$$0.301p = 1 \qquad \text{or} \qquad p = 3.32$$

The time for the population to decay to one-tenth of its original value is 3.32 times the time to decay to half its original value.

EXERCISE 7 We said it takes 10 half-lives for a population to decay to 1/1024 of itself. That's 0.00097656; awfully close to 0.001 or 10^{-3}. How many half-lives does it take to get 10^{-3}?

$$2^{-p} = 10^{-3}$$

Take logarithms and solve. But before you do—Will the answer be more or less than 10?

Semilogarithmic Graph Paper

Every 14.3 days the number of phosphorus nuclei in a sample of ^{32}P is halved. Every 14.3 days the logarithm of that number falls by 0.301. (That's log 2.) While the population decays *exponentially,* its logarithm falls off linearly. If we wanted to convert the decay curve (Fig. 19.4) into a straight-line graph, instead of plotting the fraction f versus the time, we should plot its logarithm.

A good way to do this without much labor is to use semilog graph paper (Fig. 19.5). If you look at the vertical axis, you see that the numbered lines are not evenly spaced. But if you measure the distance between the 1 and the 2, it is the same as the distance between the 2 and the 4, the same as the distance between the 4 and the 8. It is also the same as the distance between the 5 and the 10. The 10? There is no 10 marked. The line we would call 10 is just marked 1 again, the 20 is marked 2, and so on. You have to put the powers of 10 in yourself. Try measuring the distance between the 8 and the 16. Familiar?

If you look at the straight line plotted in Fig. 19.5, the upper left-hand corner might represent 100%: At time zero, nothing has decayed, all the nuclei are still there. At time $t = 14.3$ days, the fraction remaining is 0.5. It looks as if we are plotting the decay curve of phosphorus-32, with a half-life of 14.3 days. If we had called the upper left-hand corner 1 million nuclei, then at time $t = 14.3$ days we would read half a million.

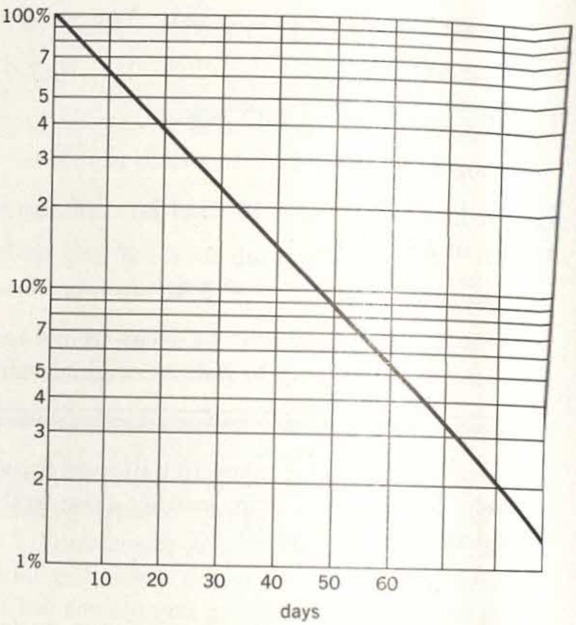


FIGURE 19.5 Decay of phosphorus-32. Semilog plot.

Notice that semilog paper has no "origin." Every time the population is divided by 10, the ordinate (the reading on the vertical scale) falls down into the same spot in the next "decade": From 100% to 10% to 1% to 0.1%, each step takes you the same distance down the vertical axis. Obviously, you never get to zero that way. From 0.1 to 0.01 is the same distance on the logarithmic scale as from 0.2 to 0.02 or from 200 to 20. That is what we call one decade.

EXERCISE 8

(a) What is the half-life of a decay curve that goes from 0.2 to 0.02 in 47.5 days?

[Answer: 14.3 days] (HINT: Use the semilog scale.)

(b) What is the half-life of a decay whose slope on the semilog scale is twice as steep?

(HINT: It drops from 0.2 to 0.02 in $\frac{1}{2} \times 47.5$ days.)

Decay Rate

If you wanted to measure the half-life of a short-lived radioisotope like phosphorus-32, you would set your sample near a counter, count for a minute or two, and repeat the procedure periodically to see how fast the activity fell. For ^{32}P it would take a few days before you saw an appreciable decline. You would plot the activity versus time on a semilog scale to verify that it was indeed decaying exponentially—giving a straight line on the semilogarithmic plot. Also, you would have no hesitation in setting the time in which the activity falls to half equal to the half-life, the time in which the number of undecayed nuclei falls to half. That the activity is proportional to the number of undisintegrated nuclei just reflects the mutual independence of the nuclei: The probability of decay of each nucleus per unit time is constant and does not depend on the presence of other nuclei.

But how is the decay rate related to the half-life? What is the proportionality constant relating the activity of a sample to the number of undecayed nuclei in it? You can read it right from the graph of Fig. 19.4: The decay rate is the rate at which the number of undisintegrated nuclei decreases. So the fraction decaying per unit time is the slope of the decay curve right at time zero, when 100% remain undecayed. Try measuring it on the graph in Fig. 19.4. Then compare your result with the formula we derive in Problem 18:

$$\text{decay rate} = 0.693\,\frac{\text{population}}{\text{half-life}} \qquad \text{(Eq. 2)}$$

With a half-life $\tau_{\text{half}} = 14.3$ days for ^{32}P, that formula gives a slope of 4.85% per day.

Don't make the mistake of multiplying 4.85% per day times 14.3 days (the half-life) and expecting to get 50%.[9] Remember, the decay curve is not linear, but exponential. The first day, about 4.85% of the sample decays. The second day, about 4.85% of what is left decays. Fewer nuclei disintegrate on the second day than on the first; the decay curve gets flatter, less steep. But you don't notice that it is curving until you get to time intervals which are an appreciable fraction of a half-life.

Decay Constant and Mean Life[10]

From Eq. 2, or from the decay curve, we obtained the proportionality constant relating the activity (or disintegration rate) to the population. For phosphorous-32, this *decay constant* was

$$\frac{\text{disintegration rate}}{\text{population}} = 0.0485 \text{ per day}$$

We could have written that as a fraction

$$= \frac{1}{20.6 \text{ days}}$$

This means that the average ^{32}P nucleus decays in 20.6 days, or the average lifetime (also called mean life) of a ^{32}P nucleus is 20.6 days. It should not surprise you that the *mean life* is longer than the *half-life*, 14.3 days. After all, 10% of the nuclei live more than 3.32 half-lives, 1% live more than 6.64 half-lives, 0.1% live more than 9.96 half-lives, and so forth: The decay curve never really ends. Look at Fig. 19.4. It has a long tail. Any statistical distribution skewed in that way is going to have its mean greater than its median. Going back to the formula, the mean life, which is the reciprocal of the decay constant, is the half-life divided by 0.693:

---

[9] You get 69.3%. OK?

[10] This section is rather mathematical.

$$\tau_{\text{mean}} = \frac{\tau_{\text{half}}}{0.693}$$

Equation 2 becomes simply

$$\text{decay rate} = \frac{\text{population}}{\text{mean life}} \qquad \text{(Eq. 2')}$$

Some books prefer to write the formula (Eq. 1) for the fraction remaining after time t not in terms of the half-life τ_{half},

$$f(t) = 2^{-t/\tau_{\text{half}}} \qquad \text{(Eq. 1)}$$

but in terms of the mean life τ_{mean},

$$f(t) = e^{-t/\tau_{\text{mean}}} \qquad \text{(Eq. 1')}$$

The number e thus defined is equal to

$$e = 2.718$$

which is known as the "base of the natural logarithms." Note that to any base, we can write

$$\log 2 = 0.693 \log e$$

In particular, the ratio 0.693 of the half-life to the mean life is $\log_e 2$.

What is the advantage of using e rather than 2, of using mean life rather than half-life? Not much, but you are likely to run across it. We should say this: If you want to measure the half-life of a long-lived species like radium, you don't do it by counting for 1620 years. You measure the activity by counting for a few minutes, measure the population by doing quantitative chemistry, and divide to find the decay constant, or its reciprocal, the mean life.

EXERCISE 9 Half-Life of Radium

To measure the half-life of a radioisotope like radium-226 ($\tau_{\text{half}} = 1620$ years), you don't have to wait around for hundreds of years watching the counting rate slowly decline. You use the proportionality of the activity and the population. The number of nuclei in a radium sample can be determined by accurate weighing, knowing the atomic weight (= 226) and Avogadro's number (= number of atoms in a mole). Here is a historic calculation, based on an activity of 3.7×10^{10} disintegrations per second (= 1 curie) from 1 gram of radium:

$$\text{no. of atoms in 1 gram of radium} = \frac{6.02 \times 10^{23} \text{ atoms per mole}}{226 \text{ grams per mole}}$$

Now use Eq. 2:

$$\text{activity} = \frac{3.7 \times 10^{10}}{\text{sec}} = 0.693 \frac{\text{no. of atoms}}{\text{half-life}}$$

and solve for the half-life. You get it in seconds and want to convert to years:

$$1 \text{ year} = 365 \text{ days} \times (24 \text{ hr/day}) \times (60 \text{ min/hr}) \times (60 \text{ sec/min})$$

You expected to get 1620 years and are a bit disappointed. But remember, we called this "historic calculation." The 2% discrepancy represents the state of the art half a century ago, when the curie was defined.

19.6 RADIATION DOSE AND SAFETY

When you visit a nuclear-reactor facility, you register upon entering, and a radiation monitor is attached to your lapel. This may be a simple film badge, in which ionizing radiation causes a photochemical reaction in the emulsion. The film has to be developed before they know how much exposure you got. Or the radiation detector may be a dosimeter, from which the radiation dose you have received can be read directly. Looking something like a fountain pen, most dosimeters are essentially electroscopes, charged up to several hundred volts. Each ion produced inside by radiation allows one electronic charge to leak off. When you leave, the charge remaining on the instrument can be measured.

Why all the safety precautions? Surely you are not going to stand in a beam of neutrons emerging from the reactor! The danger comes from other things hit by the neutrons. Neutrons can be absorbed by stable nuclei, making these radioactive. And even a microgram of a short-lived radioisotope can be a dangerous source of ionizing radiation.

Dose—Roentgen, Rad, and Rem

How much radiation do you get from a dental X ray? How much lead will protect you from a "hot" source? How much does breathing radioactive dust hurt you? How much radiation will give you cancer, leukemia, or a cataract? In the technical literature, such questions are answered in technical jargon. Radiation **dose** is measured in **roentgens, rads,** or **rem.** The ratio of the rem to the rad is the RBE (relative biological effectiveness). How big a dose it takes to kill you is expressed in MLD (mean lethal dose). The thickness of lead to shield you is measured in **HVL** (half-value layer).

Why the jargon? After all, energy flux is easy enough to measure—the number of watts striking per unit area of the beam—and we measure sunshine that way: With the sun directly overhead, we get about 1 kilowatt per square meter. Why don't we talk that way about X rays? It is because our concern is not how much energy we are absorbing but rather how much radiation damage. And damage has to do with ionization. It's not just how warm the X rays make your insides, it's how many electrons they rip off the atoms there that matters. But the various X rays and alphas and betas and gammas and neutron beams are not the same in that respect. That is where RBE comes in—to translate energy absorbed into amount of ionization produced.

The **roentgen** is the amount of X ray exposure that produces a certain density of ion pairs (2.08×10^9 per cubic centimeter in air) irrespective of the intensity of the beam, the width of the beam, or the energy of each X-ray photon. That doesn't tell you how much it ionizes human tissue, which turns out to depend on the X-ray energy. The energy deposited per gram of tissue irradiated is measured in rads. One **rad** is the radiation dose that deposits 100 ergs per gram of tissue (= 0.01 joule/kg). This definition was chosen so that, for medical X rays, an exposure of 1 roentgen represents a dose of about 1 rad. One hundred rads (= 1 joule/kg) is called 1 gray.

For other radiations, it is useful to know how effectively they produce ionization, joule for joule, compared to X rays. The ratio is called the RBE. There are tables of this ratio for the emanations of the various radioisotopes used in biology and medicine. You multiply the rads by the RBE to get dose in **rem** (radiation equivalent:man). For example, the RBE of alpha rays is 10. One rad of alphas corresponds to a dose of 10 rem. One rad of alphas does as much damage as 10 rads of X rays, gammas, or high-energy betas. This means that the RBE of X rays, gammas and high-energy betas is 1.0. Low-energy betas have an RBE of 1.7; the RBE of thermal neutrons is 3.

Now let's use our new vocabulary. If you irradiate the human reproductive organs with X rays, it takes an average exposure of 500 roentgens (written 500 R) to induce sterility. About the same dose of X rays generates a cataract in the cornea. Instead of saying exposure of 500 roentgens, we could have said dose of 500 rads or 500 rem. Those numbers probably don't depend much on whether the radiation is absorbed over a few minutes or over many years. But if the whole body is irradiated, a dose of about 400 rem has 50% probability of being lethal if absorbed over a short time. It is less lethal if it is spread over a long time. In other words, there is some healing of damaged tissue. The mean lethal dose **(MLD)** of whole-body irradiation is around 500 rem for one-shot exposure. For a whole-body dose of 100 rem the probability is about 50% that the victim will show symptoms of radiation sickness, which are nausea, fatigue, and greatly reduced white-corpuscle count in the blood. There is now considerable evidence that such doses of radiation shorten the average life span. There is no evidence that there is any threshold dose below which radiation has no harmful effects. An annual dose of 1 rem will, over a long time, induce about 2 fatal cancers in a population of 10,000 people.

Nonetheless there are so-called permissible doses, specified in laws and safety regulations. For workers in occupations using radiation, the **MPD,** or maximum permissible dose, is 5 rem per year. For exposure only to the hands, the MPD is 75 rem per year. For the general population, on the other hand, the MPD is 0.5 rem/year. Perhaps the logic is that the casual passerby is not being paid to be irradiated. You might want to compare these "legal" dose rates with those from "natural" causes, namely the cosmic rays from outer space and the radioactivity of the air, the water, and the soil. That is in the range 0.1 to 0.2 rem per year. If you live in a brick house, it can get as high as 0.5 rem/yr, because brick contains radioactive cobalt-60.

A chest X ray can give as much as 50 millirem per film. A dental X ray can give 5 millirem. A course of cancer therapy can give a total as high as 10,000 rem. Note that these are highly localized, not whole-body exposures.

Two examples of radioisotope therapies illustrate this difference. Tumors in the thyroid gland, which has a mass around 40 grams, may be treated with iodine-131, which emits 0.6-MeV betas as well as 0.4-MeV gammas. Although its radioactive half-life is 8 days, iodine is excreted at such a rate that 5.5 days after administration its activity is down to 50%. So its *effective* half-life is 5.5 days. An initial activity of 3 mCi will deliver 6100 rem to the thyroid, about 60% of which happens in the first week.

Compare with a whole-body treatment for leukemia with sodium-24, whose half-life is only 15 hours, emitting 1.4-MeV betas and 2.7-MeV gammas. Injecting 500 mCi of ^{24}Na into a 60-kg man will give him a total dose of only 0.4 rem. A much greater source activity (500 mCi *vs* 3 mCi), and greater particle energies. But because the activity is distributed over such a large volume and decays so rapidly, the total dose is quite small.

Shielding

Before the dentist takes an X ray, he may cover you with a "lead apron," to reduce your exposure to stray radiation. Is the 2 or 3 millimeter thickness of lead embedded in his rubber apron enough to shield you from the harmful X rays? Again, you won't get a yes-or-no answer to such a question if you want scientific accuracy rather than a reassuring pat on the head. This kind of question is usually answered in terms of the **HVL,** the half-value layer: How thick a barrier of lead does it take to absorb half the X rays? That is the way to ask the question, because that is the way the absorption of a beam of photons of a well-defined energy works. One HVL of lead cuts the beam intensity in half. Two HVL's cut it in half twice, so only one-quarter of the X rays get through. Through a thickness of three HVL's only one-eighth of the X-ray intensity penetrates, and so forth. Sounds familiar? Absorption is an exponential process, like radioactive decay. Ten HVL's cut the intensity to 1 part in 1024 (= 2^{10}). For 100-KeV X rays, the HVL of lead is about one-tenth of a millimeter. A one-millimeter layer would be 10 HVL's and would cut the intensity down to about one part in a thousand. Two millimeters of lead will let through about one X-ray photon in a million.

This is not written in order to make you feel good about X rays or to get you to wear lead aprons. It is included in this chapter in order to build your intuition about radiation shielding. There is no shield that absorbs 100% of the X-ray photons incident on it. But if 99.9% absorption is good enough for you, then you can get along with 10 HVL's of shielding material.

X-ray technician wearing a protective lead apron.

EXERCISE 10 Do the arithmetic to prove this.

Here are some words of caution. "Harder" X rays—higher photon energies—are more penetrating. For example, the HVL of lead for 2.5-MeV photons is about a hundred times that for the 100-KeV photons. Two millimeters of lead would absorb a negligible fraction of such rays. Yet some gamma-ray energies from radioactive substances are that high. Now you know why the lead bricks used for shielding in radiochemistry laboratories are over 2 inches thick.

Lead bricks shield against "hard" ionizing radiation in a "hot lab."

Range

The absorption of alpha and beta rays works quite differently from the absorption of photons. Unlike a photon, an alpha can lose its energy a little bit at a time. It just slows down gradually and finally grinds to a halt. As it ionizes the material it passes through, it gives up its kinetic energy along its path. The higher the kinetic energy of the alpha, the further it can go before stopping. So the energy of an alpha can be measured by its range in air, using a "range-energy" table. Here is an example:

Alpha-Particle Energy	Range in Air
5.3 MeV	3.6 cm
7.6 MeV	6.6 cm

Note that the range is not proportional to the energy: As an alpha slows down, it has time to create more ion pairs per centimeter of travel. The range-energy relation is similar in animal tissue (or water), but the ranges are much shorter, typically less than a tenth of a millimeter. Betas have longer ranges than alphas, but we are still talking about millimeters in animal tissue.

19.7 STABILITY IS RELATIVE

When you first learned about the Periodic Table of the elements in your chemistry course, you may have wanted to ask why there are no elements in nature heavier than uranium. If you did ask, you may have been told that indeed, there are transuranic elements, but they are all man-made and all highly radioactive. Now you know why: Larger nuclei fall apart. The nuclear force is short-range; the electrostatic repulsion between protons is long-range. In Sect 19.3 (Fission), you learned that certain large nuclei can break up into smaller nuclei with the release of some neutrons and a lot of energy. The nuclear fission process was described as occurring when a large nucleus absorbed a neutron to become still larger. Is there any reason fission can't occur spontaneously?

Spontaneous fission does occur. For some transuranic nuclides (nuclear species) it is the most likely decay mode. It is a possible mode of decay for most nuclei in the heavy half of the Periodic Table, including the ones marked "stable" or nonradioactive. It is possible; but mostly it is *very* slow.

Take a familiar heavy element: gold. Only one isotope of gold occurs in nature, ^{197}Au, and no one has ever observed a gold nucleus decay. But a spontaneous fission process like

$$^{197}_{79}\text{Au} \rightarrow {}^{127}_{53}\text{I} + {}^{56}_{26}\text{Fe} + 14\,{}^{1}_{0}\text{n}$$

in which gold splits into iodine plus iron plus neutrons, is theoretically possible. It is highly improbable. Its halflife may be on the order of 10^{33} years. Take all the gold in Fort Knox: The probability is tiny that even one gold nucleus there has undergone fission in all the years that gold has been stored.

You might feel that such an improbable event is uninteresting. The point is that stability is a relative concept and can be made quantitative. Again, the discussion may strike you as nitpicking or overly philosophical. Is defining stability something like trying to define honor or virtue? But bear in mind the slogan, "Anything that *can* happen *will* happen." That quality of nature is inherent in the quantum theory. Any event that does not violate a law of physics has a certain probability per unit time of occurrence. When we say that a structure is stable, we mean that it probably won't come apart for a long time. In other words, its half-life for disintegration is very long on the time scale we happen to be considering.

Would you call something stable that has a 50% chance of decaying in 18 nanoseconds ($= 1.8 \times 10^{-8}$ sec)? You probably would not, until you compared it with something else with a half-life less than 10^{-16} sec. The one is 200 million times more stable than the other. We are talking about charged and neutral pions. These are particles emitted when two nuclei collide hard enough. They are part of the "nuclear glue" we talked about at the beginning of this chapter, the *strong interaction* that binds nucleons into nuclei. In studying the strong interaction, particles have been discovered with half-lives even shorter than the neutral pions. These "elementary-particle" discoveries have given new insight into the possibility of the breakup of that very stable particle, the proton.

In the study of living things, the longest life spans we encounter are of the order of 10^3 years. The shortest time constants, characterizing the speed at which biological structures can respond to external changes, are measured in milliseconds. That seems like a large range, from 10^{-3} to 10^{10} sec: more than 13 decades, that is, 13 powers of 10. But compared with the enormously greater range of time constants found in the realm of the fundamental particles, even that seems small. Life encompasses only a small fraction of the spectrum of nature.

SUMMARY

Discovery of radioactivity.

Nuclear force and stability; why heavy nuclei may be radioactive.

Randomness of radioactive decay.

Alpha decay.

Relative stability/activity of uranium and radium.

Beta and gamma decay.

The uranium-radium series.

Number of neutrons needed for nuclear stability.

Fission of uranium-235.

Controlled and uncontrolled chain reaction.

Breeder reactors, pro and con.

Some applications of radioactivity.

Decay curve; definition of half-life.

How to solve equations involving the exponential function.

Relation between decay rate and half-life.

Half-life versus mean life.

Definition of *dose;* roentgen, rad, rem.

Shielding against high-energy photons: How many HVL's?

Range of alphas and betas in air and water.

Very long and very short half-lives.

KEY TERMS

Nucleus The part of an atom containing all the positive charge and almost all of the mass. It consists of protons and neutrons. An atom consists of a nucleus plus a number of electrons equal to the number of protons in the nucleus.

Atomic number The number of protons in the nucleus. The elements are arranged in the Periodic Table in order of atomic number, Z. Hydrogen is $Z = 1$; uranium is $Z = 92$.

Radioactivity Spontaneous disintegration of a nucleus, in which it emits an alpha particle, an electron (beta particle), a positron (β^+) or a gamma-ray photon. The radioactive nucleus is thereby changed into the *daughter* nucleus.

Nucleon Proton or neutron.

Isotopes Two or more nuclear species of the same element (same number of protons) having different numbers of neutrons.

Alpha particle A nucleus of ^4_2He, consisting of two protons and two neutrons. Emitted in alpha decay.

Beta particle An electron emitted in beta radioactivity.

Beta plus A positron. Has the same mass as the electron but an equal and opposite charge. β^+ decay is positron emission.

Gamma ray A high-energy photon emitted by a radioactive nucleus.

Nuclear fission Disintegration of a heavy nucleus into two medium-heavy nuclei (the *fission fragments*) plus some neutrons.

Radiation detector Ionization chamber, Geiger counter, scintillation counter, junction detector, film badge, dosimeter.

Half-life The time in which activity of a single species of radioisotope decreases by 50%.

Nuclear force Short-range attraction that binds protons and neutrons into nuclei. Also called *strong interaction*.

Activity The number of nuclei disintegrating per second. The SI unit is the Becquerel (Bq) = 1 disintegration per second.

Curie Unit of activity = 37 billion disintegrations per second. Approximately the activity of 1 gram of radium.

Radioactive series Chain of radioactive decays—alpha, beta, and gamma—by which a heavy unstable nucleus is gradually transformed into a lighter stable nucleus. EXAMPLE: ^{238}U goes in 14 steps to ^{206}Pb (Fig. 19.2).

Radioactive dating A procedure in which monitoring the activity of a radioisotope and thereby determining its concentration permits finding the age of a specimen from a knowledge of the half-lives.

Decay curve Plot of the activity versus time. For a single radioactive species, this is an exponential decay.

Fissile nucleus A nucleus capable of undergoing fission rapidly upon absorbing a neutron. Uranium-235, uranium-233, and plutonium-239 are examples.

Dose Measure of ionizing radiation absorbed, in terms of ionization per unit volume.

Roentgen Unit of dosage for X rays and gamma rays. 1 R makes a density of ionization in air of 2.083×10^9 ion pairs per cm^3.

Rad Unit of dosage in terms of radiant energy absorbed per gram of tissue. 1 rad = 100 ergs per gram.

Rem Unit of dosage that permits comparing different ionizing radiations in terms of their relative biological effectiveness (RBE). 1 rem = 1 rad × RBE.

MLD Mean lethal dose. Whole-body exposure is assumed.

HVL Half-value layer. Thickness of absorber that lets through half the incident intensity of radiation.

MPD Maximum permissible dose, as determined by some regulatory agency. Usually expressed in dose per month or per year.

Transuranic element An element having atomic number greater than 92. These elements are all man-made and all unstable. The one most in the news is plutonium, atomic number 94.

REVIEW QUESTIONS

- The Periodic Table of the elements has mass numbers ranging from 1 up to 238. Where in that list is uranium?
- What does $Z = 92$ tell you about uranium?
- The sum of the number of protons plus the number of neutrons in a nucleus is called its ______________________ number.
- A collection of nucleons consisting of two protons and two neutrons is stable, that is, stays together, if they all get close enough together. What is it called?
- Can a collection of 238 nucleons stick together for a long time? What is one way it could break apart?
- Nuclear fission converts a large nucleus into two medium-sized nuclei plus some leftover neutrons. What characteristic of the strong interaction tends to make medium-sized nuclei more stable than large?
- All the transuranic elements (elements with $Z > 92$) are radioactive. Can you use a property of the strong interaction to conjecture why?
- What is the probability that a uranium-238 nucleus stays together for more than 4.5-billion years?
- What is the probability that a certain ^{238}U nucleus will disintegrate during the next year?
- In chemistry we learn that no chemical process can change an atom of one element into an atom of another element. Name two radioactive decay processes that do just that.
- What happens to the proton number in alpha decay? In beta decay?
- What happens to the neutron number in alpha decay? In beta decay?
- Why would you expect to find some lead in every sample of radium?
- If a magnet deflects alpha rays to the right, how would it deflect beta rays? Gamma rays?
- What does an alpha decay followed by two successive beta decays do to the proton number? How many neutrons lighter is the granddaughter than the mother?

- Why will a large piece of plutonium explode while a small piece won't?
- Why won't a large piece of natural uranium explode?
- To put out a fire, we take heat away faster than the burning produces heat. To quench a nuclear chain reaction, we need to absorb / accelerate neutrons faster than the fission process releases them.
- To sustain a chain reaction, at least one neutron from each fission reaction must be slowed down enough to start another fission before it escapes. To have a breeder, making more fuel than it consumes, at least two neutrons must be slowed down and absorbed before they escape. Explain.
- Uranium-239 can be made in a nuclear reactor when ^{238}U absorbs a neutron. Show how two successive beta decays convert a $Z = 92$ nucleus to a $Z = 94$ nucleus: uranium to plutonium.
- Explain the veiled joke in the footnote about "two brief cases." Why is it dangerous to carry a supercritical mass of plutonium in one briefcase?
- How are neutrons used to make stable isotopes into radioisotopes?
- How can radioactivity be used to find out how rapidly a drug is being eliminated through the urine?
- Strontium-90 has a 28-year half-life. How is it that there is more of it on earth today than there was 100 years ago?
- Why does a hospital get frequent deliveries of phosphorus-32, instead of buying a lot at one time?
- During which period of time does a ^{32}P source (two-week half-life) emit more beta rays, during the first two weeks after arrival, or during the second year?
- Why do you use logarithms to plot a decay curve?
- An exponential decay curve plotted on semilog paper becomes ______________.
- On semilogarithmic graph paper 1 decade (the distance from 1 to 10) is also called 1 "cycle." To plot an exponential decay curve for three half-lives, you can get along with 1 cycle. For seven half-lives, you have to have at least ________ cycles.
- Which is longer, the half-life or the mean life?
- To measure how much radiation you have received, suppose you multiply the energy flux in the beam times the cross-sectional area of the beam, to get energy per unit time. If you multiply that times the exposure time, you get the total radiant energy that bombarded you. Why is this not as useful as knowing the radiation dose?
- Distinguish roentgen, rad, and rem.
- Which is greater, MLD or MPD?
- Which of the two treatments described in Sec. 19.6 delivers more beta-ray energy to the body, the 3 mCi of iodine-131 or the 500 mCi or sodium-24?

- How much more radiation gets through three HVL's than through four?
- How many HVL's of shielding do you need to shield out 90% of the incident radiation (approximately)?
- Why do you need thicker shielding to protect against 1-MeV gamma rays than to protect against 100-keV gammas?
- Why does the concept of "range" not apply to gammas?
- One millimeter of lead will shield against virtually all the 3.5-MeV betas emitted by iodine-132. Why is it not a good shield against the 772-keV gammas that accompany this emission?

☐ QUESTIONS FOR FURTHER THOUGHT

α. We find X rays and gamma rays so useful because they go through human tissue so much easier than visible electromagnetic rays (light). Conjecture why no living species has developed X-ray vision.

β. Most alpha and beta decays are accompanied by a gamma decay. Was it alphas or gammas that fogged Becquerel's photographic plates, which were wrapped in thick black paper?

γ. Near the low end of the Periodic Table, the stable nuclei tend to have roughly as many neutrons as protons. Nuclei with more neutrons than are needed for stability usually disintegrate by ____________________ decay. This brings the neutron and proton numbers closer to equality. Some unstable nuclei with fewer neutrons than protons may be made artificially. They usually disintegrate by converting a proton into a neutron. The particle emitted in this radioactive decay has the same mass as an electron and is called a *positron*. For example, sodium-21 ($^{21}_{11}$Na) is a positron emitter. Its daughter is stable. What is this daughter called?

δ. Do you think that the dying of animals and humans obeys an exponential decay law? Is the probability of an individual's dying in the next hour independent of his age?

ε. What about a colony of bacteria in an intense beam of ultraviolet radiation? Present an argument for the validity of an exponential decay law here.

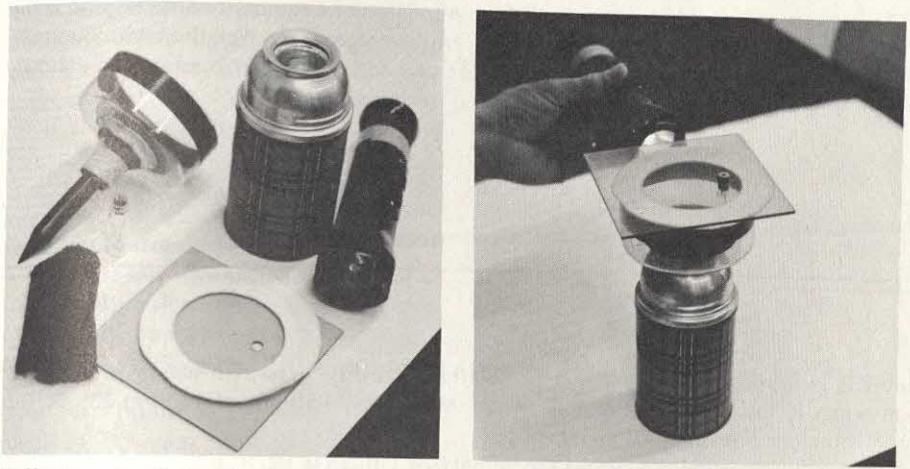
ζ. A neutron can emit an electron (beta decay) and become a proton, giving up energy in the process. The decay of a proton into a neutron with the emission of a positron (positive electron) only takes place inside a nucleus. Explain, in terms of what you know about the nuclear force.

☐ LAB AT YOUR DESK 1. Cloud Chamber

A cloud chamber allows you to see the tracks made in air by alpha and beta rays, much like a "condensation trail" made by a jet airplane. You can build your own.

Materials:

Styrofoam cup (large) or wide-mouth thermos bottle
Open metal can, small enough to fit inside (orange juice can . . .)
Transparent plastic cheese box



Diffusion cloud chamber—a classroom version. The plastic cylinder with the metal cooling foot fits into the thermos filled with dry-ice-alcohol slurry. The little hole in the plastic cover holds the stopper-fitted glass rod containing the alpha source. The wool cloth is for rubbing the cover between experiments to charge it up and collect any stray ions still in the active volume.

Blotting paper or filter paper
Dry ice
Alcohol (methyl is cheapest, but poisonous)
Flashlight
Alpha source (weak) (Your instruction will advise you about appropriate safety precautions.)

Procedure: Make a dry-ice-and-alcohol slurry in the cup and invert the can into it, so the end gets cold. Cut a hole in the bottom of the plastic box (about 2 in.) and place it over the cold metal. Use the cover of the box to hold a ring of blotting paper, soaked in alcohol.

Place the alpha source inside. As alcohol vapor condenses in the temperature gradient, you can see it diffusing down toward the cold metal surface if you shine the flashlight through the sides. Alpha tracks will become visible radiating out from the source. For better visibility it may help to blacken the metal surface with a candle flame or paint.

Experiment: How thick a piece of paper is needed to stop the alphas?

☐ LAB AT YOUR DESK 2. Luminous Dial

It used to be fashionable to make watches and clocks with numerals that glowed faintly. The luminous paint contains a "phosphor" like zinc sulfide and an alpha emitter like radium. When ZnS is bombarded with an alpha particle, it emits greenish photons. A low-powered microscope allows you to see the individual flashes in a dark room. Allow a few minutes for your eyes to dark-adapt.

Experiment: Concentrate on a small area, a fraction of a square millimeter, and count flashes. Take several one-minute counts, and average. [This was the technique used in the early experiments (1911) in nuclear physics.] Use your readings to estimate how many Curies of radium are contained in the whole dial.

PROBLEMS

✶ **1. Thorium Breeders** Uranium-233 is **fissile,** that is, it readily undergoes nuclear fission when it absorbs a slow neutron. This isotope is not found in nature but can be manufactured ("bred") in a nuclear reactor from natural thorium: $^{232}_{90}\text{Th}$ absorbs a neutron to become $^{233}_{90}\text{Th}$.

Write down the succession of beta decays by which $^{233}_{92}\text{U}$ is thus produced.

✶✶ **2. Radiocarbon Dating** Carbon-14 (half-life 5730 years) is created in the upper atmosphere when cosmic-ray neutrons impinge on nitrogen to cause the nuclear reaction $^{14}\text{N} + \text{n} \rightarrow {}^{14}\text{C} + {}^{1}\text{H}$. In the air the atomic carbon is soon oxidized to carbon dioxide. In this form it can be taken up by green plants. When a tree is cut down for wood, it stops taking on new ^{14}C. 5730 years later, the ^{14}C content of the wood will be half of that of living trees. By measuring the carbon-14 activity of old wood, its age can be determined.

(a) What fraction of the initial ^{14}C atoms will have survived 11,460 years after a tree is cut down?

(b) 2865 years?

(c) Suppose that the neutron flux in the cosmic rays is 1% greater today than it was 11,000 years ago. How would that affect the accuracy of dating an 11,000-year-old piece of wood?

NOTE: There seems to be evidence of some variation in neutron flux.

✶ **3. Half-Life** What fraction of a sample of phosphorus-32 remains undecayed six weeks after it is delivered? (Half-life = 2 weeks.)

✶✶ **4. Half-Life** The activity of a sample of ^{32}P is 10 mCi. When it was delivered, there were 160 mCi.

(a) How long ago was it delivered?

(b) It took one day (24 hours) for those 160 mCi to get from the nuclear reactor to the receiving room. What was the activity upon removal from the reactor?

✶ **5. Relative Stability** The half-life of uranium-238 is 4.47×10^9 years; the half-life of radium-226 is 1620 years.

(a) If the activity of 1 mole (6.02×10^{23} atoms) of radium is 8.17×10^{12} disintegrations per second, what is the activity of 1 mole of uranium? (HINT: Activity is inversely proportional to half-life.)

(b) One mole of carbon-14 (half-life 5730 years)?

(c) One mole of carbon-15 (half-life 2.4 seconds)?

(d) What is the ratio of the activities of carbon-15 and carbon-14?

(e) Calculate the activities of 1 gram of ^{226}Ra, ^{238}U, ^{14}C, and ^{15}C.

(HINT: Radium does not come out exactly 3.7×10^{10}/sec.)

✶✶ **6. Dating of Rocks** The earth's mantle solidifed about 3.9 billion years ago. That is about 0.87 of the half-life of ^{238}U, which is 4.5 billion years. If 1 mole of ^{238}U solidified in a rock, we would expect a fraction $(\frac{1}{2})^{0.87}$ of it to remain undecayed today, or 0.546 of a mole. (See Section 19.5 for help in checking this arithmetic.) The rest, or 0.454 of a mole, will have decayed, most of it having undergone all the steps of the radioactive series to lead-206.

(a) Verify that starting with 238 grams of molten ^{238}U, we expect to find, 3.9 billion years later, 130 grams of ^{238}U and 108 grams of ^{206}Pb.

(b) What is the ratio of the number of lead atoms to the number of uranium atom?

(c) If the rock was originally pure uranium dioxide (UO_2), which is 88 percent uranium by weight (verify!), what percent lead would you expect to find now? (HINT: Oxygen has atomic weight 16.)

✶✶ **7. Decay Rate** The activity of a radioisotope decreases 0.2% per hour. (HINT: That's one part in 500 per hour.)

(a) Show that its mean life is 500 hours.

(b) What is its half-life? Express in days.

✶ 8. **Radiation Dose** A certain therapeutic X-ray treatment delivers a dose of 1 roentgen.

(a) If the exposure time were doubled, the dose would be ________ R.

(b) If the current to the X-ray machine were doubled, keeping the voltage and the exposure time constant, the dose would be ________ R. (HINT: Power = volts × amps.)

(c) If the current were doubled but the exposure time halved, the dose would be ________ R.

(d) If a diaphragm cut the area of the beam in half, so that a smaller region of the body is exposed, the dose would be ________ R. Explain.

✶✶ 9. **Plutonium Pollution** Plutonium-239 has a half-life of 24,000 years. Suppose that 1 gram of ^{239}Pu somehow got lost in your house.

(a) How many alpha particles will it emit in 1 second?

(b) How many alpha particles will it emit in 1 year?

(c) How many curies in 1 gram? (HINT: $1\text{ Ci} = 3.7 \times 10^{10}$ disintegrations/sec.)

(d) If you happened to breathe 1 microgram of it, how many alphas will irradiate your lung during the next 40 years?

✶✶ 10. **Spent Fuel Rods** In a uranium-235 reactor, the fuel rods are roughly 97% uranium-238. The fission of ^{235}U provides neutrons that can be absorbed by ^{238}U to make plutonium. Plutonium can be chemically separated from uranium.

(a) Argue that a sophisticated terrorist might attack an old uranium-235 reactor and steal old fuel rods, as a way of obtaining plutonium for a nuclear bomb.

(b) Argue that this terrorist probably has to reprocess more than 30 kilograms of spent fuel rod to make 1 kilogram of plutonium.

(c) To see why the terrorist needs to be sophisticated, calculate the activity (= 0.693 N/half-life) of 1 kilogram of plutonium-239. (HINT: $N = 6.02 \times 10^{23} \times 1000/239$.)

✶✶ 11. **Radioactive Generators** For applications requiring short-lived radioisotopes it is obviously desirable to have the isotope produced near the location at which it is to be used. Every hospital can't have its own nuclear reactor; it can have a "radioactive cow" that can be "milked."

Such a generator contains a long-lived isotope with a short-lived daughter. The daughter, chemically different from the mother, can be separated from the "cow" by dissolving it in an eluting solution. For example, cesium-137 has a 30-year half-life for beta decay into "metastable" barium-137. The daughter decays into stable barium-137 with the emission of a gamma ray, with a 2.5-minute half-life. A slightly acid solution dissolves the barium, leaving the cesium behind. This is "milking the cesium cow."

(a) Suppose that you have 1 μCi of ^{137}Cs (= 3.7×10^4 disintegrations per second). If the barium is allowed to accumulate, its activity increases but only to the point where its rate of decay is equal to the rate at which it is replenished. Thus, in equilibrium, there will also be 1 μCi of radioactive ^{137}Ba (written ^{137m}Ba). Show that now the total activity of the generator is 2 μCi.

(b) You elute, taking away substantially all the barium. What is the activity of the solution emerging from the "cow"?

(c) What is the activity of the "cow" right after elution?

(d) What is the activity of the solution five minutes after elution?

(e) Twenty-five minutes after elution?

✶ 12. **Neutron Activation Therapy** A technique for treating brain tumors uses neutrons to produce a beta source right in the tumor. Boron-11, the more common isotope of boron, is neutron hungry: $^{11}\text{B} + n \rightarrow {}^{12}\text{B}$. Boron-12 beta-decays rapidly (half-life = 0.02 sec). Given intravenously, boron goes preferentially to the brain. A narrow neutron beam is aimed at the tumor. Every time a ^{11}B nucleus absorbs one of those neutrons, the tumor is irradiated with the 13-MeV electrons given off by ^{12}B.

(a) Suppose 1 micromole of ^{11}B (10^{-6} moles = 11 micrograms) has been absorbed by the tumor; of these boron atoms one in a million will be neutron-activated. How many beta rays will bombard the tumor?

(b) Why does therapy with beta radiation have to involve injecting something into the body? After all, we do gamma-ray therapy with external sources. (HINT: What is the range of betas in human tissue?)

✶✶ 13. **Neutron Activation** Only one isotope of aluminum is stable: $^{27}_{13}$Al. If a sample of pure aluminum is irradiated with neutrons in a reactor, the heavier isotope ^{28}Al is formed, which beta decays with a half-life of 2.3 minutes.

For a study in radiation damage, it was necessary to procure very pure aluminum and bombard it with neutrons. Several days after removal from the reactor, the sample was still "hot," emitting gamma rays with energies characteristic of scandium-46, which has an 83-day half-life. Evidently, the reputedly "pure" aluminum had some scandium-45 impurity, activated by the neutrons:

$$^{45}_{21}\text{Sc} + {}^{1}_{0}\text{n} \rightarrow {}^{46}_{21}\text{Sc}$$

(a) Suppose that the neutron irradiation was sufficient so that 1% of the ^{45}Sc impurity nuclei in a 1-mole sample of aluminum was activated and became ^{46}Sc. The ^{46}Sc activity several hours after removal from the reactor was 1.5 Ci. Express the scandium impurity content in parts per million. (HINTS: According to Eq. 2, one mole of ^{46}Sc would have an activity of $6.02 \times 10^{23} \times (0.693/83 \text{ days})$. Convert the days to seconds, and remember that $1 \text{ Ci} = 3.7 \times 10^{10}$ disintegrations/sec.)

(b) The supplier had guaranteed the aluminum to be "5 nines pure," that is, 99.999% pure. Was it?

✶ **14. Sodium-22 has a Half-life of 2.6 Years.**

(a) What fraction of a sodium-22 sample will decay in 10 years?

(b) What fraction will decay in 1.3 years?

✶ **15. Decay Probability** The probability that a radioactive nucleus decay in one half-life is $\frac{1}{2}$. Suppose that a radioactive sample is almost all decayed away. Only two undecayed nuclei are left. One half-life later, what is the probability that

(a) Both have decayed?

(b) Only one has decayed?

(c) Both remain undecayed? (HINT: The sum of those three probabilities has to equal 100%.)

✶✶ **16. Tritium Dating** Some radioactive $^{3}_{1}$H is constantly being produced by cosmic rays, so there is some tritium in the water we drink. It has a half-life of 12.26 years. Suppose that the concentration of tritium in a bottled wine is 88.7% of the natural tritium concentration found in the grapes from which the wine was made? How old is the wine?

✶ **17. How to Take Square Roots on a Four-Function Calculator** If your calculator has a square-root key, this recipe may not be useful to you—unless you need more accuracy! It starts with a guess. You guess the square root of your number, divide your number by the guess, and average the quotient and the guess to obtain a better guess. You repeat this until two successive approximations are as close to each other as you desire.

EXAMPLE: Try taking the square root of $\frac{1}{2}$ this way. Suppose that your guess is 0.8:

$$0.5/0.8 = 0.625$$

Average 0.625 and 0.8 to get 0.7125, an improved guess. Next:

$$0.5/0.7125 = 0.70175$$

Average 0.70175 and 0.7125 to get 0.70713.

Even starting with a poor guess we got four significant figures in two steps.

(a) Do the next approximation. (HINT: It turns out to be good to eight significant figures. To check that, do one more.)

(b) Take the square root of the number you got, that is, the fourth root of $\frac{1}{2}$, good to five decimal places.

The better your first guess, the faster you get there. When the fifth decimal place no longer changes, try raising your result to the fourth power to see how close that is to $\frac{1}{2}$.

✶ **18. Deriving the Decay-Rate Formula** The activity or decay rate of a radioactive sample is proportional to the population of radioactive nuclei. We want to calculate the proportionality constant, starting from the exponential decay law

$$f(t) = \left(\tfrac{1}{2}\right)^{t/\tau_{half}} \qquad \text{(Eq. 1)}$$

If we put in the time $t = \tau_{\text{half}}$ = one half-life, then the fraction remaining is 0.5, and the average decay rate during this time is

$$\frac{\text{no. decayed}}{\text{time}} = 0.5 \times \frac{\text{original population}}{\text{half-life}}$$

That is not a good estimate of the decay rate at the beginning of the time interval, which is surely greater. To get a better estimate, we should use a shorter time interval. For half a half-life, the fraction remaining will be the square root of 0.5, as shown in Sec. 19.5, so the fraction decaying will be $1 - (0.5)^{1/2} = 1 - 0.707 =$ ________. This gives an average decay rate

$$\frac{\text{no. decayed}}{\text{time}} = \frac{0.293 \times \text{original population}}{0.5 \times \text{half-life}} = \frac{0.586\, N_o}{\tau_{\text{half}}}$$

over a time interval of half a half-life.

(a) Verify.

(b) Again, you know that the initial activity is greater than this average. The task is to halve the time interval, which means taking square roots and to keep doing it until the proportionality constant "settles down" to the accuracy you need. Let's try for three significant figures. You will find that you need to carry at least six significant figures in your square-root taking.

To hasten the task, here are the successive square roots. You fill in the average decay rate.

Time in half-lives	Fraction remaining	Average decay rate
1	0.5	$0.5\ N_o/\tau_{\text{half}}$
$\frac{1}{2}$	0.70710678	$0.586\ N_o/\tau_{\text{half}}$
$\frac{1}{4}$	0.84089642	
$\frac{1}{8}$	0.91700404	
$\frac{1}{16}$	0.95760328	
$\frac{1}{32}$	0.97857206	
$\frac{1}{64}$	0.98922801	
$\frac{1}{128}$	0.99459942	
$\frac{1}{256}$	0.99729606	
$\frac{1}{512}$	0.99864711	
$\frac{1}{1024}$	0.99932333	$0.693\ N_o/\tau_{\text{half}}$

Convince yourself that going further in this process will not change the third significant figure; that is, the process converges.

(c) (Optional) What you did in part (b) is to take the derivative of the function $(\frac{1}{2})^{t/\tau_{half}}$ with respect to time t, numerically. Perhaps you know how to differentiate an exponential function formally. If so, do it, and check your result in part (b). (HINT: The number 0.693 is the natural logarithm of 2.)

✶✶✶ **19. Radiation Dose** Which creates more ion pairs, a dose of 100 rads localized in a 60-gram organ, or a 1 rad whole-body exposure to a 60-kilogram man? Explain.

✶✶✶ **20. Half-Life**

Isotope	Natural Abundance	Half-life
^{238}U	99.3%	4.47×10^9 yr
^{235}U	0.7%	7.1×10^8 yr

What were the natural abundances of the two isotopes 3 billion years ago?

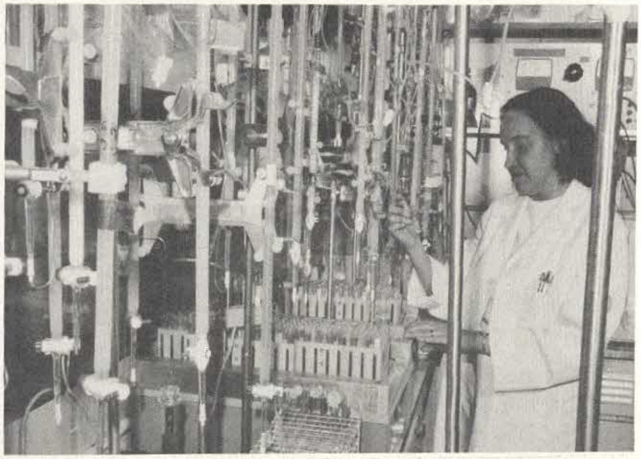
✶✶ **21. Labeling Causes Splitting** DNA molecules are long strands ("double helix") with molecular weights in the millions or billions. Some of the strongest chemical bonds in these molecules are the so-called phosphate bonds. DNA synthesized by bacteria growing on nutrients containing radioactive phosphorus ($^{32}_{15}$P) will have a molecular weight that decreases with time: When a phosphorus-32 nucleus beta-decays, it transforms into a sulfur-32 nucleus.

$$^{32}_{15}\text{P} \xrightarrow{\beta^-} {}^{32}_{16}\text{S}$$

The phosphate bond of which it was a part breaks. The molecule comes apart at the break. On the average, each disintegration cuts the molecular weight of the DNA in half.

Suppose that the average "labeled" DNA molecule contains 100 radioactive phosphorus-32 nuclei. How much will the average molecular weight have decreased by the time 1% of the ^{32}P have decayed?

By the time 2% have decayed?



Dr. Rosalyn Yalow.

✶ **22. RIA—A Diagnostic Tracer Technique** The 1977 Nobel Prize in Physiology or Medicine was awarded to Rosalyn Yalow for the development of a technique called radioimmunoassay (RIA). It uses radioactive labeling to measure the concentration of a specific hormone in an "unknown" of biological origin. Iodine-125 is a frequently used label. First, one "labels" a tiny sample of the hormone, highly purified. Meanwhile, in an experimental animal, an antibody for the specific hormone has been developed. A mixture containing the labeled hormone and the unlabeled unknown is then reacted in vitro with an appropriate dilution of the antibodies, so that the labeled and unlabeled hormones compete for binding sites on the antibodies. After a suitable incubation period to allow the hormone-antibody reaction to proceed, the bound hormone (the part that has reacted) is separated from the free (the unreacted fraction). The free fraction is placed in the radiation counter. Its activity is compared with the activity of samples prepared in the same way by mixing the same labeled hormone with a set of unlabeled "standard" hormone preparations and reacting with the same amount of antibody.

Suppose that the hormone concentration in unknown A

is greater than in unknown *B*. For which unknown would the measured activity of the free fraction be greater? Present the argument carefully.

✶✶ **23. Radioactive Heating** How much heat does the earth get from radioactivity? To get an order-of-magnitude estimate, let's suppose that there is only uranium-238. The earth contains an average of 4 grams per ton. The mass of the earth is 6×10^{24} kg, or 6×10^{21} tons. This gives us an estimate of _______ grams of uranium in the earth, or _______ moles, at 238 grams per mole. Multiply times 6×10^{23} (Avogadro's number) to get the number of U-238 nuclei. Their decay rate is $0.693/4.5 \times 10^9$ yr = _______ per second for each nucleus. Multiply times the number of nuclei to get the number of disintegrations per second. If each disintegration gives off 4 MeV, the rate of heating is _______ MeV/sec or _______ watts. (Remember, 1 MeV = 1.6×10^{-13} joules, and 1 watt = 1 joule/sec.)

Compare your answer with the energy we get from the sun, about 2×10^{17} watts.

✶✶ **24. Nuclear Fusion** In nuclei near the heavy end of the Periodic Table the nucleons are less tightly bound than in nuclei near the middle of the Periodic Table. This is what you expect, knowing about the short range of the nuclear force: The heavier nuclei are bigger. This size effect is reflected in the possibility of spontaneous nuclear fission (Sect. 15.7). In nuclei near the light end of the Periodic Table the nucleons are also less tightly bound than in those near the middle. Each nucleon does not have as many neighbors to grab on to. This size effect is reflected in the possibility of nuclear fusion: Two light nuclei can react (fuse) to make a heavier nucleus. For example, two isotopes of hydrogen can fuse to form helium, emitting a neutron:

$$^{3}\text{H} + {}^{2}\text{H} \rightarrow {}^{4}\text{He} + {}^{1}n$$

Energy is released in the process; that is, the reaction is exothermic. For each helium nucleus thus formed, 17.6 MeV of energy is released, shared between the helium and the neutron. Since the two hydrogen nuclei repel each other electrostatically, at least one of them has to be moving very fast in order for the two to get close enough to react, for the short-range nuclear force to take over.

(a) How much energy (in joules) is released from the fusion of 3 grams of ^{3}H (tritium) with 2 grams of ^{2}H (deuterium) to form 1 mole of ^{4}He?

(b) Suppose about half of that energy goes into heating up the helium gas. How hot would it get? The specific heat of helium is 3 calories per mole per degree (= 12.5 joules/mole degree).

(c) Invent a fusion bomb. Suggest you use a fission bomb as a trigger. Outline the important design features.

(d) The burning of hydrogen with oxygen to form water releases 2.5 eV of energy per molecule of water. How many joules are released with the formation of 1 mole of water vapor? We generally think of hydrogen as burning with a very hot flame. Contrast with nuclear fusion.

Centre Européen de Recherche Nucléaire (CERN) with Lac Leman and the Swiss Alps in the background. The round structure in the foreground houses the Intersecting Storage Rings (ISR). Protons are accelerated in the 28-GeV proton accelerator to speeds faster than 99.9% of the speed of light and in the ISR make collisions with protons going around the other way at the same speed.

20

RELATIVITY

SOME QUESTIONS THIS CHAPTER WILL ANSWER

What is the Principle of Relativity?

Electric charge is the source of electric and magnetic fields, which both obey inverse-square laws. How do the two universal constants involved in those laws combine to give the speed of light?

How much faster does the light signal from an approaching airplane get to you than the light signal from a receding airplane? Assume that the airplanes are traveling at 300 miles per hour.

What is the Einstein time dilation?

What is the Lorentz length contraction?

Is it true that when traveling at 97% of the speed of light your clock appears to run only one-quarter as fast as stationary clocks?

What is meant by *proper time?*

How fast do you have to move to travel 4 light-years while you age only 1 year?

Argue that repeated doubling of the momentum of a body increases its speed very little after the first few doublings.

Explain why kinetic energy is proportional to v^2 only at nonrelativistic speeds.

What is meant by $E = mc^2$?

Argue that it is unlikely that we will ever get rich enough to travel to planets outside our solar system in a few days.

20.1 THE UNIVERSAL CONSTANT c AND THE PRINCIPLE OF RELATIVITY

In Sec. 16.5, Changing Electric and Magnetic Fields, we arrived at a constant of nature having the dimensions of a velocity: meters per second. On the scale of everyday experience, that characteristic velocity c seemed incredibly fast: 3×10^8 m/sec. We showed that this is the speed at which electromagnetic waves travel in free space.

You might enjoy reviewing the argument. But you get the force of the ideas even from dimensional considerations. Only two numbers went into that speed of travel. They were the Coulomb constant k_e and the Biot-Savart constant k_m. Remember, k_e describes how strongly electric charges push on each other via their electric fields. The constant k_m describes how strongly charges in motion push on each other via their magnetic fields. The values are

$$k_e = 9.0 \times 10^9 \, \frac{\text{newtons m}^2}{\text{coulomb}^2} \quad \text{and} \quad k_m = 10^{-7} \, \frac{\text{newtons}}{\text{amp}^2}$$

respectively. Their ratio does have the dimensions of m^2/sec^2. If you take the square root of the quotient k_e/k_m, you get the familiar $c = 3 \times 10^8$ m/sec. You might want to convince yourself just by dimensional analysis that there is no other way to get meters per second from combining those two electromagnetic constants. Don't misunderstand. We are not claiming that dimensional analysis can deliver the exact number; only that it puts us in the ballpark and indicates where the number comes from.

It comes from the laws governing electromagnetic fields in free space. Those laws don't play favorites among different observers or different laboratories. An observer in a traveling laboratory expects to find—and finds—the same results from electrical experiments as the "stationary" observer. But the words *traveling* and *stationary* seem to imply that we know which one is traveling and which is stationary. Those words seem to say that there *is* some preferred frame of reference we can call "at rest." If nature really plays no favorites, then those words have meaning only relative to each observer. A laboratory on a spacecraft is described as stationary to the observer on the spacecraft. To an observer back on earth it is described as moving. If the two laboratories agree on the "universal constants" k_e and k_m, then they should agree on the speed of light.

As long as the relative velocity of the two observers is uniform, that is, without any acceleration, all those numbers turn out the same for both. The Principle of Relativity puts that even more starkly: It says: *No* experiment can distinguish between two observers in uniform relative motion.

That is a strong statement. If you have seen the movie "Frames of Reference," you may remember the situations where one man says, "You're moving," and the other answers, "No, *you're* moving." It's not just the laws of electricity and magnetism that display this remarkable lack of prejudice for one frame of reference over another. According to the Principle of Relativity, all physical laws are so even-handed. All instrument readings have a validity and a meaning that is independent of the relative motion of the instruments.

Speed of Light for Different Observers

Suppose that you are on a bus moving at 30 miles per hour down a straight road and you start walking forward at 1 mph down the aisle. You would not hesitate a moment to tell me that your speed in the reference frame of the road is 31 mph. Similarly, if you were walking toward the back of the bus at 1 mph, you would tell me that your speed, as measured by an observer standing on the road, is 29 mph. You have learned how to do vector addition of velocities to solve this kind of problem. And now we are telling you that, according to the Principle of Relativity, a light signal measured to be traveling forward at speed c by an observer on the bus is also traveling at speed c when measured by an observer on the road, *not* at $c + 30$ miles per hour!

Not surprisingly, such a dramatic symmetry principle has dramatic consequences for all of our ideas about space and time. The rest of this chapter will explore those consequences. The theoretical structure that emerges is called Special Relativity. It was discovered by Albert Einstein and published in two very readable papers in 1905. You can see why the theory is called Relativity. The word "Special" refers to the restriction that the velocity of the two observers must be uniform. In other words, Special Relativity is concerned with observers who are not accelerating. In the jargon of Chapter 2 we could say the theory is restricted to *inertial frames of reference.*

20.2 EINSTEIN TIME DILATION

Albert Einstein at nonrelativistic speed.

One of the dramatic consequences of the Principle of Relativity is that a clock in motion keeps time differently from a clock at rest: Moving clocks run slow! Before we present the proof, we should say that this relativistic effect is very small unless the speed of the clock's travel gets reasonably close to the speed of light. The "slowing-down factor" for a clock moving at speed v is

$$\left(1 - \frac{v^2}{c^2}\right)^{1/2} \qquad \text{(Eq. 1)}$$

also called the Lorentz factor. Even for something as fast as a satellite in orbit close to the earth, that square root differs from unity by only three parts in 10^{10}. (The arithmetic is done in Problem 2.) So we are not going to notice relativistic effects in household or highway situations. They have actually been verified by an atomic clock carried aboard a coast-to-coast jetliner. But there are very fast particles in the cosmic rays. Some of them are unstable—they undergo radioactive decay. Their average lifetime can therefore be used as a clock. It turns out that the faster ones live longer than the slower ones. They behave according to our **Lorentz square-root** formula.[1] That is, particles moving at speed v live longer, on the average, than the same particles at rest by a factor

$$\frac{1}{\left(1 - \frac{v^2}{c^2}\right)^{1/2}}$$

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[1] Try to see the film "Time Dilation," which shows an experiment with fast **muons** (Modern Talking Pictures, 1962.)

EXERCISE 1 Show that particles traveling at 96.82% of the speed of light live 4 times as long as the same particles at rest.

Intuitively, you may object to having the interval between two events be 4 times as long in one reference system as in another. Doesn't the Principle of Relativity say that different reference systems can't give different answers? Isn't there a violation here of the symmetry that implies? No. The method of timing the interval between two events is different in the two frames. It matters whether the interval is timed by a clock present at both events rather than by two different clocks at two different locations. We need to distinguish between a "one-clock time" and a "two-clock time." What will now be proved is that the one-clock time interval between two events is always less than the two-clock time interval.

Proof of Time Dilation—Einstein's Thought Experiment

Imagine a very simple clock, designed to time nanoseconds (10^{-9} sec). For this idealized experiment it will be a **"light clock."** At a speed of 3×10^8 meters per second, light travels 0.3 meters in 1 nanosecond—about a foot. If a light is flashed at the bottom end of a foot rule, the signal will get to a mirror attached at the top end in 1 nanosecond. It will be reflected back and arrive at its starting point in another nanosecond. To make the device seem more like a clock, you can think of saying "tick" when the signal hits the mirror and a second "tick" when it gets back, so "flash, tick, tick" represents 2 nanoseconds. Notice that the "flash" and the second "tick" are events occurring at the same place in the foot-rule frame of reference.

The thought experiment is to have a traveler carry this foot-rule clock while moving in a straight line at some uniform speed v. The foot rule is held perpendicular to the direction of travel and is used to time the trip. Think of yourself first in the traveler's reference frame and focus on the two events 2 nanoseconds apart—the emission of the light signal ("flash") and the reception of the light signal after being reflected (the second "tick"). In your frame of reference the light signal first travels up 1 foot and then, having been reflected, travels down 1 foot, retracing its path. We call that 2 nanoseconds a "one-clock time."

We now analyze those two events in the laboratory frame of reference. The foot rule moved from left to right. A glance at Fig. 20.1 shows the diagonal path of the light signal in getting from "flash" to "reflect" and again from "reflect" to "receive." If the height of each of those right triangles is 1 foot, the hypotenuse is surely more than 1 foot long. Yet those two hypotenuses represent how far the light signal traveled, and we know it travels at 1 foot per nanosecond. The light path is more than 2 feet long, so the

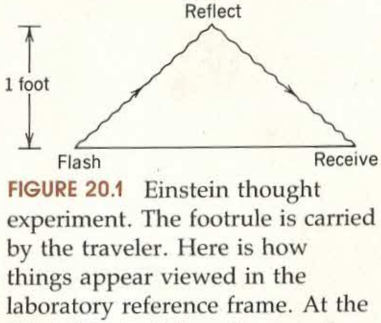

FIGURE 20.1 Einstein thought experiment. The footrule is carried by the traveler. Here is how things appear viewed in the laboratory reference frame. At the "flash" instant the mirror on the end of the footrule is just 1 foot from the source of the flash. By the time the signal is reflected, the mirror has moved to the right. When the signal is back at the source end of the footrule ("receive"), that has moved further to the right. The light path is along the hypotenuses.

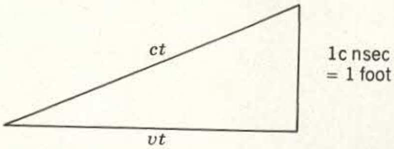

FIGURE 20.2 The left half of Fig. 20.1, with the lengths of the sides labeled. The footrule is 1 foot (= 1 light nanosecond) long in the traveler's as well as in the lab reference frame. That is because it is carried at right angles to the direction of travel.

time interval is more than 2 nanoseconds. A clock stationary in the laboratory reference frame located at the position of the "receive" event will read a time greater than 2 nanoseconds for that event.

We now calculate that two-clock time. In order to do the algebra, all we need is Pythagoras's theorem for those right triangles. Call the lab time of the "receive" event $2t$ and let the speed of the foot rule be v. Then the foot rule traveled a distance $2vt$ between the two events. The light, traveling at 1 foot per nanosecond along the hypotenuse, traveled a distance (1 foot/nsec) × $2t$. If we concentrate on only one hypotenuse, those 2's drop out (Fig 20.2). Square the hypotenuse and set it equal to the sum of the squares of the legs:

$$\left(\frac{1\ \text{foot}}{\text{nsec}}\right)^2 \times t^2 = (1\ \text{foot})^2 + (vt)^2$$

Now solve for t. Notice that we have to collect terms in t^2 from both sides of the equation:

$$\left[\left(\frac{1\ \text{foot}}{\text{nsec}}\right)^2 - v^2\right] t^2 = (1\ \text{foot})^2$$

Divide through by the expression in the brackets:

$$t^2 = \frac{(1\ \text{nsec})^2}{[1 - v^2/(1\ \text{foot/nsec})^2]}$$

Finally, take square roots:

$$t = \frac{1\ \text{nsec}}{[1 - v^2/(1\ \text{foot/nsec})^2]^{1/2}}$$

That square-root expression in the denominator is the slowing-down factor we have been talking about. Since it is less than 1, the "two-clock time" $2t$ between "flash" and "receive" is greater than 2 nanoseconds regardless of the speed v. Of course, the closer v is to the speed of light, the longer that two-clock time.

It may be useful to repeat what it is we have proved and to point out how general the proof really is. To measure our "one-clock time," we used a "light clock." That was convenient for our proof. But the conclusion has to hold for any clock. The trip was arranged so that our clock was present at both of the events. The events take place at different places in the laboratory frame of reference, so the time interval between them has to be measured using two different clocks in the laboratory frame. The conclusion is that the two-clock time is always longer than the one-clock time. The ratio is that Lorentz square root:

$$\frac{t_{\text{one-clock}}}{t_{\text{two-clock}}} = \left(1 - \frac{v^2}{c^2}\right)^{1/2}$$

EXERCISE 2 Review this proof and point out where it uses the constancy of the speed of light in different reference frames.

The technical term for the one-clock time between two events is **proper time.** This is the time someone riding along with the ruler would measure. Some people prefer to call it *local time*. Evidently, the proper time is the shortest time *any* observer measures for the interval between those two events. The farther apart those two events take place, that is, the greater the distance between the two clocks on which the events register, the longer the time interval between the events. The lab-frame observer located at the second event will always say to the traveler, "Hey, buddy, your clock is slow. I read it as you whizzed past, and it was slow by the factor $(1 - v^2/c^2)^{1/2}$."

EXERCISE 3 Suppose that our foot rule is moving at 80% of the speed of light: $v = 0.8c$. In the time t that it takes the light signal to get to the mirror at the top, the light goes a distance ct, while the ruler travels a distance $0.8ct$. Pythagoras says

$$(1 \text{ foot})^2 + (0.8ct)^2 = (ct)^2$$

Do the algebra to prove that $ct = 1 \text{ foot}/0.6 = 1.67$ feet, or $t = 1$ nanosecond$/0.6 = 1.67$ nsec.
(HINT: $(1 - 0.8^2)^{1/2} = ?$ In other words, the proper time is only three-fifths of the two-clock time at a speed of $0.8c$.)

Lorentz Contraction

Let's go on with that exercise. In the 1.67 nsec that elapsed between the two events (two-clock time) the foot rule traveled a distance $vt = (0.8 \text{ feet/nsec}) \times (1.67 \text{ nsec}) = 1.33$ feet. If you were traveling with the foot rule, how far would you say the lab traveled during a nanosecond you measure on your foot-rule clock? The Principle of Relativity tells us that the speed of the lab is also $0.8c$. If I see you moving at speed v, then you see me moving at speed v the other way. If you are moving at $0.8c$ to the right in the lab frame of reference, then the Principle of Relativity says that in your frame of reference the lab is moving to the left at $0.8c$. So in 1 nanosecond (foot-rule time) the lab travels $0.8c$ nanoseconds or 0.8 feet. The distance the foot rule traveled in the lab frame of reference was 1.33 feet. The distance the lab traveled in the foot-rule frame of reference was 0.8 feet—shorter by that Lorentz shrinking factor $(1 - 0.8^2)^{1/2} = 0.6$. Check that: $0.8/1.33 = ?$

Here is some jargon: The distance between the two laboratory clocks is called a **proper distance.** That was the 1.33 feet. By measuring that distance "in flight" at speed v, the value is reduced; the distance is contracted by the factor $(1 - v^2/c^2)^{1/2}$. The closer v gets to the speed of light c, the smaller the contraction factor.

EXERCISE 4 Verify that at 99% of the speed of light, distances measured along the line of travel are contracted to 14% of their proper lengths.

Let's review the argument that gave us the length contraction. In the process, we might warn against some common conceptual errors. The Principle of Relativity says that all inertial frames are indistinguishable. A by-product of this principle is that the speed of light c is the same in all inertial reference frames. Using a "light clock" to time the interval between two events—the sending and the receiving of a light signal—we established that the one-clock time between the two events is shorter than the two-clock time by the Lorentz square-root factor $(1 - v^2/c^2)^{1/2}$. Here v is the speed of the traveler relative to the "two clocks." But the traveler measures the two clocks to be moving at this same speed v—the other way, to be sure. So the measurement of the distance they move is shorter than their proper distance by that same Lorentz-square-root contraction factor.

Now for the warnings. It would not be correct to say that the moving observer measures the "contracted" distance between the two events. After all, in the traveler's frame of reference the two events occur at the same place. For the traveler, the distance between the two events is zero. That is exactly why the time measured between the two events by the traveler is a *proper* time. The Lorentz-contracted distance is a distance between two *moving* clocks. The left-hand clock was present at the first event; the right-hand clock was present at the second event. If the traveler's clock agrees with the left-hand clock at the time of the first event, it disagrees with the right-hand clock at the time of the second event. To measure the distance between those two clocks the traveler could not use a yardstick, because they are rushing past. He had to make a speed measurement and a time measurement and multiply them together. Of course, his time measurement has to be made on his (one) clock, not on two different "moving" laboratory clocks. That is why his result for their distance apart is less than their proper distance. Indeed it is exactly the same fraction of their proper distance as his one-clock time is of the two-clock time between the two events.

Synchronize Your Watches

If you set your clock to agree with somebody else's, will your two clocks still be in agreement some time later? If you can't answer Yes, at least one of those clocks is not very good. Well, we are not talking about defects in workmanship. Assume that you have both bought reliable instruments. What we have learned, though, is that you can only answer Yes if the other clock is not in rapid motion relative to yours. There is no problem in synchronizing clocks at different locations as long as they are not in relative motion. Just send time signals. Problems 8 and 9 show that there is no conceptual difficulty here. The only requirement is that there be some reliable way to measure distance. Being able to time light signals between two distant points satisfies that requirement.

You may wonder why we bother saying these rather obvious things here. Well, it is quite likely that your intuition suffered a bit of a jolt as you read in this section about time dilation and length contraction. We want you to be quite confident that we are not talking about any mysterious inability to measure time or distance reliably. On the contrary, you should be on the road to the conviction that space and time are inexorably woven together. You might even want to say that space and time are just two aspects of the same thing. Groping for a name, you might be tempted to call it **"space-time."** That is good scientific jargon. You are now in a position to try to impress your friends with a casual allusion to the four-dimensional space-time continuum. The risk you take is that the other guy may counter with a sly reference to world lines on the **Minkowski diagram** (Problem 14).

To aid in good, intuitive relativistic thinking, we shall try as much as possible to use easily convertible units for time and distance. For example, astronomers like to measure distances between stars in **light-years.** A light-year is the distance light travels in a year. In the same way, since light travels at about 1 foot per nanosecond, 1 foot is equal to 1 **light-nanosecond.** Repeat: 1 foot = (1 foot/nsec) × (1 nsec) = 1 c nsec.

EXERCISE 5 Verify that there are 3.1×10^7 seconds in a year. How many feet in a light-year?

Problem Solving with the Triangle Mnemonic

Having thought through the Einstein thought experiment with the light clock, you will find it convenient to use the same logic for other problems involving two events. Always let the vertical leg of your right triangle represent the proper time t_o between the two events (Fig. 20.3). Saying this more precisely, let the vertical leg represent the distance ct_o light travels in the proper time. The horizontal leg is the distance moved by the traveler. If this speed is v, that distance is vt. That is the proper distance between the two events. Note that the time t (no subscript) is a two-clock time. The time at the beginning of the trip is read on a clock at the starting point; the time at the end of the trip is read on a clock located at the destination. The hypotenuse of the triangle represents the distance ct traveled at the speed of

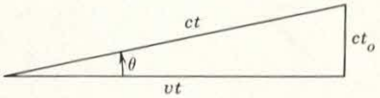

FIGURE 20.3

light in the two-clock time t. A quick look at the triangle is all it takes to convince you that the two-clock time between two events is always longer than the one-clock (proper) time: The hypotenuse is always longer than either leg of a right triangle.

The triangle is a mnemonic (memory aid) for the relationships between one-clock and two-clock times. From Pythagoras,

$$(vt)^2 + (ct_o)^2 = (ct)^2$$

t_o = proper time
t = two-clock time

we can solve for the ratio of the proper time to the two-clock time:

$$\frac{t_o}{t} = \left(1 - \frac{v^2}{c^2}\right)^{1/2}$$

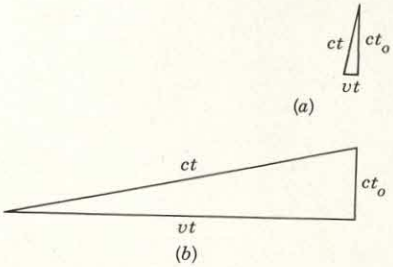

FIGURE 20.4 *(a)* Nonrelativistic (slow) traveler; *(b)* relativistic (fast) traveler.

That is the Lorentz square root. The angle θ made by the hypotenuse with the horizontal becomes smaller the faster the speed of travel v. Evidently,

$$\cos\theta = \frac{vt}{ct} = \frac{v}{c}$$

So a tall, thin triangle (Fig. 20.4a) corresponds to a slow trip, a nonrelativistic speed. A long, flat triangle (Fig. 20.4b) corresponds to a relativistic speed. As the speed v approaches c, the length of the horizontal leg approaches the length of the hypotenuse. As they both get longer, their ratio approaches unity.

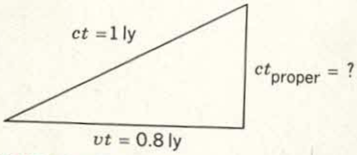

FIGURE 20.5 Triangle mnemonic for the space-traveler in Exercise 6.

EXERCISE 6 Here is another one about the space traveler going in a straight line at speed 0.8c. As shown in Fig. 20.5 above, in one year as measured on earth clocks he travels 0.8 light years.

(a) Show that his clock only registers 0.6 years at the end of the trip. Use the triangle mnemonic to do the bookkeeping. The horizontal leg is the distance traveled as measured in an earth-frame laboratory = speed × time = 0.8 ct. Note that this time t is a two-clock time. That is, the time at the start of the trip is read on an earth clock; the time at the end of the trip is read on a clock at the destination. The hypotenuse of the triangle is the distance light travels in that time t. The vertical leg is the unknown, the distance light travels in the proper time between the beginning and end of the trip.

(b) Show that, in the traveler's reference frame, the earth has receded 0.48 ly during that time. (HINT: In the traveler's frame, the earth is receding at 80% of the speed of light.)

(c) Show that the 0.48 ly is shorter than the earth-frame distance between the departure and arrival events by the factor $(1 - 0.8^2)^{1/2}$.

EXERCISE 7 How fast would a space ship have to travel to go 0.8 light-years while its own clock advances 0.6 years? Use the same triangle as in Exercise

6 to show that the answer is 0.8c. (HINT: Avoid the knee-jerk reaction that to go 0.8 ly in 0.6 years requires a speed of 0.8 ly/0.6 years = 1.33 c. That's faster than the speed of light! What is wrong? What is wrong is that you have taken a distance in one frame of reference and divided it by a time in another frame of reference.)

20.3 THE RELATIVITY OF SIMULTANEITY

The sections you have just read were intended to convince you that the results of measurement of distance and time between events depend on the reference frame of the observer. One consequence of the time dilation associated with speeds close to the speed of light c is that the *order* in which events take place need not be the same in all reference frames. If event A occurs before event B in one reference frame, it might well be described as occurring after event B in some other reference frame. That also says that events described as simultaneous in one frame are not simultaneous in another.

Talking about events like the receiving of light signals makes this particularly easy to see. The argument will again rely on the fact that light travels at the same speed c in all reference frames. In Figure 20.6 the upper sketch represents the sending of a light signal from the midpoint of the rocket. The signal goes out in all directions. We have sketched in an observer standing outside the rocket, essentially in the same place as the "sending" event. This observer can synchronize his clock with the rocket clock that records the time the signal is sent. The lower sketch represents the receipt of the signal by observers in the nose and tail of the rocket. In their (rocket) frame of reference, the forward and backward signals have traveled equal distances, so their times for the receiving events surely agree.

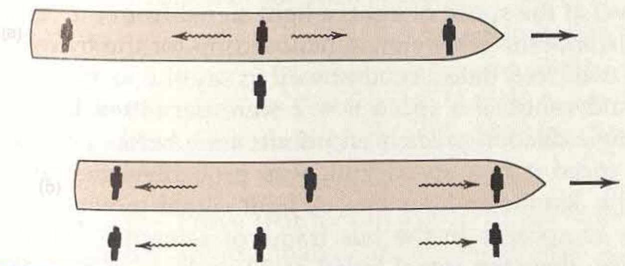

FIGURE 20.6 *(a)* Signal emitted at midpoint of rocket. The lab observer is right there. *(b)* Signal received at tail and nose. The lab observer standing at the "flash" event is still standing where he was. Note that he is closer to the tail receiver than to the nose receiver.

But how about observers outside, at rest in the lab frame of reference? We have sketched in lab observers essentially in the same place as the nose and tail observers riding the rocket. The one next to the nose just as the signal gets to the nose measures the light path as much longer than the one shown next to the tail. So his clock will read a later time for the "signal received" event. Lab-frame observers will insist that the signal gets to the tail before it gets to the nose. Of course! It doesn't have as far to go. The failure of the signals to be measured as simultaneous by the lab-frame observers is a simple consequence of the constancy of the speed of light in all reference frames.

You may well criticize the bottom sketch. It shows the rocket as being to the right of its former position. So you are tempted to think that it represents a "snapshot" taken from the lab frame. On the contrary, it shows the tail and nose events as occurring simultaneously. That is only correct in the rocket frame. A correct "rocket-frame" picture would have shown the rocket in the same place and the lab observer displaced to the left. Could we have drawn a correct "lab-frame" picture? We would have needed two: one showing the receipt of the signal at the tail and a later one showing the receipt of the signal at the nose. If you want to go through the calculation of the various times algebraically, try Problem 10, The Relativity of Simultaneity.

20.4 VELOCITY OF RELATIVE MOTION

Universal Speed Limit

You may have enough "relativistic intuition" by now to reject the possibility that an observer could travel at the speed of light c. Is it obvious that speeds greater than c are impossible? You may well have looked at the Lorentz square root $(1 - v^2/c^2)^{1/2}$ and seen that it has no physical meaning for v greater than c. All the same, if there is a universal speed limit, how is it enforced?

Let's grope toward answering that question by thinking about the Einstein thought experiment again. The light clock would break down at speed *c*! If the foot rule were to travel at the speed of light, a light signal from one end could never reach the mirror at the other end. A nanosecond for the traveler would be an eternity of two-clock time. Another way to say this is that the Lorentz square root would vanish if a speed $v = c$ were permitted by the laws of physics. So the time dilation predicts an infinite time before arrival.

Your intuition about speed c as a speed limit was probably stimulated when you read about the 30-mile-an-hour bus. A light signal moving forward in the bus travels at speed c in the bus frame of reference. In the earth's frame of reference, does the signal travel at 30 mph + c? No. An observer standing in the road also measures the speed of that signal as c. What kind of velocity addition formula would give that kind of automatic "enforcement" of the speed limit guaranteed by the Principle of Relativity?

Velocity Addition Formula

Here it is in one dimension: (For proof see Problem 16.)

$$v_{\text{rel}} = \frac{u' + v}{1 + u'v/c^2}$$

Let v be the 30-mile-per-hour velocity of the bus. Let u' be the 1-mile-per-hour velocity at which you walk forward in the aisle. Then your road velocity v_{relative} should come out 31 miles per hour.

EXERCISE 8 Verify that our formula does give 31 mph, good to 15 significant figures. (HINT: c = two-thirds of a billion mph.)

But suppose that u' is the speed of light: $u' = c$. Then the speed v_{rel} of that light signal measured by earth-frame observers also comes out equal to c.

EXERCISE 9 Verify.

Indeed that result is independent of the 30-mile-per-hour figure you put in for v. This is exactly what the Principle of Relativity requires!

To get a feeling for our velocity-addition formula try a few more exercises.

EXERCISE 10 Let $v = 0.5c$; that's an awfully fast bus! Let $u' = 0.5c$ also, to see how velocities close to the speed of light transform according to our formula. Verify that the road speed v_{rel} is $0.8c$.

EXERCISE 11 Let $v = 0.5c$ and $u' = 0.99c$. Calculate v_{rel} and express it as a fraction of c. (HINT: It comes out faster than $0.99c$ but slower than $0.999c$.) For more practice, see Problem 18, Velocity Addition. The proof of the velocity-addition formula is given in Problem 16, Proof of the Velocity-Addition Formula.

EXERCISE 12 Prove that for any choice of the velocities u' and v whose absolute values are less than c, the relative velocity v_{rel} has absolute value less than c. If you find this difficult, convince yourself with the calculator: Try values of u' and v closer and closer to $+c$ or $-c$.

20.5 RELATIVISTIC DYNAMICS

Thus far this chapter has been concerned with relativistic kinematics. That means the description of motion, the relation between space and time coordinates without reference to force or mass. You may be wondering how familiar laws like $F = ma$ and conservation of momentum have to be modified so that our universal speed limit is enforced. The answer is that in order to have momentum conserved, the definition of momentum has to be modified. Of course, the new definition has to be such that it reduces to the old one for speeds slow compared with the speed limit c. The relativistic mechanics that comes with that new definition turns out to have some elegant symmetries. In the same way in which space and time have been rolled into one comprehensive concept *space-time,* relativistic mechanics combines energy and momentum. It also includes the prediction that mass can be converted into energy according to the formula $E = mc^2$. That prediction has been accurately confirmed.

Here is the new definition of momentum:

$$\text{momentum } p = \frac{mv}{(1 - v^2/c^2)^{1/2}} \qquad \text{(Eq. 1)}$$

You can see that in the nonrelativistic limit $v << c$ this reduces to plain old mv. But for speeds approaching c, the momentum grows dramatically with increasing v. Expressed somewhat differently, it takes large increases in momentum to achieve small increases in speed. Indeed, no finite addition of momentum can ever get a body up to speed c. You can keep hammering away, adding and adding momentum, and the speed will keep increasing, getting closer and closer to c. But it will never equal c; it will never exceed c.

EXERCISE 13 Set $m = 1$ kg and $v = 1$ m/sec so that the momentum is 1 kg m/sec. If you doubled that momentum, what would be the new speed?

NOTE: The 1 kg is irrelevant, of course.

EXERCISE 14 Set $m = 1$ kg and $v = 0.1c$ so that the momentum is 0.1005 kg c. Verify. (Note how convenient those unfamiliar units are.) If you doubled that momentum, show that the new speed would be $0.197c$. (HINT: $1/v^2 = 1/c^2 + m^2/p^2$.)

EXERCISE 15 Set $m = 1$ kg and $v = 0.9c$ so that the momentum is 2.065 kg c. (Verify.) If you doubled that momentum, show that the new speed would be $0.972c$. (For more practice at seeing how much momentum increase is needed to get closer and closer to speed c, see Problem 29, Doubling the Momentum Increases the Speed by . . .)

Relativistic Energy and Momentum

Section 20.2 showed how the intimate connection between space and time—the laws of relativistic kinematics—followed from the Principle of Relativity. These laws were formulated using our convenient right triangle as a mnemonic (Fig. 20.7 *a*): The ratio of the horizontal side to the hypotenuse represents the speed of the particle in speed-of-light units. We shall formulate the laws of relativistic dynamics using the same triangle, that is, a triangle of the same shape for a particle of the same speed. The formulation will require a new concept, the relativistic energy E. Its zero-speed value E_{o} you may be tempted to call the "proper energy," by analogy with the proper time in kinematics. The convention is to call it **rest energy** E_{o}.

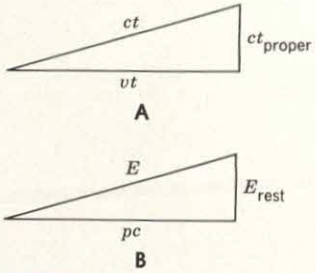

FIGURE 20.7 *(a)* Triangle mnemonic for relativistic kinematics; *(b)* triangle mnemonic for relativistic dynamics.

The law relating the **relativistic energy** E, the rest energy E_{o}, and the momentum p is formulated by assigning these quantities to the sides of our triangle (Fig. 20.7 *b*): Pythagoras's theorem says that

$$E_{\text{o}}^2 + (pc)^2 = E^2 \qquad \text{(Eq. 2)}$$

From the shape of the triangle we recall that the ratio of the horizontal side to the hypotenuse represents the speed of the particle:

$$\frac{pc}{E} = \frac{v}{c} \qquad \text{(Eq. 3)}$$

The vertical side E_{o} represents an invariant property of the particle. You guessed it: The invariant is proportional to the particle's mass. Again note the analogy to relativistic kinematics. The vertical leg represents an invariant in both triangles. Whenever a physicist sees an invariant, he knows that some law of nature is reflected in it. What law? Here, it is the Principle of Relativity.

Now let's give meaning to our new terms. You already know about the momentum. The rest energy is simply the particle's mass[2] expressed in speed-of-light units:

$$E_{\text{o}} = mc^2$$

The relativistic energy E has been defined is such a way as to give us a work-energy theorem. The theorem says that the difference between E and E_{o} is the work done on the particle to get it up to speed v. That is what we are used to calling the particle's **kinetic energy.** We do use the term kinetic energy for $E - E_{\text{o}}$ in relativistic dynamics. Just don't be surprised that it is *not* equal to $\frac{1}{2}mv^2$ except in a low-speed approximation.

EXERCISE 16 Try $v = 0.01c$, a nice nonrelativistic speed. Using Eq. 1 you find that the momentum is a bit greater than $0.01mc$. Equation 3 gives the relativistic energy E. Calculate the kinetic energy $E - E_o$. How far off is the nonrelativistic approximation $\frac{1}{2}mv^2 = 0.00005mc^2$?

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[2] Some people prefer the term *rest mass*. See end of KEY TERMS list.

EXERCISE 17 Now try $v = 0.6c$. Verify that the momentum is $0.75mc$, and the relativistic energy is $1.25mc^2$. That gives a kinetic energy $E - E_o = 0.25mc^2$. Note how far off the approximation $\frac{1}{2}mv^2$ would have been.

Notice that the kinetic energy $E - E_o$ is the difference between the hypotenuse (E) and the vertical leg (E_o) of our triangle. For nonrelativistic speeds, meaning tall, thin triangles, that difference is small compared to the rest energy. For speeds close to c, meaning long, flat triangles, the hypotenuse is much longer than the vertical leg. The rest energy represents only a small part of the total relativistic energy. Problems 25 and 26 show that the kinetic energy is proportional to the square of the momentum at these low speeds but becomes directly proportional to the momentum at relativistic speeds.

How Relativistic Dynamics Automatically Enforces the Speed Limit

Our triangle mnemonic shows how the relativistic energy E depends on speed v: The ratio of the vertical leg to the hypotenuse is the Lorentz square root:

$$\frac{E_o}{E} = \left(1 - \frac{v^2}{c^2}\right)^{1/2}$$

It is exactly the same as in kinematics, where t_o/t is equal to the Lorentz square root. If you remember that the rest energy E_o is an invariant, that is, is independent of observer, then you see that the total relativistic energy

$$E = \frac{E_o}{(1 - v^2/c^2)^{1/2}}$$

increases without bound as v approaches the speed limit c. In other words, the speed limit c is enforced simply because it would require an infinite amount of energy to accelerate any mass up to speed c.

EXERCISE 18 Rapid interstellar travel is fashionable in science fiction. Let's calculate the energies required. Our nearest star is 4 light-years away. In the pulp magazines, such trips are accomplished in hours or days. That would require shortening time by two or three orders of magnitude. For a Lorentz square root of $\frac{1}{100}$, we need a speed of $0.99995c$.

(a) Verify.

(b) For a shortening factor of $\frac{1}{1000}$, the speed has to be $0.9999995c$. Verify.

(c) Let's stick with the more conservative case (a). If the Lorentz square root is $\frac{1}{100}$, the energy of a space ship is 100 times its rest energy. That says it takes an amount of work equal to 99 times its rest energy to accelerate it from rest up to that speed. Let's be conservative again and take a small spacecraft. Make it only 1 ton ($m = 10^3$ kg). Its rest energy mc^2 is almost 10^{20} joules. (Verify!) So we need almost 10^{22} joules to attain cruising speed. That is two orders of magnitude more than the total annual energy output of all the power plants in the whole world. Note that we have neglected the work needed to accelerate the fuel, which would be necessary if we used a rocket. That would give another factor of 10 or 100.

The Rest Energy of Composite Systems

The formula relating rest energy to mass,

$$E_o = mc^2$$

may be said to be an expression of the "equivalence of mass and energy." So far we have used it in our relativistic dynamics formalism only as a definition, as the zero-speed remainder of the total relativistic energy. This gave meaning to our kinetic-energy expression $E - E_o$. Now we want to widen the scope of the rest-energy concept. For a system with internal degrees of freedom, the internal energy is a part of the rest energy. This is true whether the energy in the internal degrees of freedom is kinetic or potential.

EXERCISE 19 For example, a hydrogen atom consists of a proton and an electron, bound together by their Coulomb attraction. The rest energy of a free proton is 938.3 MeV; the rest energy of a free electron is 0.511 MeV. The sum of their kinetic energies plus the potential energy of the electrical attraction (<0) is -0.0000136 MeV. How much of an error do we make if we add the mass energies and forget to subtract the 13.6 eV? Answer in the form "1 part in ______."

Problem 28, Inelastic Collision, demonstrates how the mass of a system containing a spring is increased by compressing the spring, that is, by storing potential energy in the spring. For 10-gram marbles the effect is rather small, to be sure. For atoms or nuclei it can be significant. One experimental test involved a nucleus that emitted a gamma-ray photon. After the photon emission, the nuclear mass was measurably less.

EXERCISE 20 An iron-57 nucleus emits a 14-KeV photon. To what precision (how many parts in 10 million) must a mass measurement be made in order to detect the change in mass? (HINT: The rest energy is 57 × 931.5 MeV.) [Answer: 3 parts in 10^7] The 14-KeV "spectral line" is sharp enough (narrow enough) so that an energy change less than one part in 10^{10} can be measured. This measurement turns out to give a quantitative test of $E = mc^2$.

In your chemistry course you were probably taught that mass is conserved. Now your physics course is telling you it's not. Let's analyze a chemical reaction in which we know that light (photons!) can be emitted to see how much mass is actually lost.

EXERCISE 21 Try the burning of coal,

$$C + O_2 \rightarrow CO_2$$

The energy released in burning one carbon atom, which can go off as light and/or heat, is about 4 electron volts. To what precision would you have to weigh the carbon and oxygen before, and the carbon dioxide after the reaction in order to detect the change in mass? Here are some conversion factors; you may not need them all: 1 eV = 1.6×10^{-19} joules. One mole of carbon is 12 grams, and that is 6.02×10^{23} carbon atoms. The rest energy of 1 atomic mass unit (1 dalton) is 931.5 MeV, repeat, 931.5×10^6 eV. Carbon has mass 12 and oxygen (O_2, that is) has mass 32, expressed in atomic mass units (daltons). [Answer: The weighing would have to be done to better than 1 part in 10 billion. Weighings on laboratory balances can be good to 6 or 7 significant figures, but not to 10. So it is not surprising that the chemistry books don't say anything about such small changes in mass].

The situation is different with nuclear reactions, just because the energy released per atom is so much greater than with chemical reactions. The typical nuclear fission reaction releases on the order of 200 MeV, or about 50 million times as much energy as a chemical reaction. That puts the mass change into the measurable range.

EXERCISE 22 If the fission of uranium-235 releases 200 MeV, to what precision must you be able to weigh to detect the mass change? (HINT: The rest energy is 235 × 931.5 MeV.) [Answer: About 1 part in a thousand. Even a crude balance can measure a difference of 1 gram out of 1 kilogram.]

Now you see why people start to talk about $E = mc^2$ in connection with nuclear energy, while they forget all about it in connection with chemical energy. Here we have used the physicist's colloquial jargon "$E = mc^2$" to mean the equivalence of mass and energy.

20.6 IS RELATIVITY REALLY TRUE?

That's a funny question to ask about any scientific theory. A theory that makes correct predictions about nature and doesn't contradict some other generally accepted theory is usually accepted until such time as one of its predictions turns out to contradict experiment. Truth in science is just that temporary. We learn to be skeptical of even the oldest and most revered "laws."

With the theory of relativity the question seems to have more punch. It's not as if you asked, "Is Archimedes' Principle really true?" Once you found out about Archimedes' Principle, its truth became immediately intuitive for you. You quickly had the "I-knew-it-all-the-time"-feeling. You may have felt that if Archimedes hadn't discovered it, you would have discovered it yourself. You may not have welcomed Special Relativity in the same way. At first, it may even have seemed counterintuitive. The idea that different clocks keep time at different rates may have seemed strange. The same for the idea that different observers get different results when they measure the same length.

Certainly, the reason the ideas of relativity are not as intuitive as, say, the ideas of hydrostatics, is that in our everyday lives we don't encounter speeds near c. The fastest speeds we perceive from close up are in the hundreds-of-miles-per-hour range, many orders of magnitude slower than the speed of light. The speed of sound in air is one ten-thousandth the speed of light, and nothing in our immediate environment moves that fast.

The reason Special Relativity is just as well accepted a part of physics as Archimedes' Principle is that on the microscopic level, things do move at speeds close to c. Their motions would be completely mysterious without relativistic kinematics and dynamics. All our experience with machines like electron accelerators confirms Special Relativity. There is no mystery about the dynamics of fast electrons.

Yet relativity is a product of the twentieth century. There are still stories around—left over from 75 years ago—that "only Einstein and five other scientists understand relativity." Those stories weren't true when they were invented. Nowadays every practicing physicist in the world "understands relativity."

One of the fine fringe benefits that comes with this understanding is an added appreciation of the role that symmetry plays in physics.

20.7 THE ROLE OF SYMMETRY IN PHYSICS

Science is concerned with looking for regularity in nature. Much of nature's regularity—what people call the laws of nature—can be expressed as symmetry principles. Some people think of symmetry as meaning simply that something looks the same as its mirror image. That is one kind of symmetry. It is called right-left symmetry, or invariance under the transformation of right into left, and vice versa. Invariance under other transformations also constitutes symmetry. For example, the homogeneity of space means invariance under spatial translations. We accept that symmetry of nature almost as a matter of course, until we realize it is really an expression of the conservation of momentum. In other words, if space had "rough spots," momentum would not be conserved near them. In the same way, symmetry under time translations (setting the clock forward or back) is an expression of energy conservation. Symmetry under all rotations is an expression of the conservation of angular momentum.

In this chapter, the symmetry of nature that is being advertised is invariance under transformation from one inertial frame to another. The technical term for this symmetry is Lorentz invariance. What we have seen is that it implies a unification of space and time. For motion in one dimension, that unification was expressed by the invariance of the quantity $c^2(\Delta t)^2 - (\Delta x)^2$ between two events. That was the square of the vertical leg of our triangle, $(c\Delta t_o)^2$. Similarly, there is a unification of momentum and energy. The quantity $E^2 - p^2c^2$ is an analogous invariant under transformation to a moving reference frame.

The search for symmetry has been a fruitful point of departure for scientific research for centuries. It has been remarkably successful in elementary-particle physics during the past few decades. One might even say it characterizes the style of currently fashionable physical theory. (See Chapter 21.)

SUMMARY

The universal speed limit c can be derived from the two constants k_e and k_m.

The Principle of Relativity states that experiments performed in different inertial reference frames give the same results.

In particular, all such observers measure the same speed of light in vacuo: 0.30 meters per nanosecond.

One consequence of the Principle of Relativity is that moving clocks run slow by a factor $(1 - v^2/c^2)^{1/2}$.

A more precise way to say that is that the one-clock time (proper time) between two events is shorter than the two-clock time by the Lorentz square-root factor.

Distances parallel to the direction of travel are shortened for the moving observer by that same Lorentz square-root factor. This is called the Lorentz contraction.

The various relationships constituting relativistic kinematics can be represented on a right-triangle mnemonic. In describing two events, the vertical leg represents the proper time between them; the hypotenuse represents the two-clock time; the horizontal leg represents the distance between them in speed-of-light units. The cosine of the angle the hypotenuse makes with the horizontal is the speed in speed-of-light units.

If event A occurs before event B in one reference frame, it might occur after event B in another reference frame. Events that are simultaneous for one observer are not generally simultaneous for another if the two observers are in relative motion.

The relativistic velocity-addition formula assures that if one observer measures a velocity less than c, all other observers measure that velocity as less than c.

The relativistic definition of momentum has that Lorentz square root in the denominator. This modification of the nonrelativistic expression mv is necessary in order to have conservation of momentum in the interaction between two relativistic particles.

For a mass m, the quantity mc^2 is an energy called the rest energy. In transforming to a moving reference frame, the rest energy transforms in the same way as the proper time. That is, the relativistic energy is the rest energy divided by the Lorentz square root. It follows that the rest energy is the smallest energy that mass m can have. With the relativistic energy E defined in this way, a work-energy theorem holds: $E - mc^2$ is equal to the work done in accelerating the mass m to energy E.

If a body emits (or absorbs) energy, its rest mass decreases (or increases) by an amount Δm such that Δmc^2 is equal to the energy change. In particular, if a system emits a photon of energy $h\nu$, its mass will decrease by $h\nu/c^2$. (The particle recoils, so the decrease is slightly less.)

Because E is inversely proportional to our Lorentz square root, it increases without bound as $v \to c$. The requirement of energy conservation thus keeps the speed of any object having mass from ever attaining the value c.

KEY TERMS

Inertial reference frame Any frame of reference traveling at a constant velocity relative to some "standard" reference frame. The "standard" frame can be any frame in which momentum is conserved.

Speed of light in vacuo $c = 2.998 \times 10^8$ meters per second.

Special Relativity The kinematics and dynamics of objects described in inertial reference frames. The term is usually reserved for situations in which speeds close to the speed of light are encountered.

General Relativity A theory of gravitation and motion in accelerated reference frames. (Not mentioned in the text.)

Muon An elementary particle similar to the electron but about 250 times as massive. Muons decay into electrons with a half-life of about 2.2 microseconds.

Light clock A device in which time is measured by counting how many times a light signal traverses a given closed path.

Proper time The time between two events as measured on a clock that is present at both events and has not been accelerated. Called "one-clock time" in the text.

Proper distance The greatest distance between two events measured by any inertial observer. Colloquially called "lab distance," because it is the distance measured (in principle) with a meter stick. It is the *rest length*.

Lorentz square root $(1 - v^2/c^2)^{1/2}$, the Lorentz contraction factor for distances measured parallel to the direction of travel. Also the time-dilation quotient: Dividing the one-clock time by the Lorentz square root yields the two-clock time.

Gamma $(1 - v^2/c^2)^{-1/2}$. Used in Problems 15 and 16.

Light-year Distance light travels in one year.

Light-second Distance light travels in one second. The conversion formula

$$1\ c\ \text{nanosecond} = 1\ \text{foot}$$

is less than 2% off.

Velocity-addition formula $v_{\text{rel}} = (v_1 + v_2)/(1 + v_1 v_2/c^2)$ for combining two parallel velocities.

Relativistic theory A part of physics that remains valid at speeds close to the speed of light c.

Relativistic speed A speed that is an appreciable fraction of c. For example, $0.99c$ is obviously relativistic. On the other hand, $0.141c$ would be called nonrelativistic, because the Lorentz square root differs from unity by only 1%.

Space-time The four-dimensional continuum in which each event is represented by a point, so each mass point has a world line.

World line The graph of an object's position x versus time t is a curve (colloquially, a line) on a piece of paper. The position (x, y, z) versus time t, which requires four dimensions for plotting, is called the object's world line.

Minkowski diagram A geometric representation of relativistic transformations. If one-dimensional motion of a point is represented on perpendicular x and t axes, then the space-and-time coordinates x', t' in some other inertial reference frame are given by letting the x' axis make an angle θ with the x axis and the t' axis make the angle $-\theta$ with the t axis, where $\tan\theta = v/c$. This makes the x' and t' axes oblique, of course [Problem 14].

Total relativistic energy $E = mc^2/(1 - v^2/c^2)^{1/2}$

Rest energy $E_o = mc^2$. This concept has no nonrelativistic counterpart.

Kinetic energy $E - E_o$. Only in a nonrelativistic (low-speed) approximation does this become $\frac{1}{2}mv^2$.

Relativistic mass E/c^2. To avoid confusion, this term is not used in the text. When the text uses the word *mass* it always means the invariant mass, also called the *rest mass*.

REVIEW QUESTIONS

The Coulomb constant $k_e = 9.0 \times 10^9$ newtons m^2/coulomb^2 and the Biot-Savart constant $k_m = 10^{-7}$ newtons/A^2 can be combined in a simple way to yield a quantity having the dimensions of a speed. Explain and calculate that speed.

- The Principle of Relativity has been paraphrased, "Nature plays no favorites." Explain why this is an apt statement. In what way is it imprecise or wrong?
- If an observer on a spaceship receding from the earth at 90% of the speed of light were to measure the speed of a light signal originating on earth, what result would he obtain? Would he measure time on his clocks or on earth clocks? Would he measure distance with his meter sticks or earth-frame meter sticks? Why do we bother asking?
- Describe the Einstein light-clock thought experiment to prove the time-dilation formula. Extend the argument to derive the Lorentz distance contraction.
- How precisely would you have to measure time to detect the relativistic time dilation at a speed 1% of the speed of light c?
- Explain what is meant by the relativity of simultaneity.
- How does the velocity-addition formula "enforce" c as a universal speed limit? That is, show that a velocity measured in one frame of reference (less than c) is measured in any other frame of reference in uniform relative motion as having a value less than c.
- Explain the "triangle mnemonic" for energy and momentum. Point out the analogy to the triangle mnemonic for time and distance.
- For a slow particle (v much less than c) the distance traveled (proper distance) is much less than ct_o. That is, the horizontal leg of our triangle is much shorter than the vertical leg (the "invariant"). The analogous statement for the dynamical quantities is that pc is much less than ________.
- Present an argument that vacation travel to parts of our galaxy that are light-years away would only be possible if we were much richer. Explain the precise use of the word "richer."
- Argue that the Principle of Relativity is a symmetry principle.

PROBLEMS

✶ **1. Practice with $(1 - v^2/c^2)^{1/2}$** Calculate that expression for $v = 0.01c$, that is, a speed 1% of the speed of light.

✶✶ **2. Time Dilation for an Earth Satellite** Calculate how much the "slowing-down factor" $(1 - v^2/c^2)^{1/2}$ differs from unity for an object traveling at 7.5 km/sec. (HINT: You may find that your calculator does not keep enough significant figures to do this arithmetic. The ratio v/c is $7.5/3 \times 10^5 = 2.5 \times 10^{-5}$. Square that and you get 6.25×10^{-10}. Most calculators will only keep 9 or 10 significant figures. So if you subtract that small number from 1, your calculator may just answer "1," instead of 0.999999999375. To take the square root of a number like $1 - 6 \times 10^{-10}$, you can use the binomial theorem. Convince yourself that the answer is $1 - 3 \times 10^{-10}$. To how many significant figures is that answer correct? Try it: $(1 - 3 \times 10^{-10})(1 - 3 \times 10^{-10}) = 1 - 2 \times 3 \times 10^{-10} + 9 \times 10^{-20}$. Keeping only the first-power term in the binomial expansion resulted in an error in the twentieth decimal place.)

The answer to our problem is that the satellite's clock runs slow by 3.12 seconds in 10^{10} seconds, or about 1 second in a hundred years. Show your arithmetic neatly so you can go back and check it.

★★ **3. Light-Seconds and Light-Years** Our nearest star (the star closest to our solar system) is Alpha Centauri. It is about 4 light-years away. Light takes about 5×10^2 seconds to reach us from the sun. So the sun is about 500 light-seconds away.
(a) How many earth-sun distances (Astronomical Units) is 4 light-years?
(b) As the earth moves in its orbit, an observer on Alpha Centauri would see our position in the sky change by a maximum angle (measured in radians) of 5×10^2 light-seconds/4 light-years = ________. Show that this takes half a year. Sketch!

NOTE: That is too small an angle to resolve with the naked eye but well within the capability of our larger telescopes.
(c) How can we use measurements of stellar positions at various times of the year to find stellar distances?

NOTE: The technical term is *parallax*.

★★ **4. The Muon Movie "Time Dilation"** If you have had a chance to see the film "Time Dilation—An Experiment with Muons," then the numbers in this problem are already familiar. The particles selected for the experiment had a rather narrow range of speeds close to $0.9938c$, making the Lorentz square root equal to $\frac{1}{9}$.
(a) Verify.
(b) What the movie shows is that these muons survive about 9 times as long as muons at rest. It does this by counting muons on top of Mt. Washington (altitude 6300 feet) and again at sea level. Traveling close to 1000 feet per microsecond, the muons took about 6.3 microseconds to make the 6300-foot trip as measured on earth clocks. Yet from the statistics of muons counted at sea level, the fraction surviving corresponded to a muon time of 0.7 microseconds. Identify the one-clock time (the proper time) and the two-clock time.
(c) In the muon frame of reference, sea level approaches at speed $0.9938c$, and the trip is timed at 0.7 microseconds. In the muon frame of reference, what is the distance between the top of the mountain and sea level?
(d) What fraction is this of the proper height of the mountain?

★★ **5. More Decaying Muons** Muons at rest have a mean lifetime of 2.2 microseconds. If we were to make a "light clock" 2.2 light microseconds long, we could describe the Einstein thought experiment by drawing our triangle with an altitude of $2.2c$ microseconds. That is about 2200 feet. Let the muons have a speed $v = 0.8c = 800$ feet per microsecond. In a time t (two-clock time) they travel a distance $0.8ct$, while a light signal travels a distance ct.
(a) Draw the right triangle with the legs and hypotenuse obeying

$$(0.8ct)^2 + (2.2c \text{ microseconds})^2 = (ct)^2$$

Solve for t. Verify that it is what you get if you divide the 2.2 microseconds by the Lorentz square root (= 0.6). What you have solved is the two-clock time corresponding to a one-clock time of 2.2 microseconds.
(b) How far did the muons move during that time? Answer in thousands of feet, remembering that 1000 feet = $1c$ microsecond.
(c) A measurement in the muon frame of reference would say that the second clock is approaching at $0.8c$. Its time of travel between the two events being clocked is 2.2 microseconds, muon time (proper one-clock time). So that clock traveled $0.8c \times 2.2$ microseconds $= 1.76c$ microseconds = 1760 feet in the muon frame of reference. Compare that distance with the answer to part (b). Is it shortened by the Lorentz square-root factor?
(HINT: $(1 - v^2/c^2)^{1/2} = 0.6$.)

★ **6. Time Dilation** Here is another one about the space traveler going in a straight line at $0.8c$. In one year (earth-clock time) he travels 0.8 light-years.
(a) Show that his clock only registers 0.6 years by the end of the trip. Use our handy triangle to do the bookkeeping. The horizontal leg is the distance traveled as measured in an earth-frame laboratory = speed $v \times$ time t. Note that this time t is a two-clock time. The hypotenuse is the distance light travels in that time. The vertical leg is the unknown in this problem, the distance light travels in the proper time between the beginning and end of the trip.
(b) Show that in the traveler's reference frame the earth has receded 0.48 light-years during that time. (HINT: In his frame, the earth is receding at 80% of the speed of light.)
(c) Show that the 0.48 ly is shorter than the proper distance between the departure and arrival events by the factor $(1 - 0.8^2)^{1/2}$.

★★ **7. Time Dilation—Travel Keeps You Young** How fast would a spaceship have to travel to go 0.8 light-years while its own clock advances 0.6 years? Use the same handy triangle as in Problem 6 to show that the answer is $0.8c$. (HINT: The horizontal leg vt is 0.8; the vertical leg ct_{proper}, the invariant, is 0.6. Avoid the knee-jerk reaction that 0.8 light-years in 0.6 years requires a speed of $0.8c$ years/0.6 years = $1.33c$. That is faster than the speed of light! What is wrong?)

[Answer: What is wrong is that you have taken a distance in one frame of reference and divided it by a time in another frame of reference.]

✶ **8. Clock Synchronization** In the Einstein thought experiment with the foot-rule light clock, the imaginary laboratory was populated with clocks all over the place. At least there had to be a lab clock at the position of the second event, the receipt of the light signal. It was the reading of that right-hand clock (see the figure) that is compared with the (shorter) reading of the traveling clock. The implicit assumption is that the right-hand lab clock was previously synchronized with the left-hand lab clock, even though they are at different locations. One way to accomplish this synchronization is with a light signal. In the exercise we had $v = 0.8c$. The receipt-of-signal event occurred a distance 2.67 feet from the initial "flash." Argue that if a flash occurs at noon and the light signal travels straight to the right, it will arrive at the right-hand clock 2.67 nanoseconds after noon.

✶ **9. Another Way to Synchronize Clocks** Let a flash occur at the midpoint between the two distant clocks. Argue that in this case you don't even have to know the distance between the clocks in order to synchronize them.

✶✶✶ **10. Relativity of Simultaneity** Refer to Fig. 20.6, which shows a signal sent from the midpoint of a rocket to observers in the tail and in the nose. The "lab frame" observers outside are assumed to be "right there." This just means that longitudinal distances (right-left) may be large, while transverse distances are negligible. It means you don't have to worry about the angle of the signal path to get to the outside observers.

Let the rocket have length L—as measured in its own frame. That means L is its *proper* length. It means the tail and nose "signal-received" events occur at time $\frac{1}{2}L/c$, as measured on rocket-frame clocks. We now calculate the time t_{tail} that it takes the signal to get to the tail observer as measured on lab-frame clocks. During that time the rocket, traveling at speed v, goes a distance vt_{tail} to the right. The light signal goes a distance ct_{tail} to the left. The sum of those distances is half the length of the rocket *as measured in the lab frame,* which is

$$\tfrac{1}{2}L_{\text{contracted}} = \tfrac{1}{2}L(1 - v^2/c^2)^{1/2}.$$

(a) Write this as an equation, and solve for t_{tail}.
(b) Now do the same thing for the time t_{nose} that it takes the signal to get to the nose observer. But now the distance the rocket goes has to be subtracted from the distance the light signal goes. The difference is $\frac{1}{2}L_{\text{contracted}}$. In part (a) you had $c + v$ in the denominator when you solved for the time. Here you get $c - v$ in the denominator. Which is the later time, t_{nose} or t_{tail}?

✶✶✶ **11. The Light Cone** If the position x of an object is plotted versus the time t, the resulting graph will be a straight line for an object having constant velocity.
(a) Show that the straight line representing the arrival of a light signal at position x always has slope c if the signal travels in the $+x$ direction or slope $-c$ if the signal travels in the $-x$ direction.
(b) For a light signal emitted from $x = 0$ the coordinates of the arrivals at other values of x are given by $x^2 = c^2t^2$. Show that this equation represents two intersecting straight lines.
(c) Show that the straight lines representing the constant-velocity trips of material objects always have slopes in between those two limiting lines.
(d) Add another dimension; call it y. For a light signal emitted from the origin, the equation of the wavefront is $x^2 + y^2 = c^2t^2$. Show that this is the equation of a cone. It is called the light cone.

✶✶✶ **12. The Order of Events** Event A (receipt of the signal at the tail) occurs before event B (receipt of the signal at the nose) for lab-frame observers. For observers in a reference frame moving to the right at a speed faster than the rocket speed the order would be inverted. How about event O, the sending of the signal? Prove that the order of events O and A is the same in all reference frames. This just amounts to proving that a light signal cannot be received before it is sent.

✶✶✶ **13. Twin Paradox** Twin A takes off on a trip at a speed close to the speed of light, say $0.99875c$. When he returns home 2 years later as measured on his own clock, twin B, the stay-at-home, is old and gray, having aged by 40 years. [HINT: $(1 - v^2/c^2)^{1/2} = \frac{1}{20}$]. Now for the paradox. Doesn't the Principle of Relativity say you can't distinguish reference frames? After all, from twin A's point of view, isn't it twin B that was traveling?

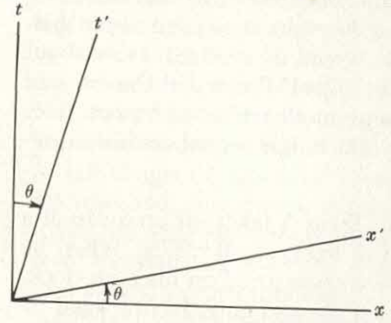
RESOLUTION: Twin A had to accelerate. Unlike B, he did not remain in an inertial reference frame. The symmetry between the twins is broken.

Now we shall formulate the problem in a way that does not involve acceleration. At time $t = 0$, twin A is already moving at velocity $v = 0.99875c$ and passes twin B, who is stationary. They synchronize their clocks at this instant. When A has traveled 1 year (as measured on his clock) he meets, by prearrangement, a look-alike A* traveling the other way at velocity $-v$. They synchronize their clocks at the time of meeting, whereupon A* travels back to "home," arriving there in one year (on his clock). Use the argument of the Einstein thought experiment to show that at the time of arrival, B is 40 years older than A was when A and B parted. Note that in this formulation, there is not even an apparent paradox. The symmetry between A and B is broken by the substitution of A* for A, which avoids having to accelerate A.

★★ **14. Minkowski Diagram** Here is a graphical representation of the transformation of the coordinates (x, t) of events in one reference frame (S) to coordinates (x', t') of those same events in a reference frame S' moving at velocity v relative to S. The oblique axes x' and t' make angles θ and $-\theta$ with the perpendicular axes x and t, respectively, where $\tan\theta = v/c$. The transformation equations are

$$x' = \frac{x - vt}{(1 - \tan^2\theta)^{1/2}}, \qquad t' = \frac{t - vx}{(1 - \tan^2\theta)^{1/2}}$$

We have chosen the same units for time and space, setting $c = 1$. We might, for example, measure t in seconds and x in light-seconds. NOTE: The dilation factor $(1 - v^2/c^2)^{-1/2}$ is often called γ = Greek gamma.



PROBLEM 14 Minkowski diagram. Note that the axes in the traveler's reference frame (S') are oblique.

(a) Show that this transformation leaves a proper time invariant, that is, $t'^2 - x'^2 = t^2 - x^2 = t_o^2$. The invariance of the quantity $c^2(\Delta t)^2 - (\Delta x)^2$ between two events described in different reference frames is logically equivalent to the invariance of the speed of light, that is, to the Principle of Relativity.

★★★(b) Give a geometrical proof of the transformation equations. (HINT: $1 - \tan^2\theta = \cos 2\theta/\cos^2\theta$.) Note that the scale on the primed axes is stretched by a factor $(1 + v^2/c^2)^{1/2}/(1 - v^2/c^2)^{1/2} = \cos^{-1/2} 2\theta$. It may be easier to prove the inverse transformation:

$$x = \gamma(x' + vt')$$
$$t = \gamma(t' + vx')$$

★★ **15. Lorentz Transformation** The Minkowski diagram (Problem 14) gives us the transformation equations:

$$x' = \gamma(x - vt), \qquad t' = \gamma\left(t - \frac{v}{c^2}x\right)$$

where $\gamma = (1 - v^2/c^2)^{-1/2}$ is the reciprocal of the Lorentz square root. In the y and z directions perpendicular to the velocity v distances transform unchanged:

$$y' = y, \qquad z' = z$$

These four equations have the property that they leave invariant the "interval" between the two events,

$$c^2(\Delta t)^2 - (\Delta x)^2 - (\Delta y)^2 - (\Delta z)^2$$

That property is required by the Principle of Relativity. (See the Einstein thought experiment in Sec. 20.2.)

(a) Do the algebra to prove the invariance.

(b) The equation of the wavefront of a light signal emitted at the origin is $x^2 + y^2 + z^2 = c^2t^2$. Argue that the distance from the origin to the wavefront is ct.

(c) Write the corresponding equation in the primed coordinates x', y', z', t'. That distance is now ct'.

★★★(d) Argue that no other algebraic expression substituted for gamma (γ) would have the invariance property proved in part (a).

★★ **16. Proof of the Velocity-Addition Formula** In reference frame S', a particle moves at speed u' to the right along the x-axis. For two close-lying events x'_1 and x'_2 we can write

$$x'_2 - x'_1 = u'(t'_2 - t'_1)$$

or, more briefly, $\Delta x' = u'\,\Delta t'$. Use the Lorentz transformation (Problem 15) to find the speed u relative to frame S moving to the left relative to S' at speed v. Proceed as follows: The coordinates transform according to the Lorentz transformation,

$$\Delta x = \gamma(\Delta x' + v\,\Delta t'), \qquad \Delta t = \gamma\,(\Delta t' + v\,\Delta x'/c^2)$$

(Note the + signs for S' traveling to the right.) Divide the two equations:

$$u = \frac{\Delta x}{\Delta t} = \frac{\Delta x' + v\,\Delta t'}{\Delta t' + v\,\Delta x'/c^2} = \frac{(\Delta x'/\Delta t') + v}{1 + v(\Delta x'/\Delta t')/c^2}$$

Show that this is equivalent to the equation of Section 20.4.

✶ **17. Velocity Addition—Baseball in a 747** Imagine pitching a ball at 90 miles per hour from the tail toward the nose on a jet airplane traveling at 600 miles per hour.
(a) Argue that the ball's ground speed is 690 mph.
(b) What is the ground speed for the same pitch from nose to tail?
✶✶(c) An electron gun pointed toward the nose shoots an electron at $0.8c$. What is the electron's ground speed? (HINT: 600 miles/hour $= 0.896 \times 10^{-6}c$.)
[Answer: $0.8000002c$]

✶ **18. Velocity Addition—Practice with the Formula** Let $v = 0.8c$ and $u' = 0.9c$. Calculate u. (HINT: If your answer is greater than c, you have made a mistake.)

✶✶ **19. Velocity Addition—Left Becomes Right** Let $v = 0.8c$ and $u' = -0.9c$. Find u. Be sure to get the sign right. Remember that + is right, − is left.

✶✶ **20. Velocity Addition in Two Dimensions** From the Lorentz transformation it is easy to prove that a velocity vector $\mathbf{u} = (u_x, u_y)$ transforms according to

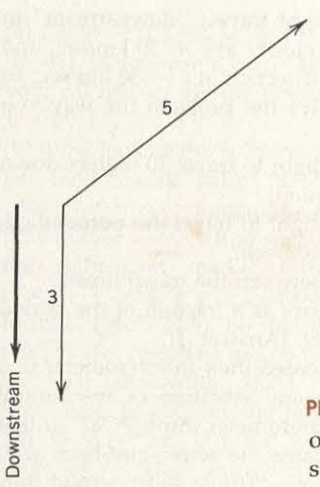
$$u_x = \frac{u'_x + v}{1 + u'_x v/c^2}, \qquad u_y = \frac{u'_y}{\gamma(1 + u'_x v/c^2)}$$

Let $v = 0.8c$ and $u' = u'_y = 0.9c$.
(a) Verify that you get $u_x = 0.8\,c$. Is that what you would have guessed without doing any algebra?
(b) Find u_y. Are you surprised that it is so much less than $0.9c$? Don't forget time dilation!

✶ **21. Velocity Addition with Light**
(a) Let $v = 0.8c$ and $u' = u'_y = c$. A light signal is traveling perpendicular to the x-axis. Find $u = (u_x^2 + u_y^2)^{1/2}$.
(b) Show that $u = c$ for all possible values of v.

✶ **22. A Michelson–Morley Boat Race** Two boats each travel at 5 miles per hour in still water. But the race is to be held on a river flowing at 3 mph. Boat A starts by going 1 mile downstream, then returning 1 mile upstream to the starting point.
(a) What is A's ground speed on the downstream leg?
(b) How long does it take?
(c) The upstream leg has a ground speed of 2 mph. Verify! How long does this take?
(d) How long does A's round trip take?

Boat B starts along a perpendicular path, traveling 1 mile across the stream. Note that to keep from being swept downstream, B actually has to head at an angle upstream.



PROBLEM 22 Vector addition of velocities needed to go straight across.

(e) Draw the vector diagram showing the addition of the velocity vectors, the 3 mph downstream and the 5 mph angling upstream. The vector sum has to point directly across the stream. What is the net cross-stream speed? (HINT: Pythagoras.)
(f) How long does it take B to go 1 mile over?
(g) How long does B's round trip take?
(h) Who wins the race?
(i) What is the ratio of B's time to A's time?
✶✶(j) Write down a general expression for this ratio in terms of the river speed v and the still-water speed c_{boat}. Note that the ratio is independent of the length of the course.

✶✶✶ **23. Michelson–Morley Experiment** (Do the Boat-Race Problem First.) A Michelson interferometer (see Problem 12.22) has two perpendicular arms with mirrors at one end. A 45-degree mirror in the middle is half-silvered so that the light beam is split and half the light goes along each arm. The reflected beams rejoin so that they can interfere coherently, making an interference pattern of concentric bright and dark rings. The locus of points for which the travel time along both arms is the same is a bright circle—an interference maximum. The locus of points for which the travel times differ by exactly half a period of the light wave is a dark circle. A time difference of a whole period gives a bright circle again, and so on. All integer multiples of a period correspond to bright fringes. All half-integer multiples correspond to dark fringes.

We know that the earth travels around the sun at roughly 30 kilometers per second. Suppose that you didn't know about the relativistic velocity-addition formula. Na-

ively, you would say that light travels "downstream" in the direction of the orbital velocity at $c + 30$ km/sec, and "upstream" in the opposite direction at $c - 30$ km/sec. In other words, you would solve the problem the way you did the boat-race problem.
(a) How long would it take light to travel 10 meters down and back? (HINT: Long division)
(b) How long would it take light to travel the perpendicular path, 10 meters across and back?
(c) Calculate the difference between the travel times.
(d) Express this time difference as a fraction of the period of visible light, 2×10^{-15} sec. [Answer: $\frac{1}{6}$]
(e) Michelson and Morley floated their interferometer in a pool of mercury. Avoiding any vibration or mechanical shock, they rotated the interferometer through 90° so that the down-and-back arm became the across-and-back arm, and vice versa. How much of a "fringe shift" would they have expected, according to your calculations? Their optics and mechanical stability were such that they could have detected a shift of one-twentieth of a fringe. In fact, they found no perceptible shift. They were disappointed. But remember, the date was 1887, 18 years before Einstein's Special Relativity Theory. Argue that, had they known the Principle of Relativity, they would have expected zero fringe shift.

✶✶ **24. Speed as a Function of Momentum**
(a) Starting from the definition of the momentum,

$$p = \frac{mv}{(1 - v^2/c^2)^{1/2}}$$

show that the speed v as a function of the momentum is given by

$$v = \frac{c}{(1 + m^2c^2/p^2)^{1/2}}.$$

(b) Sketch the function $v = v(p)$, and show that v approaches c asymptotically as p goes to infinity.

✶✶ **25. Kinetic Energy—Nonrelativistic Approximation**
(a) Calculate the kinetic energy $E - E_o$ of a mass m moving at speed $v = 0.02c$. Express in units of the rest energy $E_o = mc^2$.
(b) What % error would you have made by using the nonrelativistic approximation $\frac{1}{2}mv^2 = 2 \times 10^{-4}\, mc^2$?
(c) [More difficult] Use the binomial expansion

$$(1 - x)^{-1/2} = 1 + \tfrac{1}{2}x + \tfrac{3}{8}x^2 + \cdots$$

to verify your answer to part (b) Does the approximation $\frac{1}{2}mv^2$ give too large or too small an answer?
(d) At what speed is the $\frac{1}{2}mv^2$ formula 1% off?

✶ **26. Kinetic Energy—Relativistic Approximation**
(a) Calculate the kinetic energy of a mass m moving at speed $v = 0.9c$.
(b) Its momentum is $2.06mc$. Verify. How far off is the approximation $E - E_o = pc - E_o$ for the kinetic energy?
(c) Repeat for $v = 0.99c$.
(d) How much work is required to accelerate the mass from $0.9c$ to $0.99c$? Express as a fraction of the rest energy mc^2.

✶✶ **27. Is kinetic energy proportional to p or to p^2?**
(a) Nonrelativistically, we can write

$$\text{K.E.} = \frac{\frac{1}{2}p^2}{m}$$

How many % off is the kinetic energy this formula predicts at a speed $v = 0.02c$ from the correct relativistic kinetic energy? (HINT: $p = 0.02mc/[1 - (0.02)^2]^{1/2} = 0.020004mc$.) The kinetic energy $E - E_o$ is correctly given by

$$[E_o^2 + (pc)^2]^{1/2} - E_o$$
$$= E_o\left[1 + \left(\frac{pc}{E_o}\right)^2\right]^{1/2} - E_o$$

(b) Use the binomial expansion of that square root to verify your answer to part (a). Notice that pc/E_o is the ratio of the horizontal to the vertical leg of our triangle.
(c) At relativistic speeds the horizontal leg is much longer than the vertical. If we want a binomial expansion to converge, we have to expand in powers of a ratio *less than* 1, that is, in powers of E_o/pc. Writing

$$E = pc\left[1 + \left(\frac{E_o}{pc}\right)^2\right]^{1/2}$$

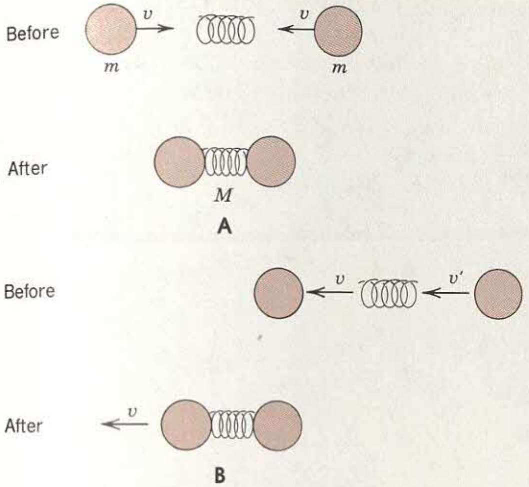
show that the kinetic energy becomes proportional to p as the ratio E_o/pc becomes negligibly small.

✶✶✶ **28. Inelastic Collision** The potential energy of a compressed spring is shown to be part of the rest energy (and hence of the mass) of a composite system of two masses plus spring [see Figure (*a*)]

The two indentical marbles (each of mass m) approach the spring (of negligible mass) with equal and opposite velocities $\pm v$. Their combined kinetic energy

$$\frac{2mc^2}{(1 - v^2/c^2)^{1/2}} - 2mc^2$$

is converted into potential energy as they stick to the



PROBLEM 28 *(a)* Inelastic collision as viewed in the center-of-mass reference frame. The balls stick to the spring and the spring has a catch to keep it compressed after the impact; *(b)* the same inelastic collision viewed in the reference frame of the left-hand ball.

spring. We assert that this potential energy is added to the rest energy of the system. So the old mass $2m$ has added to it a mass increase

$$\Delta m = \frac{\text{potential energy}}{c^2} = \frac{2m}{(1 - v^2/c^2)^{1/2}} - 2m$$

(a) Let $m = 10$ grams and $v = 100$ m/sec. Calculate Δm. [HINT: If you can't carry enough significant figures on your calculator, use the binomial expansion $(1 - x)^{-1/2} = 1 + \frac{1}{2}x + \ldots$.]

(b) Now look at the collision in the reference frame of the left-hand marble before the collision, and worry about conservation of momentum [see Figure (*b*)]. The spring now moves to the left at speed v before the collision, while the right-hand marble moves to the left at speed $v' = (v + v)/(1 + v^2/c^2)$, according to the velocity addition formula (Sect. 20.4). So its initial momentum is $mv'/(1 - v'^2/c^2)^{1/2}$. To conserve momentum, this has to be equal to the final momentum $Mv/(1 - v^2/c^2)^{1/2}$. Accordingly, the final mass M exceeds the initial mass $2m$ by the amount $\Delta m = M - 2m$. Use the conservation of momentum to solve for M. Then show that Δmc^2 is equal to the kinetic energy we wrote down at the beginning of the problem. (HINT: There is a lot of algebra here. Don't be afraid to substitute for v' inside the square-root sign. It stops looking messy quite soon after some least-common-denominatoring.)

What we have shown is that the rest energy $E_o = Mc^2$ really included the potential energy. It simply means that the terminology is consistent.

✶ **29. Increasing the momentum increases the speed by . . . .**

Set $m = 1$ kg (just to make the arithmetic easy) and $v = 0.99c$.

(a) Show that the momentum is 7.02 kg c.

(b) If the momentum is doubled, show that the speed is increased to 0.997 c.

☐ QUESTIONS FOR FURTHER THOUGHT

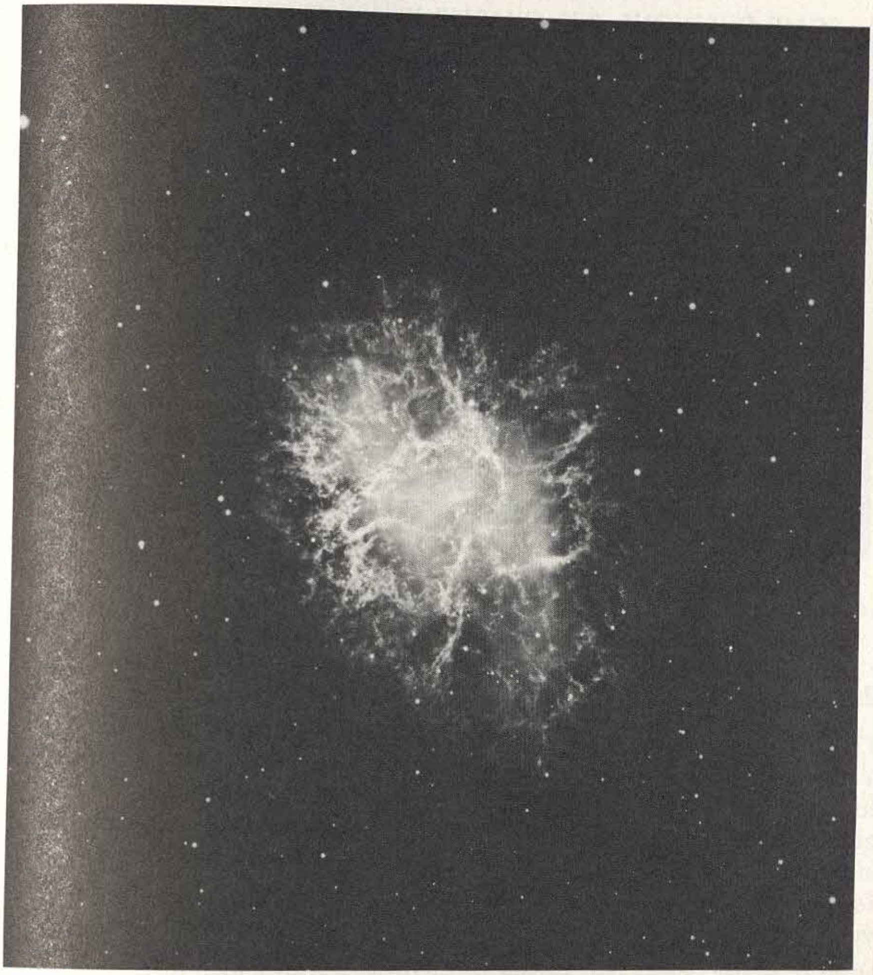
α. **Symmetry in Electromagnetic Theory—Magnetic Monopole** In Chapter 17 on Electromagnetic Waves some near-symmetries of electromagnetic theory are pointed out. For example, a changing B-field gives rise to (is equivalent to) a circulating E-field. A changing E-field gives rise to (is equivalent to) a circulating B-field. These induced E and B fields both have nonending lines, that is, field lines that close on themselves. But we also know E-lines that start and stop. Their sources and sinks are positive and negative charges, respectively. We are tempted to ask whether the E/B symmetry still holds. Are there sources and sinks for B lines? At the time of writing, such "magnetic monopoles" have not been found.

(a) Argue that their existence would make electromagnetic theory more symmetric.

(b) Why are electric forces in familiar situations usually much greater than magnetic forces? (HINT: Remember that all chemical binding is essentially electrical.

This question is intended to stimulate original thought. It is likely to engender controversy. Surely it has more than one answer.)

β. Light Cone in Four Dimensions For motion in three dimensions, the kinematic invariant is $c^2t^2 - x^2 - y^2 - z^2$. If this invariant happens to be zero (proper time between the origin and the event at x, y, z, t is zero), the two events can be connected by a light signal. The equation $c^2t^2 - x^2 - y^2 - z^2 = 0$ represents a surface in four-space called the light cone. Argue that this is an appropriate name.



Crab Nebula, the remnant of a supernova observed in 1054 A.D. Temperatures and densities in a supernova may be high enough so that the matter there is in the form of a quark-gluon plasma.

21

ELEMENTARY PARTICLES AND COSMOLOGY

SOME QUESTIONS THIS CHAPTER WILL ANSWER

The term *strong interaction* refers to the vestige of the force that binds nucleons into nuclei. What phenomena display the *weak interaction?*

Is it possible for a proton-proton collision to create a pair of pions?

The three kinds of charged leptons have masses of 0.511, 105.7, and 1784 MeV/c^2, respectively. What name is given to the neutral leptons?

What is meant by quark confinement?

Why did the quark-gluon plasma "freeze out" about 1 microsecond after the Big Bang?

The quark model and the theory of the Big Bang correctly calculate the relative abundances of the two most abundant elements in the universe. What epoch in the history of the universe determined that ratio?

Does the quark model allow magnetic monopoles? How about free quarks?

21.1 THIS CHAPTER IS DIFFERENT

This chapter looks different from all the others, different in several ways. You leaf through it and you see that there are no EXERCISES and just a handful of PROBLEMS at the end. That tells you it is going to be less "skills" oriented. Indeed, it is less quantitative, more descriptive, and admittedly a bit superficial. On the other hand, it is the first place in the book where you read about what physicists do now, rather than generations ago. It takes you to the frontier of knowledge. That also means that things are not so certain. Some "facts" presented in this chapter may turn out to change as new experiments are reported.

We choose one area of physics—elementary-particle physics—for this visit to the frontier. To be sure, only a small fraction of all physicists work in this area. Every area of physics has a frontier, a hazy boundary between the known and the unknown. Choosing elementary-particle physics was arbitrary but not random. The choice has to do with the deeply human drive to find out what happens if you take things apart. If you keep taking apart the parts of what you have taken apart, where does it end? Is there a smallest part, something that can't be taken apart? Or does it never end? Does every particle, no matter how small, have *some* structure?

We don't know the answers to these questions. But there is a model (a theory) that successfully describes the properties of all the particles that have been discovered in our giant high-energy accelerators. It is a neat picture, described in the section called Quarks, and you should feel free to go and read it right away. The next two sections supply a bit of background.

21.2 THE FOUR INTERACTIONS

We all love to take things apart to see what makes them tick. If something holds together strongly, what was the glue? In the nineteenth century, the chemists learned to take molecules apart to find out what atoms they are made of. Early in the twentieth century, physicists found out that the atom has a nucleus surrounded by a cloud of electrons. Later, "atom smashers" were able to break nuclei apart, to convert one nucleus into another. By shooting beams of fast electrons or protons at nuclei, the structure of the various nuclei was investigated. Well, as you know, all the nuclei turned out to be made of protons and neutrons (nucleons). What glue holds them together?

In Chapter 19, Radioactivity, we talked about the nuclear force. You learned that the **nuclear force**—the force that binds anywhere between 2 and 250 nucleons together—is weak at large distances and becomes a very strong attraction at distances of less than 2 fm ($= 2 \times 10^{-15}$ m). Around 0.5 fm the force suddenly becomes a strong repulsion, as though the nucleons had hard, impenetrable centers. How very different that behavior is from the simple $1/r^2$ behavior of the Coulomb force! And how much stronger the force! The nuclear force between two nucleons in a nucleus is so much stronger than the electrostatic force between two elementary charges at the same separation that the nuclear force has been named the *strong interaction*.

Of course, the main message of Chapter 19 was that many nuclear species are unstable. They decay radioactively. Indeed one of the nucleons is itself somewhat unstable. The neutron beta-decays with a half-life of around 10 minutes. The interaction responsible for beta decay is called the **weak** interaction, and its laws are now understood well enough so that many quantitative predictions are possible about the half-life of various radioactive species.

We count three different interactions:

1. The strong interaction, responsible for the attractive force between nucleons.
2. The electromagnetic interaction, responsible for all chemical forces, for adsorption, for the sticking together of flat surfaces, indeed, for the shapes of all objects we see around us.
3. The weak interaction, responsible for only a tiny fraction of the nuclear binding energies, showing itself in beta decay. It makes possible the fusion reaction of hydrogen into helium that fuels the heat of the sun.

But aren't we forgetting an interaction we have known for a long time? What about gravity? Of course, there is the gravitational interaction. We almost forgot it because it is so much weaker than the others. It shares with the electromagnetic interaction the feature that it falls off as the inverse square of the particle-particle distance. To give an idea of how weak it is, the gravitational attraction between two electrons is less than 10^{-42} of the electrostatic repulsion.

It is now believed that the quantum theory of these four interactions:

strong
electromagnetic
weak
gravitational

can explain all forces in nature. It may be useful to make a catalogue of all that is to explained. The next section may seem a bit like a tour of a zoo in which all you see is the *names* of the animals.

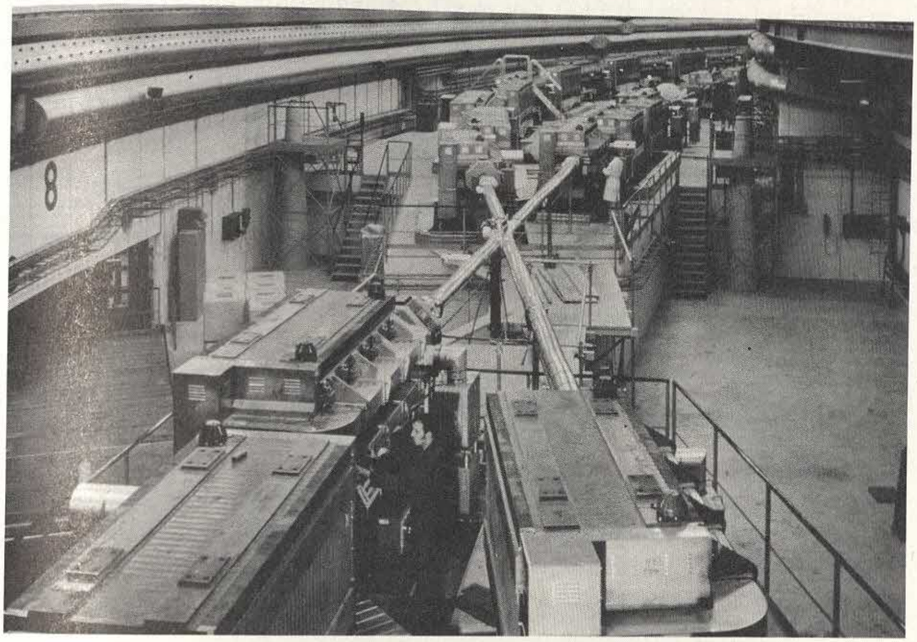
21.3 THE PARTICLE ZOO

When two protons crash into each other at high speeds, all kinds of debris are observed to come out of the collision. The more energetic the collision, the greater the number and variety of the particles observed to speed away from the crash site. At low collision energies, of course, what comes out of a proton-proton collision is just two protons. At energies of several hundred MeV, there may be additional particles called pions or pi mesons. At higher energies there are *K* mesons, or kaons. At energies of several GeV there are antiprotons and antineutrons. At still higher energies there are hyperons. The tables in Problems 6 and 7 list 17 different particles. Most of these particles are radioactive, with very short half-lives—some on the order of 10^{-23} seconds. At this time, you may not even want to bother learning their names. In the problems we list the various particles and their decay products. Some years ago, the big question seemed to be which of them were, in some sense, "fundamental" and which were "composites." How about the proton itself, is it fundamental or does it have structure? Was all that debris inside it in the first place or did the collision create it?

From Chapter 20, Relativity, you know that mass can be created out of energy. So the kinetic energy of the colliding protons can be converted into the rest energy of the particles created in a collision. For example, a pion has a rest energy of 140 Mev. If two colliding protons each have a kinetic energy greater than 140 Mev, a pair of pions (π^+ and π^-) can be created in the collision. The greater the available energy, the more massive the particles that can be created. The same is true of collisions of electron beams and of beams of the various particles created in collisions.

Classification of Particles

The so-called elementary particles are of two kinds: **hadrons** and **leptons.** The hadrons interact via the strong interaction; the leptons don't. On the other hand, for reactions governed by the weak interaction, at least two leptons are involved: one with mass and one massless one. The electron is a lepton. You already know that it has an **antiparticle,** the positron. Each particle in the zoo has an antiparticle, having the same mass and opposite



Inside the ISR (Intersecting Storage Rings) building at CERN. (For an overview, see the photograph at the beginning of Chapter 20.) In the background are magnets to make the proton paths curve. In the middle of the picture is an intersection of the two evacuated tubes containing the two beams of high-energy protons going around in opposite senses. The proton bunches may be timed to miss each other at the intersection or timed to collide.

charge. When a particle and its antiparticle come close, they can annihilate each other, creating at least two photons, or possibly other particle-antiparticle pairs.

Other leptons are labeled mu (μ) and tau (τ), in order of increasing mass. Muons decay into electrons with the simultaneous emission of an electron antineutrino and a muon neutrino. Those neutrinos have no electric charge and are massless, or nearly so. The decay of tau leptons (or tauons) results, analogously, in the emission of tau neutrinos. What all six leptons have in common is that they seem to have no structure; they are essentially point particles. We say six leptons:

$$\begin{array}{ccc} e^- & \mu^- & \tau^- \\ \nu_e & \nu_\mu & \nu_\tau \end{array}$$

If you counted the antiparticles, you would say 12.

Now for the hadrons. There are two hadron families: **baryons** and **mesons.** The proton is a baryon; so is the neutron. The other baryons are more massive (therefore called hyperons) and rather short-lived. The mesons, as a group, are less massive than the baryons. The pion, for example, has oneseventh the mass of the proton. Problem 6 gives a table of mesons and talks about their decay.

How about structure? Electron beams have been used to explore the proton, and it does have structure. It has what you might call a size. In other words, it makes sense to talk about the way the proton's charge is distributed in space. The proton is not a point particle.

So now let us ask those fundamental questions: "Is the electron made up of some smaller-scale entities?" "Does the electron have constituent parts?" The present-day answer is No. Leptons have no structure. On the other hand, ask: "Is the proton made up of some smaller-scale entities?" "Does the proton have constituent parts?" The answer is Yes. Hadrons do have structure. The section entitled Quarks is about their constituent parts.

Does It Ever End?

Great fleas have little fleas
upon their backs to bite 'em,
And little fleas have lesser
fleas and so ad infinitum.

A. DEMORGAN

Some people feel that physicists have no business trying to answer the "Does-it-ever-end?" type of question. They say that predicting the future of our science is not part of our job. As the tools are developed for investigating ever smaller scales, isn't it pretty certain that more structure will be discovered? An answer like, "No, we have reached the final constituents of matter" seems arrogant, presumptuous. Is it possible that the quarks and leptons and the fields that describe their interaction are "fundamental" in the deep sense that they have no structure?

A century ago, physicists thought that all that was left to discover was some details, that the basic questions had all been answered. That was before relativity, before quantum theory! So maybe these "presumptuous" present-day theories will turn out to be just as wrong. It seems just as exciting to speculate that they may be right.

21.4 QUARKS

We are now ready to describe the structure of the proton and all the other hadrons.

The fundamental constituents of all strongly interesting particles are six **quarks.** They are called u, d, s, c, t, and b. But it does no good to ask, "What are quarks?" That is because quarks are never found alone but always is either groups of three to make a baryon or groups of two to make a meson. Now you can play combinatorial games like figuring out how many different baryons there can be. (See Problem 5.) However, there is more to the quark model than combinatorial games. There is a field theory for the interaction among the quarks.

About Field Theories and Gauge Particles

It may be useful to say a few words about quantum field theories in physics. You know all about the electromagnetic field, and in the chapter on Quanta you read about photons, the quanta of the electromagnetic field. The electromagnetic field is the concept we use to describe how charges act on each other at a distance. The idea of action at a distance was viewed as a paradox two centuries ago. What is it that reaches out? What is it that grabs? Our answer is "the field." With the field quantized we want to ask, "How does it do it?" Our answer is "by exchanging quanta." The interacting charges are constantly exchanging photons. One charge emits a photon. A distant charge absorbs the photon. Don't we get in trouble with conservation of energy? After all, the photon carries energy! The answer is No, there is no trouble; the uncertainty principle permits just the right amount of violation. The in-flight time of the photon allows just the required energy uncertainty needed to create the photon—for the short time between its emission and its absorption.

When people ask how the exchange of a gauge particle (the photon is the gauge particle of the electromagnetic field) can result in a force, the tempting picture is that of the two skaters on very slippery ice. One skater throwing a package to the other is pushed backward when she imparts momentum to the package. The skater catching the package is pushed backward when he absorbs the momentum of the package. One trouble with this kind of explanation of how the exchange of gauge particles transmits forces is that it explains repulsive forces only. Another trouble is that it is just too simplistic. Quantum field theory is not quite that simple. But it is universal, in the sense that all forces are indeed "action at a distance" and all fields are quantized.

But let's get back to the field that describes the interaction of quarks. The quanta of this field are appropriately called **gluons.** The field theory is more elaborate than, say, the theory of the electromagnetic field. The requirements that have to be met by such a field theory are symmetry requirements at every point in space and time. To the extent that such symmetry requirements are man-made, there is, of course, some arbitrariness. From the symmetry requirements plus some good guessing about the number of dimensions in which the fields are defined (not necessarily four = number of dimensions of space-time), the properties of the quarks and gluons can be calculated. The kinds of questions that are still being answered about, say, a 10-dimensional gauge field is how the extra six dimensions manage to hide themselves from us. After all, our space-time does seem quite four-dimensional.

QCD

To describe the nature of gauge field theories requires some fancy mathematics, so no details are presented here. But one feature of a theory called quantum chromodynamics **(QCD)** is particularly elegant and illustrates how symmetries appear. In the quantum theory of the electromagnetic field (quantum electrodynamics, or **QED**) electric charge was a conserved quantity. Processes that violate charge conservation are not allowed. In QCD, another kind of charge is defined, also indestructible. Instead of there being two kinds of charge (called + and −) as in QED, this field theory requires three kinds of charge. The three have arbitrarily been called red, blue, and green, and the new kind of charge is called *color*. Please don't think that a red quark *looks* red. Remember, you can't see quarks. This is just a way of naming a quantum number that takes on three values. The rule is that any object made of quarks (any hadron) has to be *color neutral*.

Here is what that means. All the quarks can come in red, blue, or green. Antiquarks come in antired, antiblue, or antigreen. An observable (i.e., a free) particle could be composed of a red quark, a blue quark, and a green quark. That would be color neutral (red + blue + green = neutral). The particle would be a baryon. Or an observable particle could be composed of a red quark and an antired quark. That would also be color-neutral (red + antired = neutral). The particle would be a meson. The requirement of color neutrality restricts the number of possible hadrons. It also explains why we never observe free quarks. One of the requirements on the field theory (in this case, QCD) is that at every point in space and time there be symmetry with respect to permuting the colors. It is actually stronger than that: There must be symmetry with respect to mixing the colors. Mathematically, that means there is a three-dimensional "color space," and the theory has to be invariant under rotations in that space.

Isn't all this pure fantasy? You can't observe the quarks. You can't observe the gluons that bind them. You can't observe the thing we have called color. What lends legitimacy to a theory that is so distant from observable phenomena, so removed from experiment? The answer is that the theory *was* able to predict new phenomena. It was able to make a table of particles and their properties, a table with some entries for particles that had not been observed. When experiments were designed to look for the particles with the predicted properties, the particles were found. That is, after all, what theory is for—to organize and explain existing observations and predict new ones.

The Role of the Leptons

Let's repeat the "neat picture" of the fundamental constituents of all matter, as it has been roughly described above. There are six kinds of quarks. The word for

$u \quad d \quad c \quad s \quad t \quad b$

is six quark *flavors.* Each flavor comes in three *colors.* The quarks interact via a field whose quanta are the *gluons.* There are also six flavors of leptons,

$$\begin{array}{ccc} e^- & \mu^- & \tau^- \\ \nu_e & \nu_\mu & \nu_\tau \end{array}$$

They interact via a field whose quanta are called vector bosons. Even though they are quite massive, these vector bosons have now been observed in experiments at the huge proton accelerators.

What happens in a weak interaction is that a quark can change its flavor. In other words, interaction with leptons can cause a quark of one flavor to turn into a quark of another flavor. Take, for example, the beta-decay of the neutron:

$$n \rightarrow p + e^- + \bar{\nu}_e$$

quark composition

neutron: *udd* proton: *uud*

The decay involves the conversion of a *d*-quark into a *u*-quark. This is made possible by the two leptons, e^- and $\bar{\nu}_e$ (electron antineutrino), which are created in the weak decay. Note that the neutron mass is just a bit greater than the proton mass so that not only is there enough energy to create the electron, but also some energy is left over for all three product particles to carry off in three different directions.

Asymptotic Freedom

Quark confinement, by the requirement that an observable particle be color-neutral, presented us with a tricky solution to the "Does-it-ever-end?" question. "Do the quarks have structure?" is a hard question to answer if it is not possible to observe a free quark. Remember, the only free particles that QCD permits are three-quark particles (baryons) and two-quark particles (mesons). But there is a way out of quark captivity. The "ransom" is energy—a lot of it. Built right into QCD is the prediction that when quarks get very close, the interaction forces become color-blind. The jargon is **asymptotic freedom,** asymptotic as energies get high enough to allow the quarks to get close enough. When lots of quarks are brought very close, it is no longer possible to say to which hadron a given quark belongs. What results is a soup, called quark-gluon plasma.

It is considered likely that such a quark soup actually exists in places that are hot enough and dense enough: supernovas and neutron stars. But the interest in quark soup is even more fundamental. According to the cosmologists, the early universe was nothing but quark soup.

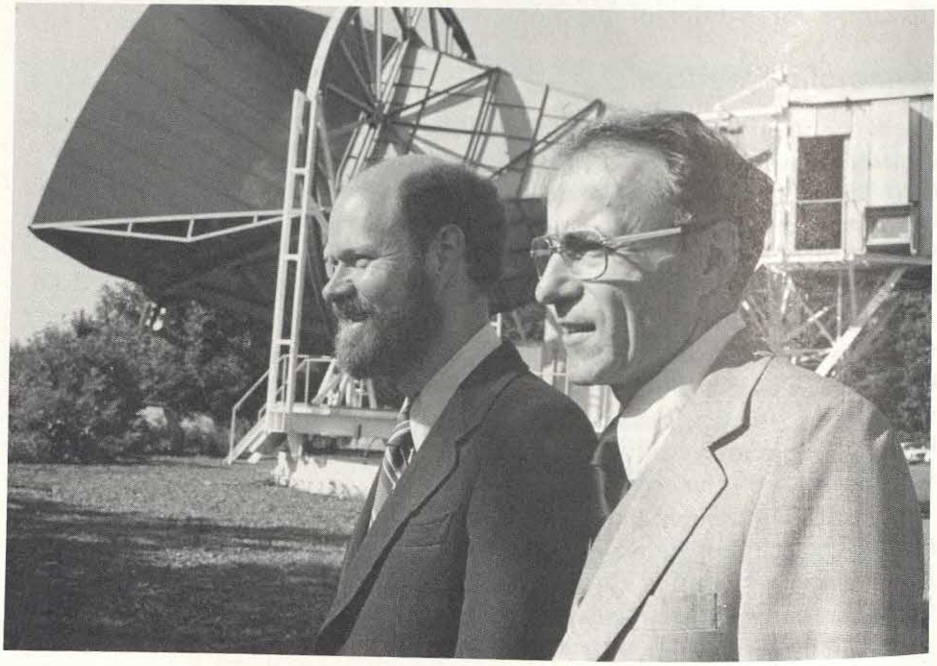
That conjecture brings us to the meeting-ground between elementary-particle physics and **cosmology.**

21.5 COSMOLOGY—THE ORIGIN OF THE UNIVERSE

Some of the old textbooks tell about the quandary posed by the "birth of the universe." If there was a beginning, what was there before? The answer "nothing" seemed implausible. How could something arise out of nothing? The idea of a "steady-state" universe was therefore very appealing. "On the cosmic scale, things have always been pretty much the way they are today," that was an idea people could live with. It even went well with the expanding-universe picture with which the astronomers explain the red shift of spectral lines from distant galaxies. The more distant galaxies recede from us faster than the nearer galaxies. How does that fit with a steady-state density of matter? People used to say that as the distance between stars increases, new matter is created everywhere, feeding on the gravitational energy of its interaction with distant matter. Just another example of $E = mc^2$!

Then, in 1965, came a discovery that gave steady-state cosmology the kiss of death. The microwave-astronomers discovered the **"three-degree background radiation."** You might think of it as the fossil record of something very, very hot that happened very, very long ago. That something was the birth of the universe, commonly called the Big Bang.

How long ago? Here is one way to date the event. The **red shift** of a distant galaxy turns out to be proportional to its distance from us. Red shift? We are talking about the Doppler effect, familiar from sound radiated by a



Robert W. Wilson (with beard) and Arno A. Penzias discovered the three-degree cosmic background radiation by using the radio telescope shown. Their Nobel Prize in physics is in recognition of this discovery, which supports Big-Bang cosmology.

moving object. The frequency of the horn of a receding car is lower than the on-board frequency of that horn. The frequency lowering is proportional to the speed of the car. In the same way, the red shift (frequency lowering) of an identifiable spectral line from a distant galaxy measures the speed of the fleeing galaxy. The quotient of distance/speed gives a time: how long ago that galaxy was *here*. There is pretty good agreement (±20%) on those times for different galaxies. The answer is around 13 billion years. That gives us an approximate date for the Big Bang.

After the Big Bang

Right after the Big Bang, the entire universe occupied a volume many times smaller than, say, the size of a proton. That means we are talking about enormous mass density and enormous energy density. Ever since, the universe has been expanding and cooling.

The rate of expansion and the rate of cooling can be calculated from theory. In particular, during most of the first microsecond of its existence, the universe was quark soup, with most of the energy in the form of photons. At age 1 microsecond, it had an average temperature around 1.5×10^{13} K. As it cooled and expanded further, the quarks coalesced into baryons and mesons. No nuclei yet! It is still too hot.

At age 3 minutes, the protons and neutrons started to fuse. First, **deuterium** formed:

$$\mathrm{H} + n \rightarrow {}^{2}\mathrm{H}$$

Then **tritium,** the radioactive isotope of hydrogen:

$${}^{2}\mathrm{H} + n \rightarrow {}^{3}\mathrm{H}$$

Then helium:

$${}^{3}\mathrm{H} + {}^{1}\mathrm{H} \rightarrow {}^{4}\mathrm{He}$$

Helium-4 is very stable, and the nucleon density was great enough so most of the deuterium and tritium was fused into helium. Heavier nuclei—the rest of the Periodic Table—were later formed by further fusion, but their abundance in the universe is much less than that of hydrogen and helium.

The relative abundance of hydrogen and helium can be calculated from the model using the kind of statistical ideas developed in Chapter 9, Entropy and Information. The prediction is that there should be 12.7 times as many protons as helium nuclei. Astronomical data (i.e., optical spectroscopy) from various parts of the universe—nearby stars, distant stars, and distant galaxies—all cluster close to that ratio. This agreement of theory with observation is one of the triumphs of theoretical physics. A theory that was formulated to explain the physics of the very small—the elementary particles—turns out to make correct predictions about an observable universe 10^{10} light-years in radius.

Atoms, Galaxies, and Stars

The correct prediction of the abundance of helium in the universe is such a spectacular achievement that there is some temptation to end the book here. But you might be interested in what happened after the formation of the nuclei. When did the universe get cool enough so that electrons would stick to those nuclei and make neutral atoms that could survive for a while? The answer is that the proton-helium-electron soup we call **plasma** persisted for about half a million years. By that time, the universe had expanded enough and cooled enough so that an atom, once formed, would not immediately be ionized by a high-energy photon. It is actually this transition from an ion-electron plasma to a gas of uncharged atoms that released the energy we now observe as the 3-degree microwave background radiation. Repeat: It took a microsecond to freeze the quark-gluon plasma. It took half a million years to freeze the proton-helium plasma.

After this time the gravitational attraction of the swirling gas could take advantage of density fluctuations to make **galaxies**—regions of space with higher gas densities separated by enormous regions with very low gas densities. Within the galaxies, further fluctuations resulted in the formation of stars, clumps containing upward of 10^{50} atoms, collapsing under their mutual gravity. As a star collapses, the gravitational acceleration heats up the gas again. Soon it reverts to the plasma state and glows brightly, because near its center it is millions of times hotter than most of the volume of its galaxy.

21.6 SUCCESSES AND LOOSE ENDS

The neat theory outlined in Sec. 21.4 has had a number of remarkable successes. The particle zoo of several dozen diverse "elementary" particles has been "explained" in terms of quarks and leptons and their interactions. Also, the model predicted the properties of new particles that were later observed.

Aesthetically, the theory has given physicists the satisfaction that it got simpler as it developed. First, there was the unification of the electromagnetic and the weak interactions into the **electroweak** interaction. Using gigantic accelerators, the vector bosons predicted by this unification were found! Then came the proposed unification of the electroweak with the strong interaction, into what is called Grand Unified Theory (abbreviated **G.U.T.**). Both of these unifications were the result of a clever use of gauge theory. The last unification, that of G.U.T. with a quantum theory of gravity, is in progress. The current jargon terms (which here will remain undefined) are supersymmetry, **supergravity,** and superunification. The prediction is that at higher and higher energies, which also means shorter and shorter distances, the strong interaction gets weaker and the gravitational interaction gets stronger. At sufficiently high energy the distinction between interactions vanishes. At those high energies, so does the distinction between quarks and leptons. We are left with one kind of particle and one kind of force: the ultimate in simplicity and elegance.

This simplicity is attained high in the energy scale, to be sure. It is much too high to ever be available from a man-made accelerator. But it actually corresponds to an epoch in the history of the universe. Supersymmetry presumes to describe the events between 10^{-43} and 10^{-35} seconds "after creation."

We have mentioned the triumph of the theory in calculating the hydrogen-to-helium ratio of the universe. The theory does make some predictions that have not been confirmed. When we listed the quanta of the interactions governing the quarks and leptons, we mentioned the photon, the gluons, and the vector bosons. We failed to mention the Higgs bosons, necessary to the theory but as yet unobserved. They may be too massive to be produced in the laboratory. The theory also predicts that there should be some very massive particles that are magnetic monopoles. (See Sec. 15.3 and Chapter 20, Question for Further Thought α.) It permits rough estimates of their density, which turn out to be so small that these particles should be observed very rarely. So the search for magnetic monopoles is justified not merely by a longing for symmetry in the equations of electromagnetism!

As for the probability of finding free quarks, the theory is pretty silent on that. At the temperatures available in the laboratory, quarks cannot escape their confinement inside the baryons and mesons. But might there be relic quarks, left over from that first microsecond before the quarks froze out? Quarks carry a characteristic signature: charge ⅔ or ⅓ of an elementary charge. This section is called "Successes and Loose Ends." Maybe by the time you read this, the question of the existence of free quarks will no longer be a loose end.

Among the many loose ends not mentioned here so far, there is one that may turn into a major success. It has to do with the existence of antimatter in the universe. The ratio of the number of photons to the number of baryons in the universe is pretty well known. Most of the photons are in the 3-degree background radiation. Most of the baryons are protons. The ratio of photon density to proton density in the universe is somewhere around 10^9, give or take a few powers of 10. There are very few antiprotons, at least in our galaxy. Any antiproton has an awfully good chance of meeting a proton and being annihilated, usually with the emission of two oppositely charged pions. When these meet, they in turn annihilate, with the emission of two or more photons. Now suppose that during that first microsecond there were more baryons than antibaryons. There would have been a flurry of annihilations, leaving behind the excess baryons and lots of photons from all the annihilations. An asymmetry of about one part in a billion in the baryon/antibaryon ratio at that early time would explain the observation. Actually, G.U.T. calls for such a fundamental asymmetry between matter and antimatter. Part of that story is the prediction that the proton is itself unstable and can decay. A *u*-quark turns into an anti-*u*-quark! It is not supposed to happen very often, perhaps once every 10^{30} years in the life of a proton. Experiments looking for traces of proton decay are in progress. If successful, they might give us an estimate of the half-life of the proton. There is, of course, another possibility. Suppose there was no baryon-antibaryon asymmetry. Will we find that some distant galaxies are actually made of antimatter?

SUMMARY

The four interactions are called strong, weak, electromagnetic, and gravitational. The last section of the chapter, Sec. 21.6, describes their mathematical "unification."

The so-called elementary particles are classified into leptons and hadrons. The hadrons are either baryons (three quarks) or mesons (quark + antiquark).

There are six quark flavors, and each comes in three colors. Free particles are color-neutral.

One role the leptons play is to change the flavor of a quark. This is what happens in beta decay (weak interaction).

The three-degree microwave background radiation is conclusive evidence for Big-Bang cosmology.

In cooling after the Big Bang, the first freezing-out was the quarks condensing into baryons and mesons (age 1 microsecond).

The second freezing-out was the nucleons fusing into helium-4 (age 3 minutes). The theory predicts the observed proton/helium abundance ratio.

The third freezing-out was the formation of neutral atoms (age 5×10^5 years). Only after that would the gas form galaxies and, later, stars.

The photon-to-proton ratio of 10^9 indicates that there were more baryons than antibaryons around during that first microsecond. Is this evidence for a fundamental lack of symmetry?

KEY TERMS

Nuclear force The force that keeps the nucleons (protons and neutrons) together in the nucleus. Arises from the *strong interaction*.

Radionuclide Any species of atomic nucleus that is radioactive.

Weak interaction One of the four fundamental interactions, the one involved in the interaction of leptons. The weak interaction is responsible for beta decay, in which a neutron decays into a proton. More generally, the weak interaction can change a quark's flavor.

Antiparticle Every particle has an antiparticle having all the same properties but opposite sign of charge. When particle and antiparticle meet, they annihilate each other in a burst of two or more photons (or particle-antiparticle pairs).

Leptons Literally, "light ones." The family (of six) consisting of the electron, the muon, and the tau, and their respective neutrinos. The leptons interact via the weak interaction. Charged leptons also interact via the electromagnetic interaction.

Hadrons Literally, "strong ones." The family consisting of the mesons and baryons. They interact via the strong interaction. Charged hadrons also interact via the electromagnetic interaction.

Baryons Literally, "heavy ones." The nucleons and hyperons. Made of three quarks.

Mesons Literally, "medium ones." Particles exchanged between nucleons to give rise to the nuclear force. The earliest mesons to be discovered were the pions: π^+, π^-, and π°. A meson is made of a quark and an antiquark.

Quarks Point particles having a charge $+\frac{2}{3}e$ or $-\frac{1}{3}e$, of which all hadrons are composed. There are six quark *flavors,* called u, d, c, s, t, and b. Quarks have another quantum number called *color,* which has three values called red, blue, and green.

Gluons The quanta of the interaction between quarks.

QCD Quantum chromodynamics. A gauge field theory of quarks, which explains their interaction and their confinement. The prefix chromo comes from the term *color.*

QED Quantum electrodynamics. Quantum field theory of the electromagnetic field, which explains the interaction of photons with charged matter.

Asymptotic freedom A property of the way quarks interact, which reduces their attractive force at high-enough energies (close-enough crowding).

Cosmology The study of the origin and fate of the universe.

Steady-state cosmology A theory that, on the cosmic scale, the universe looks much the same today as it ever did or ever will.

Big-Bang cosmology The theory that the universe began at an instant, all contained in a very small volume, and has been expanding and cooling ever since.

Three-degree background radiation Microwave radiation from all parts of the sky having the characteristic spectrum of a body at a temperature of 3 K.

Red shift Doppler shift, or frequency lowering due to a receding source of light.

Deuterium Heavy hydrogen, ^{2}H, being the bound state of a proton plus neutron.

Tritium The radioactive isotope of hydrogen, ^{3}H, being the bound state of a proton plus two neutrons.

Helium-4 The bound state of two protons plus two neutrons. The second most-abundant element in the universe.

Galaxy A region containing many stars, separated from other galaxies by extensive regions of almost empty space devoid of stars. Our galaxy is the Milky Way.

Plasma A state of matter which is so hot that all atoms are completely ionized.

Electroweak interaction The unification of the electromagnetic and the weak interaction.

G.U.T. Grand Unified Theory. The unification of the electroweak and the strong interaction.

Supergravity The unification of G.U.T. and quantum gravity.

Planck time The shortest time that has any meaning in a quantum theory of gravity $= 5 \times 10^{-44}$ sec.

Planck length The shortest distance that has any meaning in a quantum theory of gravity $= 1.6 \times 10^{-35}$ m.

Boson A particle that does not obey the Pauli exclusion principle. Quantum theory distinguishes between two kinds of particles: those symmetric under exchange of identical particles (bosons), and those antisymmetric (fermions). "Antisymmetric" means the sign of the wavefunction changes when identical particles are interchanged. Fermions obey the Pauli principle; bosons do not. Electrons, nucleons, and neutrinos are fermions. Mesons, photons, and gluons are bosons.

REVIEW QUESTIONS

- All forces are due to one of the four interactions: strong, weak, electromagnetic, and gravitational. Which of these four governs the interaction of a neutrino with matter?
- All hadrons are composed of either two or three quarks. Each baryon consists of two / three quarks, and each meson consists of two / three quarks. Explain how all hadrons can be color-neutral.
- Explain how the red shift of light from distant galaxies makes it possible to assign a date to the Big Bang.
- Explain how protons and neutrons can fuse into helium in three steps.
- Why does the fusion reaction $^3H + {}^1H \rightarrow {}^4He$ not take place at low temperatures? (HINT: What is the charge of a tritium ion?)
- At a temperature of 3 K, you would expect much more helium than hydrogen in thermal equilibrium. Then why are there 13 times as many protons as helium-4 nuclei in the universe?

PROBLEMS

★★ 1. Mesons Are Quark Pairs

(a) Show that four different mesons can be made out of u and d quarks. The rule is that a meson always contains one quark and one antiquark so that the combinations $u\bar{d}$ and $d\bar{u}$ are different mesons.

(b) The $u\bar{d}$ is the pi-plus meson, while the $d\bar{u}$ is the pi-minus. Argue that the π^+ and π^- are each other's antiparticles.

(c) How many different mesons can be made out of the six quark flavors u, d, c, s, t, and b?

Table of Quarks

Name	Charge
u	⅔
d	−⅓
c	⅔
s	−⅓
t	−⅓
b	−⅓

✶✶ **2. How Many Baryons?** How many baryons can be made out of the two quark flavors *u* and *d*? Make a list. Remember the rules: A baryon has three quarks. To be color-neutral, there are either three quarks or three antiquarks (making an antibaryon), never a mixture.

COMMENT: Specifying a quark combination does not uniquely describe the particle. For example, *uud* can be either a proton or a delta-star-plus. They differ in a quantum number this chapter has not mentioned.

✶✶ **3. How Many Baryons?** How many baryons can be made out of the three quark flavors *u*, *d*, and *s*? Make a list.

COMMENT: Both the lambda-zero and the sigma-zero hyperons correspond to the quark combination *uds*. So there are actually more different baryons than quark combinations. There are other quantum numbers (beside flavor) this chapter has not mentioned.

✶✶ **4. How Many Baryons?** How many baryons can be made out the four quark flavors *u*, *d*, *s*, and *c*? Make a list. (See the COMMENTS in Problems 2 and 3.)

✶✶✶ **5. How Many Baryons?** How many baryons can be made out of all six quark flavors, *u*, *d*, *s*, *c*, *t*, and *b*? Don't bother with a list. Just show your calculation.

✶ **6. Table of Mesons (Incomplete)** Even the longest-lived mesons have mean lifetimes of only a few dozen nanoseconds. The eta mesons have lifetimes less than 10^{-18} sec. If a proton-proton collision creates lots of meson pairs, few survive to be detected in counters or spark chambers.

An accelerator with a storage ring can have two oppositely directed proton beams colliding. In that way the entire energy of both colliding protons is available for conversion to rest energy of the products.

(a) Argue that at a beam (kinetic) energy of 139.6 MeV, it would be possible to produce a pair of charged pions at rest.

(b) At a beam energy of 150 MeV, how much kinetic energy would each pion of a pair carry off? Assume that they carry off equal energies and the protons are brought to rest, that is, collide head-on.

(c) As the beam energy is increased above 500 MeV, some charged pions are seen to travel away from points at which there was no charged-particle track. What is happening?

(d) Refer to Problem 1 for the charges of the four quarks involved in these mesons. Verify that the charges of the quark-antiquark pairs add up correctly for all nine pairs.

NOTE: Nine? Count them: pi-plus, pi-minus, *K*-plus, *K*-minus, and so on.

✶ **7. Table of Baryons (Incomplete)** The table below lists the two kinds of nucleons (protons and neutrons) and eight different kinds of hyperons. Since each baryon also has an antiparticle, this should be considered a table of twenty particles.

(a) The Ξ^o hyperon, which you might think would be called the chi-zero, is usually called the cascade particle, because its decay products are themselves so short-lived. It decays with a mean life of 0.3 nanoseconds, which indicates a "weak" decay. Argue that baryon number is conserved in this decay, but meson number is not.

NOTE: The conservation of baryon number is an old and respected principle that is currently being subjected to serious doubt. At present writing, no violation has been reported. But note the question mark after the word "stable" for the proton "decay mode." Meson number is *not* a conserved quantity. Since mesons are quark-antiquark pairs, this lack of conservation is not surprising.

Table of Mesons

Name	Symbol	Antiparticle	Mass (MeV/c^2)	Quark Combination	Decays into
Pion	π^+	π^-	139.6	$u\bar{d}$	$\mu^+\nu_\mu$
	π^o	Self	135.0	$u\bar{u}$	$\gamma\gamma$
Kaon	K^+	K^-	493.7	$u\bar{s}$	$\mu^+\nu_\mu$ or $\pi^+\pi^o$
	K_s^o	K_s^o	497.7	$d\bar{s}$ and $\bar{d}s$	$\pi^+\pi^-$ or $\pi^o\pi^o$
	K_L^o	K_L^o	497.7	$d\bar{s}$ and $\bar{d}s$	$\pi^+e^-\nu_e$ or $\pi^-e^+\nu_e$
Eta	η^o	Self	548.8	$d\bar{d}$	$\gamma\gamma$ or $\pi\pi\pi$
	η'	Self	957.6	$s\bar{s}$	$\eta\pi\pi$

Table of Baryons

Name	Symbol	Mass (MeV/c^2)	Charge	Decays into
Proton	p	938.3	+1	stable ?
Neutron	n	939.6	0	$pe^-\bar{\nu}_e$
Lambda-zero	Λ^o	1115.6	0	$p\pi^-$
Sigma-plus	Σ^+	1189.4	+1	$p\pi^o$
Sigma-zero	Σ^o	1192.5	0	$\Lambda^o\gamma$
Sigma-minus	Σ^-	1197.3	−1	$n\pi^-$
Cascade-zero	Ξ^o	1314.9	0	$\Lambda^o\pi^o$
Cascade-minus	Ξ^-	1321.3	−1	$\Lambda^o\pi^-$
Delta-star	Δ^*	1385	various	$p\pi$, $n\pi$
Omega-minus	Ω^-	1672.2	−1	$\Lambda^o K^-$

(b) Here is the principal decay mode,

$$\Xi^o \rightarrow \Lambda^o + \pi^o$$

written in terms of quark combinations:

$$uss \rightarrow uds + u\bar{u}$$

Show that one quark changes flavor in the decay.

NOTE: Flavor changes occur only in "weak" decays, never in "strong."

(c) The delta-star hyperon can have four different charge states, +2, +1, 0, and −1. For example, the Δ^{*++} has quark makeup uuu. It decays with a mean life of 6×10^{-24} seconds,

$$\Delta^{*++} \rightarrow p + \pi^+$$

written in terms of quark combinations

$$uuu \rightarrow uud + u\bar{d}$$

The short lifetime indicates a "strong" decay. Show that this decay does not involve any quark changing flavor.

✶ **8. Planck Units** The "natural" units for cosmology are combinations of the universal constants for relativity, for quantum theory, and for gravitation:

Universal speed limit	c	$= 3.00 \times 10^8$ m/s
Planck's constant	h	$= 1.24 \times 10^{-6}$ m eV/c
Gravitational constant	G	$= 6.67 \times 10^{-11}$ Nm²/kg²

defined by the Universal Law of Gravitation:

force of attraction between two mass points at a distance r

$$= \frac{Gm_1m_2}{r^2}$$

NOTE: $\hbar = h/2\pi$

Verify the arithmetic for the following units:

(a) Planck mass $M_P = (\hbar c/G)^{1/2} = 1.2 \times 10^{28}$ eV/c^2
(b) **Planck time** $t_P = \hbar/M_Pc^2 = (\hbar G/c^5)^{1/2} = 5 \times 10^{-44}$ sec
(c) **Planck length** $L_P = ct_P = (\hbar G/c^3)^{1/2} = 1.6 \times 10^{-35}$ m

NOTE: When the average particle energy of a collection of quarks and leptons is greater than 10^{28} eV, the distinction between quark and lepton disappears. Similarly, there is no distinction between strong, weak, electromagnetic, and gravitational interactions. Applying the Uncertainty Principle, it becomes meaningless to inquire into the structure of the universe less than 10^{-43} seconds after its creation. If it turns out that quarks and leptons have structure, this "earliest time" may be rolled back.

✶✶ **9. Helium Abundance** The text states that almost all of the nuclei in the universe are hydrogen and helium, in the ratio 12.7 to 1. Less than 1% of the mass of the universe consists of heavier nuclei—our own surroundings notwithstanding! Show that this implies that the mass of the universe is 24% helium.

APPENDIX A: Vectors

The intuitive way to think about a vector is as an arrow. The length of the arrow represents the magnitude of the vector. Think about a displacement vector. If I walk 1 mile east and then 1 mile north, I get to the same place as by walking $\sqrt{2}$ miles northeast (Fig. A.1). Remember Pythagoras?

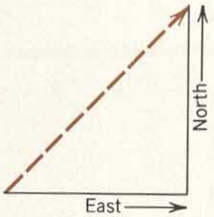
$$(1 \text{ mile})^2 + (1 \text{ mile})^2 = (\text{displacement})^2$$

or $|\text{displacement}| = \sqrt{2}$. We say that the **magnitude** (or absolute value) of the displacement vector is 1.414 miles. Its direction is 45° north of east, or 45° east of north. That description is equivalent to the description in terms of the **components**[1]

$$(x,y) = (1 \text{ mile}, 1 \text{ mile})$$

Of course, there has to be previous agreement that x means east and y means north. For any x and y, the magnitude of the displacement vector would be

$$\sqrt{x^2 + y^2}$$



A.1 If I walk 1 mile east and 1 mile north, my net displacement is. . . .

EXERCISE 1 If I go 3 miles east and 4 miles north, we write

$$(x,y) = (3 \text{ miles}, 4 \text{ miles})$$

The magnitude of the displacement is ______ miles. Note that the displacement is the same going 4 miles north and *then* 3 miles east. Is it the same as going 3 miles north and then 4 miles east? Yes or No? Note that there the magnitudes of the displacements are the same but the displacement vectors are not.

EXERCISE 2 If I go 2 miles east and 2 miles north, this is a displacement twice that of the (1 mile, 1 mile) displacement. I end up

$$\sqrt{2^2 + 2^2} = \sqrt{8} \text{ miles}$$

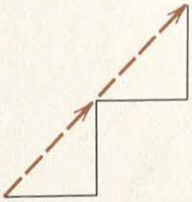
northeast. Is $\sqrt{8}$ equal to twice $\sqrt{2}$? Yes or No?

I have added the two displacements by placing the tail of the second arrow at the head of the first one (Fig. A.2). That turns out to be the same as adding the respective components.

EXERCISE 3 If I go 3 miles west and 4 miles south, I could write that

$$(x,y) = (-3 \text{ miles}, -4 \text{ miles})$$

Verify that this displacement vector is indeed the negative of the one in



A.2 If I go 2 miles east and 2 miles north. . . .

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[1] KEY TERMS defined in Chapter 4.

Exercise 1, (3 miles, 4 miles). Do this by putting the two arrows tail-to-head and showing that the resultant is zero.

EXERCISE 4 Add the displacement (1 mile, 1 mile) to the displacement (1 mile, −1 mile). Show that the resultant of two 1.4-mile displacements is a 2-mile displacement in this case. It is toward: the East / the North

Velocities add in the same way as displacements. If I am on a bus going east at 4 miles/hour and walk forward at 3 miles/hour relative to the bus, then in the road's frame of reference I am moving at 7 miles/hour. We just add the velocities. But if I walk sideward, say north, at 3 miles/hour relative to the bus, then my resultant velocity is

$$\sqrt{4^2 + 3^2} \text{ miles/hour} = 5 \text{ miles/hour}$$

EXERCISE 5 You are on a bus traveling east at 12 mph and move north relative to the bus at 5 mph. What is the magnitude of your velocity relative to the road?

VECTORS VERSUS SCALARS

You will soon develop a sense of which physical quantities are *vectors* and which are **scalars.** A vector has a direction; a scalar does not. Displacement is a vector. Velocity is the rate of change of the displacement vector, so it is also a vector. By the same kind of reasoning, acceleration is a vector. It is the rate of change of velocity. Force is proportional to acceleration, according to Newton's Second Law. Indeed, force is also a vector. Mass has no direction associated with it. It is a scalar. Momentum is mass times velocity; it is a vector.

The foregoing statements are intuitively good. You believe them easily. That is not what makes them true. They are actually statements about the physical universe. There is no philosophical guarantee that an object has to have the same mass for northward accelerations as for eastward accelerations, nor that this mass is independent of velocity and independent of the object's history. It is simply a fact of experience, what we call a law of nature. It also makes the mathematics simple: The three components of the displacement vector can be treated independently of each other. The same holds for the velocity vector and the acceleration vector.

COMPONENTS OF A VECTOR

To read this section, you should be familiar with enough trigonometry so that "cosine of an angle" immediately says to you something like "adjacent over hypotenuse" (see Appendix B). Look at Exercise 1 again. Suppose that you are given that the displacement s is 5 miles in a direction making an angle θ with (north of) the x-axis whose cosine is 0.8. Then you can immediately say that the x-component of that displacement vector is

$$\begin{aligned} x = s \cos \theta &= 5 \text{ miles} \times 0.8 \\ &= 4 \text{ miles} \end{aligned}$$

Similarly, the y-component is

$$\begin{aligned} y = s \sin \theta &= 5 \text{ miles} \times 0.6 \\ &= 3 \text{ miles} \end{aligned}$$

You need that kind of reasoning, for example, in Sec. 4.5, to calculate the work done in climbing stairs. Work is force times displacement, to be sure, but "displacement" here really means the component of the displacement parallel to the direction of the force. In that example the force is a vertical force, the force needed to support my weight. So the relevant component of the displacement is the vertical component. If the stairs go up at a 45° angle, then $\sin \theta = \sin 45° = 0.707$, and the vertical component of the displacement is 70.7% of the displacement.

EXERCISE 6 Suppose that the stairs rise 7 inches per step and each step is 10 inches deep.

(a) Show that the displacement vector of a person climbing the steps makes an angle of 35° with the horizontal.

(b) Show that the vertical component of a 1-foot displacement along the staircase is 0.57 feet.

In Sec. 15.2 the force of a charge q moving in a magnetic field B is written

$$F = qBv_{\perp}$$

where $v_{\perp}$ is the component of the velocity vector $\mathbf{v}$ perpendicular to the direction of the B-vector. The force F is itself a vector, of course, and there is an elaborate prescription for finding its direction (right-hand rule and all that). This example is quite different from the previous one. Work is a scalar; force is a vector. The mathematicians have given us a jargon for the two different ways of "multiplying two vectors." The product of a force times a displacement that gives work is called scalar product. The product of magnetic field times velocity that gives force per unit charge is called vector product. This book avoids using this terminology. What it emphasizes is that in writing the formula for work, it is the component of the displacement *parallel* to the force that counts; in writing the formula for magnetic force per unit charge it is the component of the velocity *perpendicular* to the field that counts.

ADDITION OF VECTORS

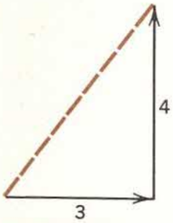


FIGURE A.3

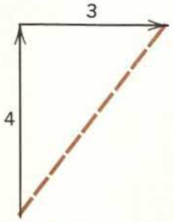


FIGURE A.4

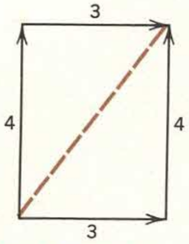


FIGURE A.5

If you think of a vector simply as a row (or column) of numbers, like the

$$(x, y) = (3 \text{ miles}, 4 \text{ miles})$$

of Example 1, then the recipe for addition of vectors is pretty obvious. You just add the components:

$$(A_x, A_y) + (B_x, B_y) = (A_x + B_x, A_y + B_y)$$

The extension to three dimensions is trivial:

$$\begin{aligned}(A_x, A_y, A_z) &+ (B_x, B_y, B_z)\\ &= (A_x + B_x, A_y + B_y, A_z + B_z)\end{aligned}$$

But if you think of a vector as something having a magnitude and a direction, then the recipe no longer looks so obvious. Here it is. Represent the vector as an arrow. Remember that the arrow has magnitude and direction; it does not have a location. So the arrow can be moved around. To add A + B, move the tail of B to the head of A and connect the tail of A to the head of B to make a new arrow. This new arrow is the resultant, the vector sum A + B.

EXERCISE 7 By drawing an appropriate sketch, convince yourself that A + B is the same as B + A, that is, vector addition is commutative. The sketch you have drawn is a parallelogram, so the rule for adding vectors is called the *parallelogram rule*. The vector sum is the diagonal of the parallelogram.

In adding an eastward displacement of 3 miles to a northward displacement of 4 miles, as in Exercise 1, both ways of looking at the addition give easy numbers. In x and y components, we have

$$(3 \text{ mi.}, 0) + (0, 4 \text{ mi.}) = (3 \text{ mi.}, 4 \text{ mi.})$$

Figure A.3 represents the tail-to-head recipe for the eastward northward addition. Figure A.4 shows it for the northward + eastward addition. The parallelogram (Fig. A.5) is a 3 × 4 rectangle.

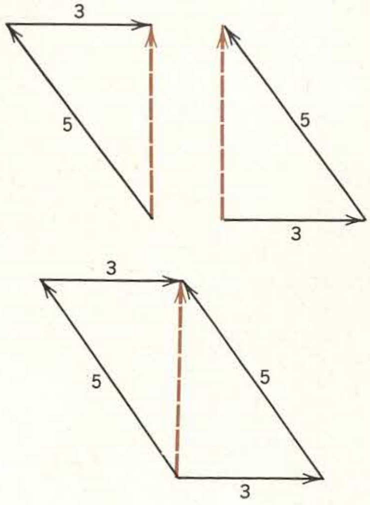


FIGURE A.6

In Problem 20.22 (see also Problem 4.25) a boat with still-water speed of 5 mi/hr crosses a river flowing at 3 mi/hr. Its heading is such that the boat actually moves straight across the river, that is, in a direction perpendicular to the flow direction. Figure A.6 shows the two ways in which the two velocity vectors can be added. The resultant velocity vector is shown in color. Here is an example where the tail-to-head recipe is much easier than the add-the-components recipe, because it was the direction of the resultant that was given. Your experience with right triangles and Pythagoras is surely sufficient so that you had no trouble arriving at 4 mi/hr for the magnitude of the resultant.

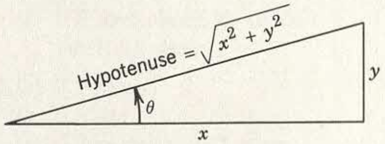
APPENDIX B: Trigonometry

If your work is going to involve the mechanics of teeth or bones, you will need to know some trigonometry. It is very likely that the box (below) labeled All the Trig You Need will really suffice. To do Problem 4.29, all you need is the definition of the sine of an angle. That problem derives an expression for the range of a throw in terms of the initial speed v_0 and the angle θ the initial velocity makes with the horizontal.

ALL THE TRIG YOU NEED

$$\text{hypotenuse} = \sqrt{x^2 + y^2}$$

$$\tan\theta = \frac{y}{x} = \frac{\text{opposite leg}}{\text{adjacent leg}}$$



$$\sin\theta = \frac{y}{\sqrt{x^2 + y^2}} = \frac{\text{opposite leg}}{\text{hypotenuse}}$$

$$\cos\theta = \frac{x}{\sqrt{x^2 + y^2}} = \frac{\text{adjacent leg}}{\text{hypotenuse}}$$

ANSWERS TO ODD-NUMBERED PROBLEMS

Chapter 1

1. (b) 25 cm/sec
(c) 25 cm/sec
(d) 47.5 cm for 0.19 sec, or 25.0 cm/sec, but don't trust last significant figure.

3. (a) 3600 m
(b) 60 m
(c) 75 steps/min
(d) 0.80 sec/step
(e) 1.25 steps/sec

5. (a) 2.5 m
(b) 12.5 m

7. (a) 25 sec
(b) 37.5 sec
(c) 2.7 m/sec
(d) 33.3 sec < 37.5 sec
(e) 20.9 sec

9. 660 m

11. (a) 80 cm/sec
(b) 1.9 sec
(c) 1 m/sec
(d) 5 cm
(e) 4
(f) $(2)^{1/2}$ of 1.41

13. (a) 0.5 hr
(b) 40 mi/hr

15. (a) 1.6 m/sec
(b) 1.6 m/sec^2
(c) 8 m/sec
(d) 4 m/sec
(e) 20 m

17. (a) 6 m/sec
(b) -2 m/sec^2
(c) -2 m/sec
(d) 3 m/sec
(e) 15 m

19. (a) 2.3×10^{-3} sec
(b) 2.9×10^5 m/sec^2 = 3×10^4 g's

21. (a) 0.0833 mi

23. (a) Yes, 2.23 m/sec^2.
(b) No, not an acceleration.

25. (a) $\frac{1}{2}(v_i + v_f)t$
(c) $2v_i/a$
(d) $v_f = -v_i$, no

27. (a) 5.0 sec
(d) 10.0 sec
(e) 4.0 sec
(h) 117.6 m

29. (a) 11.4 sec
(b) 101
(c) 637 km, about $\frac{1}{10}$ of the earth's radius

31. (a) 2.5 m/sec^2
(b) 0.8 sec
(c) 0.2 m down, 0.2 m up.
(d) 0.2 sec

33. (a) 136 lb
(b) 0.37 mi/hr sec
(c) 0.55 mi/hr sec

35. (a) 6.4 ft/sec
(b) 3.2 ft/sec
(c) 3.2 ft
(d) 1.6 ft/sec
(e) 0.8 ft
(f) 4.8 ft
(g) 2.4 ft
(h) 4

37. (a) 50 lb
(b) 100 lb
(c) 1 g
(d) zero

39. (a) 10%
(b) 0.22 mi/hr sec

41. (a) 10^6 sec, or 12 days
(b) 500 km, or 300 miles
(c) Fermi questions don't have exact answers.

43. (a) 6250 lb sec^2/ft
(b) 7.04×10^4 lb ft/sec^2

45. (a) 0.27 sec
(b) -5.7 g
(c) 5.8 mg
(d) 80°

47. (a) 0.32 m/sec
(b) 0.32 m
(c) -0.05 g
(d) 118 N
(e) The springs decouple the patient + mattress from the frame, so the practical answers will be smaller.

49. (a) 5×10^5 m/sec^2
(b) 5×10^4

51. (a) 4 g's
(b) 4

53. 20 g's

55. (b) 20
(c) 34%

57. (a) 1 cm/sec^2
(b) 0.5 g

59. (b) 233 lb
(c) 369 lb

Chapter 2

1. (a) 1 m/sec^2
(b) 1 m/sec
(c) 1 kg m/sec
(d) same

3. (a) 150 kg m/sec, no change.
(b) tripled

7. (a) 0.5 m/sec
(b) 0.67 m/sec
(c) zero
(d) 0.5 m/sec north

11. 72 newtons

13. (b) forward

15. $v_1 - v_2 = v_2^* - v_1^*$

19. -5 cm/sec, $+5$ cm/sec

21. (a) zero
(b) $\frac{1}{3}v_1$

23. 72

25. (a) 5.0×10^4 leg m/sec
(b) 16.7 m/sec

27. (a) 12 m from the 50 or 8 m from the 75
(b) 3 m/sec^2, 2 m/sec^2
(c) 3 m/sec, 2 m/sec
(d) 10.5 m

29. (a) −6.68 kg m/sec
(b) 2930 N
(c) 0.19 cm
(d) The shoulder is soft.

31. Some copies have typo: Jacket thrown at 5 m/sec.
(a) No external forces.
(b) 5 kg m/sec down
(c) 4.25 kg m/sec down
(d) 0.057 m/sec down
(e) 18% too big

33. (a) 1.0 m
(b) 0.45 sec

35. (b) 2Mg(M − m)/(M + m)
(f) ⅓g, ⅔Mg

Chapter 3

1. (a) 3%
(b) 3%

3. (a)).79 cm^2
(b) 0.033 cm^2
(c) 96
(d) 1.41 cm
(e) 1600 lb
(f) 1.41 cm

5. (a) 2%
(b) 3%
(c) 1%
(d) 44%
(e) 73%
(f) 20%

7. $\frac{1}{2}f = 1$, so $f = 2$

9. (a) 800 kg
(b) 1250 cm^3
(c) 14.3%

11. (a) stress × $80/3^2$.
(b) 13.5 × its weight.

13. (a) 0.98 mm
(b) 9.8×10^{-4}
(c) 2×10^{11} N/m^2

15. (a) 9.8×10^7 N/m^2
(b) 4.9×10^{-4}

17. (a) 2×10^3 dynes

19. (a) 4
(b) 1.41
(c) 2%
(d) 0.5%

23. 5.8 psi

25. (a) 0.252 kg/m
(b) 184 N
(c) 0.75 g

29. (b) 0.2%, no

33. $F^{1/2}M^{-1/2}L^{-1/2}$

Chapter 4

1. 33 lb

3. (a) 1000 J
(b) 1000 J

5. 10 ergs or 10^{-6} joules

7. 450 N

9. (c) 0.56 m/sec

11. (a) 4.9×10^{-2} J

13. (a) 0.20 m
(b) 0.20 sec
(c) 0.20 sec
(d) 2.0 m/sec

15. (b) -98 m/sec^2 = -10 g's
(c) 11
(d) 4410 J
(e) 4410 J

17. 221 W

19. (a) 184 W
(b) 284 W
(c) 2.5 m

21. (a) 2.24×10^4 J
(b) 5.36 kcal = 5.36 Cal
(c) 102 m

23. (a) 24.5 kW
(b) 14.7 kW

25. (a) 0.25 hr
(b) no
(c) 5 mi/hr
(e) 0.8

27. (a) 9.8 J
(b) 4.9 m/sec^2
(c) 0.90 sec
(d) 4.41 m/sec
(e) 9.8 J

31. 100.4 m, 97.0 m

Chapter 5

3. 10^5

5. (a) Clockwise
(b) 2

7. (a) 8.0 km/sec or 17,700 mph
(b) 85 minutes

9. (a) $4\pi^2$ (1 AU)^3/(1 yr)^2
(b) 0.72 AU

13. (c) 0.5%

15. (d) 3.1×10^7 J/kg, or 8.6 kWH/kg

17. (b) 893 J; lost 217 J

21. 22°, I haven't

23. (c) 2.2 rad/sec or 0.35 rev/sec
(d) yes
(e) 38.9 J

27. 668 N or 150 lb

29. 0.35%

33. (a) 4.55×10^{-48} kg m^2
(b) 2.72×10^{-13}

Chapter 6

1. (a) 1000 kg
(b) 10 m^3
(c) 0.018 l
(d) 3.0×10^{-23} cm^3
(e) 3.1×10^{-8} cm

3. (a) 1.025 kg
(b) 0.976 l

5. (b) 9.1% by mass, 15.5% by volume

7. (a) 1.051
(b) 0.951 cm^3
(c) good to 2.5 parts in 10^9.

9. (a) 0.15
(b) 1.5 cm
(c) 0.15 kg g
(d) 0.85 kg g

11. (a) 1.75 cm
(b) 6.9 N

(c) 0.0173 N/cm^2, note mixed units!
(d) 700 cm^3, no

13. 0.903 or 90.3%

15. zero

17. 1.06

19. 0.98 N or 0.22 lb

23. (a) 7.9 atm
(b) 0.9 tons

25. (a) 1.1 tons

29. (a) 14.6 atm

33. 10 N/m^2 or 10^{-4} atm

35. (a) 8 cm
(b) Small wetting angle

39. (a) Yes
(b) 0.4 atm
(c) 0.25 atm

41. 17%

43. (a) 1%
(b) 1%
(c) 0.1%
(d) 2%
(e) 33%
(f) 100%
(g) 50%
(h) 20%
(i) 33%

45. 0.06% occupied, 99.94% empty.

51. 27.3%

53. (b) −0.76% *Hint:* H_2O is 18; air is 29.

55. (a) 3.1×10^{-23} kg m/sec
(b) 6.1×10^{-3} sec
(c) 5.1×10^{-21} N
(d) 2.7×10^{25}
(e) 0.45 atm

57. Assume speed of sound proportional to rms speed.

59. (a) 96 atm
(b) 0.48%
(c) 1027 cm/atm

61. (a) 22.5 lb

63. (a) 20.1 layers
(b) 210.6 layers
(c) 1386 layers

65. (a) 0.0126%

67. (a) 3.34 km

Chapter 7

1. (a) 400 cm^3/sec
(b) 2.5 sec
(c) 2.5 min

3. (a) 2×10^4 cm^3/sec
(b) 20 kg
(c) 400 kg m/sec
(d) 400 N
(e) 90 lb? Sure!

5. (a) to the floor
(b) yes
(c) It warms the shoulder.
(d) 0.5 watts/jet

7. (a) 300 cm^3/sec
(b) 3 m/sec
(c) 4
(d) 1.125×10^{-3} J/cm^3 = 1.125×10^3 J/m^3 = 1.125×10^4 ergs/cm^3
(e) 3375 N/m^2 less

9. -2×10^4 N/m^2

11. (a) 12.5 m/sec
(b) 1.77 atm

13. (c) Clockwise

17. 0.6 N/m^2

21. 4×10^{-5} watts

23. (a) 1.1 watts

25. (b) 30 m/sec

27. (a) m^2/sec
(b) *vd*
(c) 41 cm/sec, 3420

29. (b) 0.8 N viscous force

31. 0.3 N

37. (a) 1.41
(b) 0.5%

43. (a) Increasing
(c) 1

47. (a) × 0.71
(d) 23.1 cm/sec

Chapter 8

3. 90 cm

7. Brick house loses heat faster, but has larger heat capacity.

9. (a) 7/cm
(b) 0.2/cm
(c) 0.7/cm

11. (a) 37 J
(b) 110 watts, 20 min

13. (a) 30°C
(b) 33.3°C
(c) 36.8°C
(d) 34.5°C

15. 61 W for 75-kg walker

17. (a) 116 kcal
(b) 1.6°C

19. (b) 0.03 cal/gm°C
(c) 0.33 cal/gm°C

23. 1.7×10^3 liters

27. (a) 1.000024 cm
(b) 0.0024%
(c) 0.0048%
(d) 0.0072%
(f) $\beta = 3\alpha$
(h) Most of the mercury is in the bulb.

29. (a) 822
(b) 0.32 cm^3

31. Positive

33. (c) 19.9 kW/m^2, 100
(d) 0.1°C/mm
(e) 21.2°C
(f) 200.1°C
(g) 9.5°C

Chapter 9

1. (a) 1001
(b) 16
(c) 1001 + 10 = 1011
1001 + 110 = 1111
(d) 1111101000

5. (b) 1014

7. 65,536, 2

9. (a) 8192 bits
(b) 2^{23} bits
(c) 512 pages

11. (a) 4 bits
(c) $1/16$

13. (b) 0.037 J/K

15. 1.16×10^{-5} J/K

17. (a) 1437 cal
(b) Yes
(c) Solid

21. 3.5×10^{30} J

23. $10^{3.612} \times 10^{23}$

25. (a) -4.7×10^{23} J/K mole

27. (a) 22,400 cm^3
(b) 6.02×10^{23}
(c) 1.01×10^4 N
(d) 10.1 J
(e) 1630 J vs. 1570 J exact

Chapter 10

1. $c_s = 330$ m/sec, $\lambda = 0.33$ m

3. (a) No
(b) $c_s = \text{const.} \times \sqrt{T}$

5. (a) 0.75 m
(b) 3.39 m
(c) 13.1 m

7. (a) 0.45 m
(b) 33 kHz

9. 10 kHz

11. (a) 3.48 m/sec °C
(b) 3.8% low

13. (a) 1.4983, 0.11% low
(c) 0.7 beats/sec

15. 0.9% low

19. (a) 2
(b) 4
(c) 250, 500, 750, 1000 Hz

21. 13 db

23. (a) Teenager
(b) Eardrum

27. (a) 0.25 Hz

31. (a) 2.3°

33. (b) 5% high

35. (a) 0.1 W
(b) 0.025 W
(c) 25%

37. (a) 532 Hz
(b) 6%, 6%

39. 0.0083, yes
(b) 0.44

43. (c) No

45. (a) 0.94 m/sec
(b) 660 kW/m^2
(c) 1.4×10^6 N/m^2

Chapter 11

1. (a) 1.05 cm from lens, or 0.5 mm below the book.
(c) Yes
(d) Yes, 15 cm from book.

3. (b) 50 cm

5. Double

13. (c) *f*/2.8
(d) *f*/16
(e) 35.7 mm

15. (b) *Hint:* offset $= d(\sin i - \cos i \tan r)$

17. (a) No
(b) Yes, $i = 81°$

21. (b) 3.0 m
(c) 2.51 m

25. Less

27. (a) 52.0 mm
(b) 22.6°
(c) 0.92° = 0.016 radian

31. (a) 48.8°
(d) 63.4° from vertical.

Chapter 12

1. 0.8 octave

5. (a) 1.5×10^{-4} rad
(b) 15 km or 9 miles

7. 0.029 mm

9. (b) 18.1°
(c) 38.3°
(d) 68.4°

11. (b) 2.5 mm
(c) 5 parts in 10^9
(d) Less

15. (b) $\theta_{\text{1st min}}$ stays same, $\theta_{\text{1st max}}$ moves out.
(d) $\theta_{\text{1st min}} = \lambda/w$.

17. (b) 2.915 mrad
(c) 2.08 mrad
(d) 0.8 mrad

19. (b) 0.03°

21. (b) 0.21 μm

25. (b) 120°, 240°
(e) Yes

27. (a) 50%

Chapter 13

1. (a) 2 eV
(b) 2 eV

3. (a) 90 eV
(b) 9 eV
(c) 10 eV
(d) 5 cm
(e) 1.11 cm from −plate

5. (a) −1 eV
(b) 1.11 mm
(c) 1.44×10^{-16} N
(d) Same

7. (a) 4.4×10^{-15} cm
(b) No, too small.

13. (a) *V* is halved.
(b) *Q* is doubled.

15. (b) 8.85×10^{-6} C/m^2
(c) No

17. (a) 2.7×10^{-5} C/m^2
(b) 1 mm

19. (a) 1.76×10^{14} m/sec^2
(b) Same
(c) 1.87×10^6 m/sec

21. (a) 6.1×10^7 V/m
(b) Yes

23. (a) Remove $9/10$, keep $1/10$
(b) 1000 V
(c) 1900 V

25. (b) 5 pF
(d) Yes

27. (a) 9×10^4 V/m
(b) 18×10^4 V/m
(c) 2

29.(a) $0.0855\ \mu\text{F}$
(b) $100\ \text{cm}^3$

31.(a) $900\ \mu\text{C}$
(b) 18 V
(e) In series, + to −.

33. 100 capacitors in parallel.

35.(b) 16%

39.(b) $4.52 \times 10^{-23}\ \text{cm}^3$, $10^{-4}\ \text{C/m}^2$
(c) 10^{11} V/m

41.(a) 7.8×10^{6} V/m
(b) 0.21 pC, 1.3×10^{6} ions

43.(b) 2.6×10^{21}
(d) 0.01
(e) 5.8×10^{-9} N

45.(a) 3.6×10^{-4} N
(b) 10^{10}

49. 2.9×10^{-7} V/m

53. $8k_e q/d^2$

Chapter 14

1.(a) 1.5 joules

3.(a) 2.6 A
(b) 44.2 ohms
(c) 5.2 A, 1200 W

5.(c) $\text{Na} + \text{H}_2\text{O} \rightarrow \text{NaOH} + \frac{1}{2}\text{H}_2$

7.(a) 6 Ω
(b) 4 Ω
(c) 1 A
(d) 0.5 A

9.(b) 1440 W

11.(a) 0.6 A
(b) 3.6 W

13.(a) 0.8 A
(b) 1.6 W, 1.6 W, 6.4 W, 9.6 W total
(c) 4 V, 4 V, 8 V
(d) V^2

15.(c) 6 Ω

17. 6 hr

19.(a) 1.3×10^{3} moles
(b) 8

21.(a) 2430 J
(b) Alcohol stores 75×

25.(b) 200 Ω, 60 mA, 0.36 W
(c) 50 Ω, 240 mA, 1.44 W
(d) You have the idea now.

31.(a) $12R_x/(8\Omega + R_x)$
(b) 4 V
(c) $R_x = 4\ \Omega$

33.(b) 0.404 A
(c) 4.79 V
(d) 11.86 Ω
(g) First circuit: $R = 542\ \Omega$;
Second circuit: $R = 1201\ \Omega$.

Chapter 15

1. 7.5×10^{-5} N

3.(a) 3.9×10^{-3} N/m west; wind is stronger, same direction.
(b) Yes

5.(a) East
(b) More near the poles.

7.(a) 3.6×10^{-7} sec
(b) 21.2 m

9.(a) Independent
(b) Double it.

11.(b) 1.22 MHz
(d) 0.51 m

13.(a) 1.28×10^{-20} N east
(c) 1.25 T
(d) 1.25 V
(e) $80\ \mu\text{V}$

15.(b) N
(c) Down
(d) Up

17.(a) Attract
(b) Repel

19.(b) East
(c) No

23. 1.3×10^{-8} tesla

25.(a) 6.3×10^{-11} T
(b) 3.4×10^{19} electrons
(c) 28.7 A

27.(a) 0.0126 T
(b) 10
(c) 0.05 N

29.(a) 2.6×10^{-33} N west
(b) 9×10^{4}
(c) 3×10^{8} m/sec

31.(a) 8×10^{-8} T
(b) $10^{-3}\ \text{B}_{\text{earth}}$

33.(a) 1 statcoulomb = 3.336×10^{-10} C

37. 1.26×10^{-3} T

39.(a) 0.020 H
(b) 0.039 H
(c) 0.079 H

45.(a) 0.04 J
(b) ×4
(c) 2% increase

49. $4.4 \times 10^{-8}\ \text{J/m}^3$ vs. $2.5 \times 10^{-3}\ \text{J/m}^3$

51.(a) Bin
(b) $p = 2u$

Chapter 16

3.(a) 0.71 V

5.(a) 5 V
(b) 7 V
(c) 350 W
(d) Yes, >250 W
(g) Less

9.(a) 1.3×10^{-6} N/A
(b) 1.13×10^{11} V m/sec
(c) 3×10^{8} m/sec

Chapter 17

1.(a) 7.5×10^{14} to 4.3×10^{14} Hz

7. 0.999997, 0.3679, 0.3487, 0.3660, 0.1353

9. 0.04 eV << 2 eV (photon)

11.(a) 2 mA
(b) 1.5 mW

13. 240°C

15.(a) 9.3 N
(b) $3.4 \times 10^{-3}\ \text{m/sec}^2$, yes

21. 5.1°

Chapter 18

1. 2.1×10^{-9} m

3.(a) 656, 486, 434, 410 nm

5. 75.6% ^{35}Cl, 24.4% ^{37}Cl

9.(a) 4.1×10^{-7} eV
(b) 1×10^{8} Hz
(c) 160 in 10^{9}

11. $(k_e e^2)^5 m^4 h^{-8} = 1.2 \times 10^7$ N/m^2

13.(c) (1,1,2), (2,1,1), (1,2,1)
(d) 3

Chapter 19

1. ${}_{40}\mathrm{Th}^{233} \rightarrow {}_{91}\mathrm{Pa}^{233} + e^-$, ${}_{91}\mathrm{Pa}^{233} \rightarrow {}_{92}\mathrm{U}^{233} + e^-$

3. ⅛

5.(a) 2.96×10^6 Bq
(b) 2.31×10^{12} Bq
(c) 1.74×10^{23}
(d) 7.53×10^{10}
(e) 3.61×10^{10} Bq, 1.24×10^4 Bq, 1.65×10^{11} Bq, 1.16×10^{22} Bq

7.(b) 346 hours = 14.4 days

9.(a) 2.31×10^9/sec
(b) 7.28×10^{16}/yr
(c) 0.062 Ci
(d) 2.91×10^{12}

11.(b) 1 μCi
(c) 1 μCi
(d) 0.25 μCi
(e) 1 nCi

13.(a) 96 ppm
(b) No, barely 4

15.(a) ¼
(b) ½
(c) ¼

17.(a) 0.707106781
(b) 0.8408964

19. 1 rad whole-body

21.(a) to ½
(b) to ⅓

23. 2.4×10^{22} gm $= 1.0 \times 10^{20}$ moles, 2.9×10^{26}/sec, 1.2×10^{27} MeV/sec $= 1.9 \times 10^{14}$ W, about 10^{-3} of solar power.

Chapter 20

1. 0.99995

3.(a) 2.5×10^5 AU
(b) 7.9×10^{-6} rad

5.(a) 3.67 μsec
(b) 2.93×10^3 ft
(c) 1760/2933 = 0.6

15.(c) $x'^2 + y'^2 + z'^2 = c^2t'^2$

17.(b) 510 mi/hr

19. $-0.357c$

21.(a) c

23.(a) 66.66666733 nsec
(b) 66.66666700 nsec
(c) 0.00000033 nsec

25.(a) 2.0006×10^{-4} mc^2
(b) -0.03%
(c) ½mv^2 is always too small
(d) $0.12c$

27.(a) 0.03% low

Chapter 21

1.(c) 36

3. 10

5. 56

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TABLE OF PHYSICAL CONSTANTS

Speed of light in vacuo, c	2.99792458×10^{8} meters/sec
Avogadro's number	6.022045×10^{23} mole^{-1}
Electron charge, e	$1.6021892 \times 10^{-19}$ coulomb
Planck's constant, h	4.135701×10^{-15} eV-sec
Atomic mass unit, u	931.5016 MeV/c^2
Electron mass	0.5110034 MeV/c^2 = 9.109534×10^{-31} kg
Proton mass	938.2796 MeV/c^2
	= 1836.15152 m_{electron}
	= 1.007276470 u = $1.6726485 \times 10^{-27}$ kg
Bohr magneton, μ_B	$0.57883785 \times 10^{-10}$ MeV/tesla
Electron magnetic moment	1.00115965241 μ_B
Nuclear magneton, μ_N	$3.1524515 \times 10^{-14}$ MeV/tesla
Proton magnetic moment	2.7928456 μ_N = 0.001521032209 μ_B
Rydberg	13.605804 eV
Coulomb constant, $k_e = 1/4\pi\epsilon_o$	8.987552×10^{9} newton m^2/coulomb^2
Faraday	9.648456×10^{4} coulombs/mole
Ideal gas constant, R	8.31441 joules/mole kelvin
Boltzmann's constant, k	8.61735×10^{-5} eV/Kelvin
Gravitational constant, G	6.6720×10^{-11} m^3-kg^{-1}-s^{-2}

Conversion Factors

1 GeV = 10^3 MeV = 10^9 eV = $1.6021892 \times 10^{-10}$ joules
Wavelength of 1-eV photon = 1.2398520 μm
Temperature for 1 eV per particle = 11604.50 kelvin (from $E = kT$)
1 year (sidereal) = 365.256 days = 3.1558×10^7 seconds
1 coulomb = 2.99792458×10^9 statcoulombs
1 inch = 2.54 cm
1 gallon = 3.785 liters
1 pound = 0.4536 kg

Some Astronomical Data (Rough)

Radius of earth	6.37×10^{6} m
Mass of earth	5.98×10^{24} kg
Radius of sun	6.96×10^{8} m
Mass of sun	1.99×10^{30} kg
Earth-sun distance	1.5×10^{11} m

The Greek Alphabet

alpha	Α	α
beta	Β	β
gamma	Γ	γ
delta	Δ	δ
episilon	Ε	ϵ
zeta	Ζ	ζ
eta	Η	η
theta	Θ	θ
iota	Ι	ι
kappa	Κ	κ
lambda	Λ	λ
mu	Μ	μ
nu	Ν	ν
nu	Ν	ν
omicron	Ο	ο
pi	Π	π
rho	Ρ	ρ
sigma	Σ	σ
tau	Τ	τ
upsilon	Υ	υ
phi	Φ	ϕ
chi	Χ	χ
psi	Ψ	ψ
omega	Ω	ω

Powers of 10

Equals	**Prefix**	**Symbol**
10^{-15}	femto	f
10^{-12}	pico	p
10^{-9}	nano	n
10^{-6}	micro	μ
10^{-3}	milli	m
10^{-2}	centi	c
10^{-1}	deci	d
10^{0}		
10^{1}	deka	da
10^{2}	hecto	h
10^{3}	kilo	k
10^{6}	mega	M
10^{9}	giga	G
10^{12}	tera	T

PERIODIC TABLE OF ELEMENTS

The symbol for each element is preceded by the atomic number; below is the atomic mass of the element as it occurs naturally on the earth. The atomic mass unit (u) is defined so the mass of a ^{12}C atom is exactly 12 u. The mass of carbon is listed as 12.01115 u because naturally occurring carbon is 98.89% ^{12}C and 1.11% ^{13}C. For artificially produced elements, the approximate atomic mass of the most stable isotope is given in brackets.

Group→		I	II	III	IV	V	VI	VII	VIII			O
Period	Series											
1	1	1 H 1.00797										2 He 4.0026
2	2	3 Li 6.939	4 Be 9.0122	5 B 10.811	6 C 12.01115	7 N 14.0067	8 O 15.9994	9 F 18.9984				10 Ne 20.183
3	3	11 Na 22.9898	12 Mg 24.312	13 Al 26.9815	14 Si 28.086	15 P 30.9738	16 S 32.064	17 Cl 35.453				18 A 39.948
4	4	19 K 39.102	20 Ca 40.08	21 Sc 44.956	22 Ti 47.90	23 V 50.942	24 Cr 51.996	25 Mn 54.9380	26 Fe 55.847	27 Co 58.9332	28 Ni 58.71	
	5	29 Cu 63.54	30 Zn 65.37	31 Ga 69.72	32 Ge 72.59	33 As 74.9216	34 Se 78.96	35 Br 79.909				36 Kr 83.80
5	6	37 Rb 85.47	38 Sr 87.62	39 Y 88.905	40 Zr 91.22	41 Nb 92.906	42 Mo 95.94	43 Tc [99]	44 Ru 101.07	45 Rh 102.905	46 Pd 106.4	
	7	47 Ag 107.870	48 Cd 112.40	49 In 114.82	50 Sn 118.69	51 Sb 121.75	52 Te 127.60	53 I 126.9044				54 Xe 131.30
6	8	55 Cs 132.905	56 Ba 137.34	57–71 Lanthanide series*	72 Hf 178.49	73 Ta 180.948	74 W 183.85	75 Re 186.2	76 Os 190.2	77 Ir 192.2	78 Pt 195.09	
	9	79 Au 196.967	80 Hg 200.59	81 Tl 204.37	82 Pb 207.19	83 Bi 208.980	84 Po [210]	85 At [210]				86 Rn [222]
7	10	87 Fr [223]	88 Ra [226.05]	89–Actinide series**								

*Lanthanide series:	57 La 138.91	58 Ce 140.12	59 Pr 140.907	60 Nd 144.24	61 Pm [145]	62 Sm 150.35	63 Eu 151.96	64 Gd 157.25	65 Tb 158.924	66 Dy 162.50	67 Ho 164.930	68 Er 167.26	69 Tm 168.934	70 Yb 173.04	71 Lu 174.97
**Actinide series:	89 Ac [227]	90 Th 232.038	91 Pa [231]	92 U 238.03	93 Np [237]	94 Pu [242]	95 Am [243]	96 Cm [247]	97 Bk [247]	98 Cf [249]	99 Es [254]	100 Fm [257]	101 Md [256]	102 No [253]	103 Lw [260]
	104 [261]	105 [262]	106 [263]												